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ON MEASURES OF NONCOMPACTNESS IN TOPOLOGICAL VECTOR SPACES

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Abstract: In this paper we give some axiomatic concept for measure of noncompactness which is useful in applications to the finite or infinite system of equations. In particular, fixed point theorems of Darbo type (cf.[5]) are proved.

Key words: Fixed point, measure of noncompactness, spectral radius, system of ordinary differential equations.

Classification: 47H10, 34G20

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1. Introduction. Let $L(X)$ be the algebra of continuous linear operators from a normed space $(X, \|\cdot\|)$ into itself with the standard norm $|\cdot|$, and let $r(A)$ $(= \lim_{n\to\infty} |A^n|^{1/n})$ be the spectral radius of $A \in L(X)$. It is known (see [9]) that for A in $L(X)$ and $\varepsilon > 0$ there exists a norm $\|\cdot\|_\varepsilon$ on X equivalent to $\|\cdot\|$ and such that $|A|_\varepsilon$ $(= \sup\{\|Ax\|_\varepsilon : \|x\|_\varepsilon \leq 1\}) \leq \varepsilon + r(A)$.

This note had been inspired by the above result. We present an axiomatic approach to the measure of noncompactness of sets and establish theorems of Darbo type (cf.[5]). In particular, a fixed point result for a system of k-set contractions (cf.[5]) is proved. Some applications are given.

2. Main theorems. For an arbitrary bounded subset X of a Banach space M, the measure $\alpha(X)$ of noncompactness of X, introduced by K. Kuratowski, is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of X by sets of diameter $\leq \varepsilon$. (For properties of Kuratowski function α, see e.g. [7] or [9].) There are some other definitions of measure of noncompactness (cf. [15]). Next, we give some axiomatic concept which is useful in applications to the finite or infinite systems of equations.

Throughout the rest of this section E will denote a Hausdorff locally convex topological vector space. For subset X of E, we shall denote their closure by $\overline{X}$ and their closed convex hull by $\overline{\mathrm{conv}}\,(X)$, and $F[X]$ will denote the image of X under a self-map F of E.

Definition. Let B be a Banach space and let S be a cone in B generating the partial order $\leq_S$. Assume that S_∞ is some set containing S. In S_∞ we introduce the relation $\leq$ by

$$x \leq y \text{ means} \begin{cases} x,\ y \in S \text{ and } x \leq_S y; \\ x,\ y \in S_\infty \setminus S \text{ and } x = y; \\ x \in S \text{ and } y \in S_\infty \setminus S. \end{cases}$$

We call a function $\Phi_S : 2^E \to S_\infty$ (2^E denote a family of all nonempty subsets of E) an S-generalized measure of noncompactness on E if for every point x in E and every subsets X, Y of E we have (1) $\Phi_S(X \cup \{x\}) = \Phi_S(X)$, (2) if $X \subset Y$ then $\Phi_S(X) \leq \Phi_S(Y)$, and (3) if $\Phi_S(X) = \emptyset$ ($\emptyset$ denotes the zero of B) then X is relatively compact subset of E. If B is an Euclidean space with the cone $S = [0,\infty)$ and $S_\infty = [0,+\infty]$, then our function Φ_S be called a generalized measure of noncompactness on E.

<u>Theorem 1</u>. Let K be a nonempty convex closed subset of E and let Φ_S be an S-generalized measure of noncompactness on E such that $\Phi_S(K) \in S$ and $\Phi_S(\overline{\text{conv}}\ X) = \Phi_S(X)$ for each subset X of K. Let $A \in L(B)$ be an operator with the spectral radius less than 1 and the property that $A[S] \subset S$. Assume, moreover, that F is a continuous mapping of K into itself and $\Phi_S(F[X]) \leq A(\Phi_S(X))$ for each subset X of K. Then the set of fixed points of F is nonempty and compact.

