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# FINITELY GENERATED RELATIONS AND THEIR APPLICATIONS TO PERMUTABLE AND n-PERMUTABLE VARIETIES

Ivan CHAJDA and Jaromír DUDA

Abstract: The present paper is a continuation of the systematic study of compatible binary relations. This part deals with finitely generated compatible relations on universal algebras, their relationship and connections with permutability and n-permutability $(n>1)$ of congruences. Various modifications and simplifications of methods frequently used in the theory of Mal'cev conditions, polynomial conditions etc. are derived.

Key words: Algebraic function, congruence, compatible diagonal relation, Mal'cev condition, polynomial, polynomial condition, quasiorder, tolerance, variety of algebras.

Classification: 08A25

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The objective of this paper is to give connections among some recent and old trends in universal algebra from the point of view of principal congruences. Since various different ways for these investigations are used by many authors we shall first try to find a common base for their results by means of a detailed study of compatible binary relations. This approach enables us to obtain also some new characterizations of varieties of algebras.

1. Paraphrases of the Mal'cev lemma. Characterizations of a principal congruence $\Theta(a,b)$ for some elements a, b of an algebra $\mathcal{A}$ play an important role in universal algebra,

in particular in the theory of Mal´cev conditions, polynomial conditions etc. In the original Mal´cev description of $\Theta(a,b)$, see [10], there appears the set-theoretical condition $\{\varphi_i(a), \varphi_i(b)\} = \{z_i, z_{i+1}\}$; however, such condition is not too convenient for purely algebraic purposes, namely for deriving identities. Thus, the aim of this section is to remove the above mentioned set-theoretical equality; it was first done by G. Grätzer [2], further possibilities may be found in [13]. Making full use of the connections among congruences, tolerance and compatible diagonal relations we obtain Grätzer´s original result and, further, we give here a new purely algebraic description of $\Theta(a,b)$.

Let $\mathfrak{A} = \langle A, F\rangle$ be an algebra. A binary relation C on is called <u>compatible</u> if it satisfies the Substitution Property with respect to all operations from $F$, in other words, C is a subalgebra of the direct product $\mathfrak{A} \times \mathfrak{A}$. A binary relation R on is called <u>diagonal relation</u> if $\omega_A \subseteq R$ where $\omega_A = \{\langle a,a\rangle; a \in A\}$. By a <u>tolerance</u> on $\mathfrak{A}$ is meant a compatible diagonal and symmetric binary relation on $\mathfrak{A}$. Obviously, all tolerances as well as all compatible diagonal relations on $\mathfrak{A}$ form complete lattices with respect to the inclusion, see e.g. [5]. Consequently, for any $S \subseteq A \times A$ there exist the least compatible diagonal relation or the least tolerance on $\mathfrak{A}$ containing S, denote it by R(S) or T(S), respectively. Without risk of confusion we will use R(a,b) to denote $R(\{\langle a,b\rangle\})$ and $R(\langle a_1,b_1\rangle, \dots, \langle a_n,b_b\rangle)$ to denote $R(\{\langle a_1,b_1\rangle, \dots, \langle a_n,b_n\rangle\})$; analogously for T(a,b) and $T(\langle a_1,b_1\rangle, \dots, \langle a_n,b_n\rangle)$.

We begin with the following two lemmas; they will be useful in the sequel.

Lemma 1. Let $\mathfrak{A}$ be an algebra and let $x, y, a_1, \ldots$ $\ldots, a_n, b_1, \ldots, b_n$ be elements of $\mathfrak{A}$. Then

(a) $\langle x,y\rangle \in R(\langle a_1,b_1\rangle, \ldots, \langle a_n,b_n\rangle)$ if and only if there exists an n-ary algebraic function $\varphi$ over $\mathfrak{A}$ such that $x = \varphi(a_1,\ldots,a_n)$, $y = \varphi(b_1,\ldots,b_n)$ (briefly: $\langle x,y\rangle =$ $= (\varphi\times\varphi)(\langle a_1,b_1\rangle,\ldots,\langle a_n,b_n\rangle)$);

(b) $\langle x,y\rangle \in T(\langle a_1,b_1\rangle, \ldots, \langle a_n,b_n\rangle)$ if and only if there exists a 2n-ary algebraic function $\psi$ over $\mathfrak{A}$ such that $x = \psi(a_1,\ldots,a_n,b_1,\ldots,b_n)$, $y = \psi(b_1,\ldots,b_n,a_1,\ldots,a_b)$ (briefly: $\langle x,y\rangle = (\psi\times\psi)(\langle a_1,b_1\rangle,\ldots,\langle a_n,b_n\rangle, \langle b_1,a_1\rangle\ldots$ $\ldots, \langle b_n,a_n\rangle)$).

