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Oldřich Kowalski

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THE VOLUME CONJECTURE AND FOUR-DIMENSIONAL HYPERSURFACES

Oldřich KOWALSKI

Abstract: In this note we prove the volume conjecture by A. Gray and L. Vanhecke for the four-dimensional hypersurfaces of E^5 with the exception of a subclass of hypersurfaces satisfying a non-trivial geometric inequality.

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Let us consider the following "volume condition":

(V): For an analytic Riemannian manifold (M,g), suppose that any geodesic ball in (M,g) of sufficiently small radius $r>0$ has the same volume as the Euclidean ball of the same dimension and radius.

The volume conjecture by A. Gray and L. Vanhecke, [2], then says that (M,g) should be locally Euclidean. The volume conjecture has been proved in many important situations, for example, for all manifolds of dimension $n\leq 3$, for manifolds with non-positive, or non-negative Ricci curvature, for the products of surfaces, for the products of classical symmetric spaces, and so on. Little is known about the 4-dimensional Riemannian manifolds with the exception of the case when the metric is Ricci-parallel.

In all these results, what has been really used is not the strong condition (V) but only the information contained in the second order - and the fourth order term of the power-series expansion for the volume of a geodesic ball (with respect to its radius r). In other words, the following weaker condition has been used as the start point:

(V′): The volume of any small geodesic ball in (M,g) coincides with the volume of the corresponding Euclidean ball upto a remainder term of the form $O(r^5)$.

The purpose of this Note is to prove the following:

<u>Theorem</u>. Let M_4 be a four-dimensional analytic hypersurface of E^5 satisfying the weak volume condition (V′). Then either M_4 is locally Euclidean, or we have the inequality

$$(1) \qquad -6,9456\ldots \leqq (K/h^4) \leqq -3,9288\ldots$$

where K, or h, denotes the Gauss-Kronecker curvature, or the mean curvature of M_4, respectively.

<u>Proof</u>. We shall start with some preparations. For any Riemannian manifold (M,g), let us denote by R, ϱ, τ the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of (M,g), respectively. According to [1],[2], the condition (V′) is equivalent to the following couple of conditions:

$$(2) \qquad \begin{aligned} &a) \quad \tau = 0, \\ &b) \quad 3\|R\|^2 = 8\|\varrho\|^2, \end{aligned}$$

where $\|R\|$ and $\|\varrho\|$ denotes the norm of R and ϱ, respectively.

Consider a hypersurface $M \subset E^{n+1}$ $(n \geq 4)$ equipped with the induced Riemannian metric. At any fixed point $p \in M$, let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of the second fundamental form, and $s_1, s_2, \dots, s_n$ the corresponding elementary symmetric functions.

<u>Lemma</u>. At any point $p \in M$, the conditions a),b) from (2) are equivalent to the following conditions for the elementary symmetric functions:

a′) $s_2 = 0$,

b′) $s_1 s_3 = 7 s_4$.

<u>Proof</u> of the Lemma. Let p_k, $k=1,2,\dots$, denote the sum of the k-th powers of the eigenvalues λ_i. We shall use the following formulas by Newton (cf. [4]):

$$p_1 = s_1$$
$$p_2 = s_1 p_1 - 2s_2$$
$$p_3 = s_1 p_2 - s_2 p_1 + 3s_3$$
$$p_4 = s_1 p_3 - s_2 p_2 + s_3 p_1 - 4s_4.$$

Hence we obtain immediately

$$p_2 = (s_1)^2 - 2s_2,$$
$$p_3 = (s_1)^3 - 3s_1 s_2 + 3s_3,$$
$$p_4 = (s_1)^4 - 4(s_1)^2 s_2 + 4s_1 s_3 + 2(s_2)^2 - 4s_4.$$

Let us choose an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space T_pM which diagonalizes the second fundamental form (the "shape operator") S; then $S_{ij} = \delta_{ij} \lambda_i$ for $i,j = 1, \dots, n$. We have the Gauss equations

$$R_{ijk\ell} = S_{ik} S_{j\ell} - S_{i\ell} S_{jk}, \quad i,j,k,\ell = 1, \dots, n.$$

Hence $R_{ijij} = R_{jiji} = -R_{ijji} = -R_{jiij} = \lambda_i \lambda_j$ for any $i \neq j$, and $R_{ijk\ell} = 0$ whenever at least 3 indices are different. Further,

$$\rho_{ii} = \sum_{j=1}^{n} R_{ijij} = \sum_{\substack{j=1 \\ j\neq 1}}^{n} \lambda_i \lambda_j = \lambda_i s_1 - (\lambda_i)^2, \quad i=1,\dots,n$$

and $\rho_{ij} = 0$ for all $i,j = 1,\dots,n$, $i \neq j$. Finally,

$$\tau = \sum_{i=1}^{n} \rho_{ii} = (s_1)^2 - p_2.$$

From the Newton's formulas we see that $\tau = 0$ is equivalent to $s_2 = 0$. Now, we have