<u>Proof</u>. Suppose that X is a subset of K such that $\Phi_S(F[X]) = \Phi_S(X)$. Then $\Phi_S(X) \in S$ and $\Phi_S(X) \leq A(\Phi_S(X))$; thus $\Phi_S(X) \leq A^n(\Phi_S(X))$ for n = 1,2,... . Since the equation y = Ay has exactly one solution and $A^n(\Phi_S(X)) \longrightarrow 0$ as $n \longrightarrow \infty$ (cf. Th. I.2.2.9 in [11]), we conclude that $-\Phi_S(X) \in S$ and therefore $\Phi_S(X) = \emptyset$. Thus X is relatively compact.

Let us put $X_0 = \{x_m : m = 0,1,\dots\}$ with x_0 in K and $x_n = Fx_{n-1}$ for $n \geq 1$. We have $\overline{X}_0 \subset K$ and $\Phi_S(X_0) = \Phi_S(F[X_0])$. Since $\overline{X}_0$ is compact and F maps $\overline{X}_0$ into itself there exists a nonempty subset Z_0 of $\overline{X}_0$ such that $F[Z_0] = Z_0$ (see [2], Th. VI.1.8).

Now, let $\mathcal{V}$ denote the class of all subsets V of K such that $Z_0 \subset V$, $\overline{\text{conv}}\ (V) = V$ and $F[V] \subset V$. We have $K \in \mathcal{V}$ and $\overline{\text{conv}}\ (F[V]) \in \mathcal{V}$ whenever $V \in \mathcal{V}$. Further, put $V_0 = \bigcap \{V : V \in \mathcal{V}\}$. Then $V_0 \in \mathcal{V}$, and we get $V_0 = \overline{\text{conv}}\ (F[V_0])$. Consequently F is a continuous mapping of the convex compact V_0

into itself and by Schauder-Tychonoff fixed point theorem, F has a fixed point. This finishes the proof.

Theorem 2. Let Φ be a generalized measure of noncompactness on E with the property that condition (1) is replaced by (1'): $\bigcap_{n\geq 1} X_n$ is nonempty compact whenever $\{X_n: n = 1,2,\ldots\}$ is a family of nonempty closed subsets of E such that $X_{n+1}\subset X_n$ for $n\geq 1$ and $\Phi(X_n)\to 0$ as $n\to\infty$. Let K be a nonempty convex closed subset of E, $\Phi(K)<\infty$ and $\Phi(\overline{\mathrm{conv}}\ X) = \Phi(X)$ for each subset X of K. Assume, moreover, that F is a continuous mapping of K into itself and $\Phi(F[X])\leq p(\Phi(X))$ for every subset X of K, where p is a right continuous function on $[0,\infty)$ with $p(t)<t$ for $t>0$. Then the set $\{x\in K: Fx = x\}$ is nonempty and compact.

The proof of this result, although more difficult, resembles that of Darbo theorem (cf. Th. IV.3.2 in [9]) and therefore will be omitted.

The above result has interesting applications, whose basic ideas are illustrated by the examples below.

Example 1. First, we consider a "discrete measure of noncompactness" σ on our space E with the weak topology $\sigma(E,E^*)$. Define

$$\sigma(X) = \begin{cases} 0, & \text{iff } X \text{ is } \sigma(E,E^*)\text{-relatively compact;} \\ 1, & \text{iff } X \text{ is not } \sigma(E,E^*)\text{-relatively compact} \end{cases}$$

for subsets X of E. Obviously, σ is monotone and the assumption (1') is satisfied for $\Phi = \sigma$. Therefore from the above theorem we deduce a fixed-point result whenever all bounded sets of E are $\sigma(E,E^*)$-relatively compact. More precisely, we get the following result:

Suppose K is a nonempty convex closed bounded set in E. If E is a semireflexive space and if F is a weakly-weakly continuous mapping (i.e., the topology on both the domain and the range is the weak topology $\sigma(E,E^*)$) of K into itself, then the set of fixed points of F is nonempty and $\sigma(E,E^*)$-compact.

Example 2. Here we define a measure of weak noncompactness on a Banach space M (see De Blasi [6]): Denote by U the norm unit ball in M. The measure of weak noncompactness $\beta(X)$ of a nonempty subset X of M is defined as the infimum of all $\varepsilon > 0$ such that there exists a weakly compact subset C of M with $X \subset C + \varepsilon U$.