For the proof, see [5].

Lemma 2. Let $\mathfrak{A}$ be an algebra and let a, b be elements of $\mathfrak{A}$. The following conditions hold:

(a) $\Theta(a,b) = \bigcup_{n<\omega} \underbrace{T(a,b)\circ\ldots\circ T(a,b)}_{\text{n-times}}$;

(b) $\Theta(a,b) = \bigcup_{n<\omega} \underbrace{R(a,b)\circ R(b,a)\circ\ldots\circ R(a,b)}_{(2n-1)\text{-times}}$,

where $\circ$ denotes the relational product.

The proof is straightforward and hence omitted.

Theorem 1. Let $\mathfrak{A}$ be an algebra and let a,b,x,y be elements of $\mathfrak{A}$. The following conditions are equivalent:

(1) $\langle x,y\rangle \in \Theta(a,b)$;

(2) BINARY SCHEME: There exist and integer $n \geq 1$ and binary algebraic functions $\beta_1,\ldots,\beta_n$ over $\mathfrak{A}$ such that

$x = \beta_1(a,b)$

$\beta_i(b,a) = \beta_{i+1}(a,b)$ for $1 \leq i < n$

$y = \beta_n(b,a)$;

(3) GRÄTZER SCHEME: There exist an integer $n \geq 1$ and unary algebraic functions $\alpha_0, \ldots, \alpha_{2n-2}$ over $\mathfrak{A}$ such that

$$x = \alpha_0(a)$$

$$\left.\begin{array}{l} \alpha_{2i}(b) = \alpha_{2i+1}(b) \\ \alpha_{2i+1}(a) = \alpha_{2i+2}(a) \end{array}\right\} \text{ for } 0 \leq i \leq n-2$$

$$y = \alpha_{2n-2}(b).$$

Proof. The equivalence $(1) \Longleftrightarrow (2)$ follows directly from Lemma 1(b) and Lemma 2(a); the equivalence $(1) \Longleftrightarrow (3)$ (the original Grätzer's result, see [2; p. 342]) is a consequence of Lemma 1(a) and Lemma 2(b).

Remark 1. Lemma 2 gives rise to a problem: under which conditions does $T(a,b) = R(a,b) \circ R(b,a)$ follow? The subsequent Theorem 2 gives a solution for varieties of algebras in the form of polynomial conditions.

Theorem 2. Let V be a variety. The following conditions are equivalent:

(1) For each $\mathfrak{A} \in V$ and every two elements a, b of $\mathfrak{A}$, $T(a,b) = R(a,b) \circ R(b,a)$;

(2) For every pair of n-ary polynomials s, t and of (n + 1)-ary polynomials p, q there exists an (n + 2)-ary polynomial r such that: if $p(t(x_1,\ldots,x_n), x_1,\ldots,x_n) =$ $= q(t(x_1,\ldots,x_n), x_1,\ldots,x_n)$ then

$$p(s(x_1,\ldots,x_n),x_1,\ldots,x_n) = r(s(x_1,\ldots,x_n), t(x_1,\ldots,x_n), x_1,\ldots,x_n)$$

$$q(s(x_1,\dots,x_n),x_1,\dots,x_n) = r(t(x_1,\dots,x_n),\ s(x_1,\dots,x_n),\ x_1,\dots,x_n).$$

<u>Proof</u>. Clearly $T(a,b) \subseteq R(a,b) \circ R(b,a)$ for every algebra $\mathfrak{A}$ and each a, b of $\mathfrak{A}$. Hence, we shall proceed only to prove the equivalence of (2) with the converse inclusion:

(1) $\Rightarrow$ (2). Let $\mathfrak{A} = F_n(x_1,\dots,x_n)$ be the free algebra in V with free generators $x_1,\dots,x_n$ and let a, b be elements of $\mathfrak{A}$. Then there exist n-ary polynomials s, t with $a = s(x_1,\dots,x_n)$, $b = t(x_1,\dots,x_n)$. Suppose $\langle c,d\rangle \in R(a,b) \circ R(b,a)$. By Lemma 1(a), there exist $(k+1)$-ary polynomials p, q of V such that

$$c = p(a,u_1,\dots,u_k)$$
$$p(b,u_1,\dots,u_k) = q(b,v_1,\dots,v_k)$$
$$d = q(a,v_1,\dots,v_k).$$