$$\|R\|^2 = 4 \sum_{1\le i<j\le n} (R_{ijij})^2 = 4 \sum_{1\le i<j\le n} (\lambda_i \lambda_j)^2 = 2(p_2^2 - p_4),$$

i.e.,

(3) $$\|R\|^2 = 8s_4 + 4s_2^2 - 8s_1 s_3,$$

and

$$\|\rho\|^2 = \sum_{i=1}^{n} (\rho_{ii})^2 = \sum_{i=1}^{n} (\lambda_i^2 s_1^2 - 2\lambda_i^3 s_1 + \lambda_2^4) =$$

$$= s_1^2 p_2 - 2s_1 p_3 + p_4 = -2s_1 s_3 + 2s_2^2 - 4s_4.$$

The relation $8\|\rho\|^2 = 3\|R\|^2$ then yields $s_4 = \dfrac{s_1 s_3}{7} + \dfrac{s_2^2}{14}$.

Hence the result follows.

<u>Proof of the Theorem</u>. Let us recall the definition of the Gauss-Kronecker curvature and the mean curvature for a hypersurface $M_4 \subset E^5$ (cf. [3]). Here we have $K = s_4 = \lambda_1 \lambda_2 \lambda_3 \lambda_4$, $h = s_1/4$. From (3) we get $\|R\|^2 = -48K$, and because $K = \frac{4}{7} h s_3$, we see that any of the relations $K = 0$, $h = 0$ implies that M_4 is locally Euclidean.

Suppose now that M_4 is not locally Euclidean and consider the characteristic equation of the second fundamental form:

$x^4 - s_1x^3 + s_2x^2 - s_3x + s_4 = 0$. We shall recall in brief the theory of a biquadratic equation. Consider the equation

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \tag{4}$$

and put

$$p = a_2 - \frac{3}{8}a_1^2$$

$$q = a_3 - \frac{1}{2}a_1a_2 + \frac{1}{8}a_1^3$$

$$r = a_4 - \frac{1}{4}a_1a_3 + \frac{1}{16}a_1^3a_2 - \frac{3}{256}a_1^4$$

The so-called <u>cubic resolvent</u> of the equation (4) is given by

$$t^3 + 2pt^2 + (p^2 - 4r)t - q^2 = 0.$$

The <u>discriminant</u> D of the equation (4) can be written in the form

$$D = 16p^4r - 4p^3q^2 - 128p^2r^2 + 144prq^2 + 256r^3 - 27q^4.$$

We have $D = \prod_{1 \leq i < j \leq 4} (\lambda_i - \lambda_j)^2$.

Now, the general theory (see [6]) says that the equation (4) has 4 simple real roots if and only if

$$D > 0,\ p < 0,\ p^2 - 4r > 0.$$

The equality $D = 0$ corresponds to the case of a multiple root.

In our case we have

$$p = -\frac{3}{8}s_1^2,\ q = -s_3 - \frac{1}{8}s_1^3,\ r = -\frac{3}{4}s_4 - \frac{3}{256}s_1^4.$$

Hence $p < 0$ iff $h \neq 0$, and $p^2 - 4r > 0$ iff $\frac{1}{16}s_1^4 + s_4 > 0$, i.e., $-16 < (K/h^4)$.

After a long but routine calculation we get

$$D = -\frac{243}{16}s_4^2 s_1^4 + \frac{81}{8}s_4 s_3 s_1^5 - \frac{27}{16}s_3^2 s_1^6 - 108 s_4^3 + \frac{81}{14}s_4 s_3^2 s_1^2 -$$

$$- \frac{27}{2}s_3^3 s_1^3 - 27 s_3^4.$$

Substituting now $s_1 = 4h$, $s_3 = \frac{7K}{4h}$, $s_4 = K$, we get

$$D = -27K^2h^4 \left[(7/16)^4(K/h^4)^2 + (102/(16)^2)(K/h^4) + 1\right]$$

and hence the condition $D \geq 0$ implies

$$(7/16)^4(K/h^4)^2 + (102/(16)^2)(K/h^4) + 1 \leq 0.$$

This is the case if and only if

$$-(51 + \sqrt{200})(16/49)^2 \leq K/h^4 \leq -(51 - \sqrt{200})(16/49)^2$$

which is the wanted inequality (1). The relation $-16 < K/h^4$ is a consequence of the above, thus it cannot bring in any new restrictions for our invariants. It can be also checked that in the case $D = 0$ our equation (4) has only real roots, too.

<u>Remark</u>. The inequality (1) in our theorem has an intrinsic meaning. In fact, because $K \neq 0$, the second fundamental form is non-degenerate and thus, following [5], it is uniquely determined by the metric of M_4 (upto a sign). Thus K/h^4 is a Riemannian invariant of M_4. It remains an open problem whether the inequality (1) is compatible with the strong volume condition (V), or not.

R e f e r e n c e s

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Matematicko-fyzikální fakulta, Universita Karlova, Sokolovská 83, 18600 Praha 8, Czechoslovakia

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