The functions α and β are examples of generalized measures of noncompactness with the properties listed in Theorem 2. In particular this theorem yields the following corollary:

Let M be a Banach space, let K be a nonempty convex closed bounded set in M, and suppose F is a weakly-weakly continuous mapping of K into itself such that $\beta(F[X]) \leq p(\beta(X))$ for every subset X of K and with a function p as in Theorem 2. Then the set of fixed points of F is nonempty and weakly compact.

Example 3. To conclude this section we give an example showing how generalized measures of noncompactness can be used to obtain a result of Nashed and Wong type (cf. e.g. Th. 2 in [10]).

Let B be a Banach space with the measure of noncompactness α, and let K be a nonempty convex closed bounded subset of B. Suppose we are given: L - a linear continuous mapping of B into itself such that some iterate L^n of it is a k-set contraction on K (i.e., $\alpha(L^n[X]) \le k \cdot \alpha(X)$ for each subset X of K, and f - a continuous mapping from K into a compact subset of B.

For a bounded subset X of B, $\Phi(X)$ is defined as $\sum_{i=1}^{n} k^{(n-i)/n} \alpha(L^{i-1}[X])$; if the diameter of X is infinite, then let $\Phi(X)$ be equal $+\infty$. It is not hard to see that Φ is a generalized measure of noncompactness on B such that $\Phi(\overline{\text{conv}}\ X) = \Phi(X)$ and $\Phi(X + Y) \le \Phi(X) + \Phi(Y)$. Now for any subset X of K, $\Phi(\{Lx + fx : x \in X\}) \le \Phi(L[X]) = \sum_{i=1}^{n} k^{(n-i)/n} \alpha(L^{i}[X]) \le k^{1/n} \Phi(X)$ and therefore, we have the following corollary to Theorem 1:

If $k < 1$ and $Lx + fx \in K$ for each x in X, then the equation $Lx + fx = x$ has a solution in K.

3. <u>An existence theorem for system differential equations</u>. In this part we assume that $(E_i, \|\cdot\|_i)$ $(i = 1,2,\ldots,n)$ is a Banach space with the measure of noncompactness α_i, and nonempty convex closed bounded subsets K_i.

First, we prove a fixed point result for a system of k-set contractions:

Let F_i $(i = 1,2,\ldots,n)$ be a continuous mapping from $K = K_1 \times K_2 \times \ldots \times K_n$ into K_i such that $\alpha_i(F_i[X_1 \times X_2 \times \ldots \times X_n]) \le \sum_{j=1}^{n} k_{ij} \alpha_j(X_j)$ for each subset X_j of K_j $(j = 1,2,\ldots,n)$. Assume, moreover, that $[k_{ij}]$ $(i,j = 1,2,\ldots,n)$ is a matrix with the spectral radius less than one (i.e. equivalently

$$\begin{vmatrix} 1-k_{11} & -k_{12} & \cdots & -k_{1i} \\ -k_{21} & 1-k_{22} & \cdots & -k_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ -k_{i1} & -k_{i2} & \cdots & 1-k_{ii} \end{vmatrix} > 0$$

for all $i = 1,2,\ldots,n$. Then there exists a point $(x_1,x_2,\ldots,x_n)$ in K such that $x_i = F_i(x_1,x_2,\ldots,x_n)$ for $i = 1,2,\ldots,n$.

To prove this claim let $F = (F_1,F_2,\ldots,F_n)$ and let L denote the linear operator generated by the matrix $[k_{ij}]$. Let us put $\Phi_S(X) = (\tilde{\alpha}_1(X_1), \tilde{\alpha}_2(X_2),\ldots, \tilde{\alpha}_n(X_n))$ for each subset $X = X_1\times X_2\times\ldots\times X_n$ of $E = E_1\times E_2\times\ldots\times E_n$, where $\tilde{\alpha}_i(X_i)$ is equal to $\alpha_i(X_i)$ or $+\infty$ if diameter of X_i is finite or infinite, respectively. Then Φ_S is an S-generalized measure of noncompactness on E which has the properties listed in Theorem 1 where B is the n-dimensional Euclidean space with the cone S of nonnegative coordinates and $S_\infty = \{(q_1,q_2,\ldots,q_n):0\le q_i\le +\infty\}$. This completes the proof.