Since $\mathfrak{A} = F_n(x_1,\dots,x_n)$, we can suppose $k = n$ and $v_i = u_i = x_i$ for $1 \le i \le n$, i.e. we get

$$c = p(s(x_1,\dots,x_n),\ x_1,\dots,x_n)$$
$$p(t(x_1,\dots,x_n),x_1,\dots,x_n) = q(t(x_1,\dots,x_n),x_1,\dots,x_n)$$
$$d = q(s(x_1,\dots,x_n),x_1,\dots,x_n).$$

Further, $\langle c,d\rangle \in T(a,b)$ yields (see Lemma 1(b)) the existence of a binary algebraic function $\wp$ over $\mathfrak{A}$ with

$$c = \wp(a,b),\quad d = \wp(b,a).$$

Consequently, there exists an $(n+2)$-ary polynomial r of V such that $\wp(w_1,w_2) = r(w_1,w_2,x_1,\dots,x_n)$ and, by replacing a,b,c,d by these polynomials, condition (2) immediately follows.

(2) $\Rightarrow$ (1). Let $\mathfrak{A} \in$ V, a,b,c,d be elements of $\mathfrak{A}$ and $\langle c,d\rangle \in R(a,b) \circ R(b,a)$. Then $\langle c,e\rangle \in R(a,b)$ and $\langle e,d\rangle \in R(b,a)$ for some element e of $\mathfrak{A}$, i.e., by Lemma 1(a), there exist

polynomials p, q of $\mathfrak{A}$ with

$$c = p(a,z_1,\dots,z_k)$$
$$e = p(b,z_1,\dots,z_k) = q(b,v_1,\dots,v_m)$$
$$d = q(a,v_1,\dots,v_m).$$

By applying the hypothesis, we get an (n + 2)-ary polynomial r of V such that n = k + m + 2 and

$$p(a,z_1,\dots,z_k) = r(a,b,a,b,z_1,\dots,z_k,v_1,\dots,v_m)$$
$$q(a,v_1,\dots,v_k) = r(b,a,a,b,z_1,\dots,z_k,v_1,\dots,v_m).$$

By Lemma 1(b), we conclude $\langle c,d\rangle \in T(a,b)$.

Remark 2. Although the condition (2) from Theorem 2 looks rather hard to be satisfied, it does hold in every permutable variety. This follows directly from the well-known fact that congruences, tolerances and compatible diagonal relations coincide on any algebra in a permutable variety, see [12], [4] and also the following Theorem 3.

## 2. Finitely generated compatible diagonal relations and n-permutable varieties.

Several important characterizations of n-permutable varieties ($n>1$) were derived by J. Hagemann and A. Mitschke. Making full use of their results, see [4] or [3] , we get the following description of n-permutable varieties in terms of finitely generated relations.

Theorem 3. Let $n \geq 1$ be an integer. Then for any variety V the following conditions are equivalent:

(1) V has (n + 1)-permutable congruences;

(2) For every $\mathfrak{A} \in$ V and each two elements a, b of $\mathfrak{A}$,

$$\theta(a,b) = \underbrace{R(a,b)\circ\dots\circ R(a,b)}_{n\text{-times}} .$$

**Proof.** (1) $\Longrightarrow$ (2). The inclusion $\Theta(a,b) \supseteq R(a,b) \circ \dots \circ R(a,b)$ is clear. Prove the converse inclusion. By [4], $(n+1)$-permutability of V implies $R^{-1} \subseteq \underbrace{R \circ \dots \circ R}_{n\text{-times}}$ and $\underbrace{R \circ \dots \circ R}_{(n+1)\text{-times}} \subseteq \underbrace{R \circ \dots \circ R}_{n\text{-times}}$ for every compatible diagonal relation R on $\mathfrak{A} \in V$. Hence $\underbrace{R \circ \dots \circ R}_{n\text{-times}}$ is a congruence relation on $\mathfrak{A}$. In particular, $\underbrace{R(a,b) \circ \dots \circ R(a,b)}_{n\text{-times}}$ is a congruence on $\mathfrak{A}$ collapsing the pair $\langle a,b \rangle$ thus $\Theta(a,b) =$
$= \underbrace{R(a,b) \circ \dots \circ R(a,b)}_{n\text{-times}}$ and (2) is proved.