Now, let us put $I = [0,1]$, $B_i = \{x\in E_i: \|x\|_i\le r\}$ for $i = 1,2,\ldots,n$ and $B = B_1\times B_2\times\ldots\times B_n$. Let f_i ($i = 1,2,\ldots,n$) be a bounded continuous function from $I\times B$ into E_i such that $\sup\{\|f_i(t,x)\|_i:(t,x)\in I\times B\}\le r$, and $\alpha_i(f_i[I\times X]) \le \sum_{j=1}^{n} k_{ij}\alpha_j(X_j)$ for any subset $X = X_1\times X_2\times\ldots\times X_n$ of B with $X_j\subset B_j$.

By (+) we shall denote the problem of finding the solution of the system of differential equations

$x_i' = f_i(t,x_1,x_2,\ldots,x_n)$ ($i = 1,2,\ldots,n$)

satisfying the initial conditions $x_i(0) = \emptyset_i$ ($\emptyset_i$ denotes the zero of E_i) for $i = 1,2,\dots,n$. The following theorem can be deduced from the above result:

<u>Theorem 3</u> (cf. [17]). Suppose that

$$k_{ij} = \begin{cases} 0 \text{ for } 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n \text{ with } j \neq i+1; \\ (n - i + 1)^{-1} \text{ for } 1 \leq i \leq n-1 \text{ and } j = i+1; \\ a_j \text{ for } i = n \text{ and } 1 \leq j \leq n \end{cases}$$

and $q = \sum_{j=1}^{n} ((n - j + 1)!)^{-1} a_j < 1$. Then there exists a solution of problem (+) defined on I.

<u>Proof</u>. Let us denote by $C(I,E_i)$ the space of continuous functions from an interval I to E_i, with the usual supremum norm $|||\cdot|||_i$ and the measure of noncompactness α^*_i. Moreover, let K_i be the set of all functions g in $C(I,E_i)$ such that $g(0) = \emptyset_i$ and $\|g(t) - g(s)\|_i \leq r|t - s|$ for $t,s \in I$, and let us put $K = K_1 \times K_2 \times \dots \times K_n$.

Define a continuous mapping $F = (F_1,F_2,\dots,F_n)$ as follows:

$(F_i x)(t) = \int_0^t f_i(s,x(s))ds$ for x in K. Assume that $X = X_1 \times X_2 \times \dots \times X_n$ with $X_i \subset K_i$. By the integral main-value theorem we have

$$\int_0^t f_i(s,x(s))ds \in t \cdot \overline{\mathrm{conv}}\,(\{f_i(s,x(s)) : 0 \leq s \leq t\})$$

for x in X. Hence

$$\sup_{t \in I} \alpha_i(\{\int_0^t f_i(s,x(s))ds : x \in X\}) \leq$$

$$\leq \alpha_i(f_i[I \times \bigcup\{x[I] : x \in X\}]) \leq$$

$$\leq \sum_{j=1}^{n} k_{ij}\, \alpha_j(\bigcup\{x_j[I] : x_j \in X_j\})$$

and, using the Ambrosetti lemma ([1], Lemma 2.2°), we obtain

$$\alpha_i^*(\{F_i x : x \in X\}) = \sup_{t \in I} \alpha_i(\{\int_0^t f_i(s,x(s))ds : x \in X\}) \leq$$
$$\leq \sum_{j=1}^{n} k_{ij}\, \alpha_j(\cup\{x_j[I] : x_j \in X_j\}) =$$
$$= \sum_{j=1}^{n} k_{ij} \sup_{t \in I} \alpha_j(\{x_j(t) : x_j \in X_j\}) =$$
$$= \sum_{j=1}^{n} k_{ij}\, \alpha_j^*(X_j).$$

The matrix $L = [k_{ij}]$ is such that the matrix $1 - L$ has the form

$$\begin{bmatrix} 1 & -k_{12} & 0 & \cdots & 0 & 0 \\ 0 & 1 & -k_{23} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -k_{n-1,n} \\ -k_{n1} & -k_{n2} & -k_{n3} & \cdots & -k_{n,n-1} & 1-k_{nn} \end{bmatrix}$$

with $k_{i,i+1} = (n - i + 1)^{-1}$ for $1 \leq i \leq n-1$ and $k_{ni} = a_i$ for $1 \leq i \leq n$. Since $\det(1 - L) = 1 - q > 0$, so L has spectral radius less than 1. Consequently, by our result there exists $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)$ in K such that $x_i^0(t) = (F_i x^0)(t)$ $(i = 1, 2, \ldots, n)$ for t in I.

4. Final remarks. Let I, $E = E_1$, $B = B_1$ and $C(I,E)$ be as in Theorem 3. We follow here the terminology of [3]. Denote by β the measure of weak noncompactness on E. Assume that f is a weakly-weakly continuous function from $I \times B$ into E, $\|f(t,x)\| \leq r$ on $I \times B$, and $\beta(f[I \times X]) \leq k \cdot \beta(X)$ for every subset X of B.

We study the Cauchy problem for the equation $x_w' = f(t,x)$ (here x_w' denotes the weak derivative) applying the method of

Euler polygons. Using this method we prove that if E is weakly sequentially complete, then there exists a weakly differentiable function x in C(I,E) such that $x(0) = \emptyset$ and $x_w'(t) = f(t,x(t))$ for t in I (see [3] and [16]), moreover, the set of weak solutions of our problem is compact and connected set in the space $C_w(I,E)$ of weakly continuous functions from I to E endowed with the topology of weak-uniform convergence. Further, we obtain a theorem on the existence of extremal weak integrals and a theorem on continuous dependence of extremal weak integrals on initial data. The idea of the above results is contained in [12] - [14] and [17]. Next, for the convenience of the reader we sketch a proof of the first of these results.

First, applying an argument analogous to [4] or [17] we conclude that there exist some h and a weakly compact set X_0 such that $X_0 = \overline{\bigcup_{0 \le \lambda \le h} \lambda \cdot \operatorname{conv}\,(f[J \times X_0])}$ with $J = [0,h] \subset I$. Denote by S_ε ($\varepsilon > 0$) the set of all functions v in C(J,E) having the following properties: $v(0) = \emptyset$, $\| v(t) - v(s) \| \le$ $\le r|t - s|$ for t and s in J, $\sup_{t \in J} \| v(t) - \int_0^t f(s,v(s))ds \| < \varepsilon$ (here the integral being taken in the weak Riemann sense), and $v(t) \in \bigcup_{0 \le \lambda \le t} \lambda \cdot \operatorname{conv}\,(f[J \times X_0])$ for t in J.

We call a polygon Euler line for our problem a function $v_\varepsilon : J \longrightarrow B$ $(0 < \varepsilon \le h)$ defined in the following manner: $v_\varepsilon(t) =$ $= \emptyset$ for $0 \le t \le \varepsilon$, $v_\varepsilon(t) = v_\varepsilon(t_i) + (t - t_i) f(t_i, v_\varepsilon(t_i))$ for $t_i \le t \le t_{i+1}$, where $t_i = i\varepsilon$ with $i = 1,2,\ldots$. We prove that for $n \ge 1$ there exists an Euler polygon line v_η such that $v_\eta \in S_{1/n}$ whenever $0 < \eta < \eta_0(n)$. Further, modifying the proof of Szufla [18] one can prove that the set $S_{1/n}$ is connected in $C_w(J,E)$. For $u \in \overline{S}^w_{1/n}$ (the closure of $S_{1/n}$ in $C_w(J,E)$) we have $u[J] \subset X_0$, and therefore $\overline{S}^w_{1/n}$ is compact in $C_w(J,E)$. Consequently $\bigcap_{n \geq 1} \overline{S}^w_{1/n}$ is a nonempty compact and connected subset of $C_w(J,E)$, whence our assertion follows easily.

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