(2) $\Longrightarrow$ (1). Let $F_2(x,y)$ be the free algebra of V with free generators x, y. By hypothesis, $\langle x,y \rangle \in \Theta(y,x) =$
$= \underbrace{R(y,x) \circ \dots \circ R(y,x)}_{n\text{-times}}$ holds, i.e. there are elements $a_1,\dots,a_{n+1} \in F_2(x,y)$ such that $x = a_1$, $y = a_{n+1}$ and $\langle a_i,a_{i+1} \rangle \in R(y,x)$ for $1 \leq i \leq n$. So, by Lemma 1(a), there exist unary algebraic functions $\varphi_1,\dots,\varphi_n$ over $F_2(x,y)$ satisfying $\langle a_i,a_{i+1} \rangle = (\varphi_i \times \varphi_i)(\langle y,x \rangle)$ for $1 \leq i \leq n$. Writing this separately in each variable, we get

$$x = \varphi_1(y)$$
$$\varphi_i(x) = \varphi_{i+1}(y) \text{ for } 1 \leq i < n$$
$$y = \varphi_n(x).$$

Since $\varphi_1,\dots,\varphi_n$ are algebraic functions over $F_2(x,y)$, there exist ternary polynomiasl $q_1,\dots,q_n$ of V with $\varphi_i(t) =$
$= q_i(x,t,y)$, $1 \leq i \leq n$, and

$$x = q_1(x,y,y)$$
$$q_i(x,x,y) = q_{i+1}(x,y,y) \text{ for } 1 \leq i < n$$
$$y = q_n(x,x,y);$$

i.e. we have the Mal'cev condition for $(n+1)$-permutable varieties, see [4] or [3], which completes the proof.

By a __quasiorder__ on an algebra $\mathfrak{A}$ is meant a compatible diagonal relation on $\mathfrak{A}$ which is also transitive. Clearly, also all quasiorders on $\mathfrak{A}$ form a complete lattice with respect to the inclusion, see e.g. [5], thus there exists the least one quasiorder on $\mathfrak{A}$ containing the pair $\langle a,b\rangle$ of elements of $\mathfrak{A}$ ; it will be denoted by $Q(a,b)$. Similarly the symbol $Q(\langle a_1,b_1\rangle,\dots,\langle a_n,b_n\rangle)$ denotes the least quasiorder on $\mathfrak{A}$ containing the pairs $\langle a_1,b_1\rangle,\dots,\langle a_n,b_n\rangle$. It is easily seen that $Q(a,b) = \bigcup_{n<\omega} \underbrace{R(a,b)\circ\dots\circ R(a,b)}_{n\text{-times}}$ ( = the transitive hull of $R(a,b)$) and so, forming the countable disjunctions of equivalent conditions from Theorem 3, we immediately get:

__Corollary 1.__ For a variety V, the following conditions are equivalent:

(1) V is $(n+1)$-permutable for some integer $n\geq 1$;

(2) $\theta(a,b) = Q(a,b)$ for any $a,b \in \mathfrak{A} \in V$.

Following [6], an algebra $\mathfrak{A}$ is called __Principal Tolerance Trivial__ (briefly: PTT) if $\theta(a,b) = T(a,b)$ for each a, b of $\mathfrak{A}$ .

A variety V is PTT if each $\mathfrak{A} \in V$ has this property. Notice that the PTT varieties form a very important class of varieties because it contains:

(i) all permutable varieties, see [12];

(ii) the variety of all distributive lattices, see [7];

(iii) all varieties of p-algebras, see [9].

The PTT-property is essentially used in the following

__Corollary 2.__ Let $n\geq 1$ be an integer. Then for any variety V the following conditions are equivalent:

(1) V is PTT and $(n + 1)$-permutable;

(2) For each $\mathfrak{A} \in V$ and every a, b of $\mathfrak{A}$,

$$T(a,b) = \underbrace{R(a,b) \circ \dots \circ R(a,b)}_{n\text{-times}}.$$

Proof. $(1) \Rightarrow (2)$. By Theorem 3, $\Theta(a,b) = \underbrace{R(a,b) \circ \dots \circ R(a,b)}_{n\text{-times}}$.

Since V is PTT, we have $\Theta(a,b) = T(a,b)$ proving (2).

$(2) \Rightarrow (1)$. Take $\mathfrak{A} = F_2(x,y) \in V$. Since the tolerance $T(y,x)$ is symmetric, we have $\langle x,y\rangle \in T(y,x)$ and thus, by hypothesis, $\langle x,y\rangle \in \underbrace{R(y,x) \circ \dots \circ R(y,x)}_{n\text{-times}}$. However, as was shown in the proof of Theorem 3, this condition implies the $(n + 1)$-permutability of V.

Further, by Theorem 3, the $(n+1)$-permutability of V implies $\Theta(a,b) = \underbrace{R(a,b) \circ \dots \circ R(a,b)}_{n\text{-times}}$ for every $a,b \in \mathfrak{A} \in V$. Combining this equality with (2), we get $\Theta(a,b) = T(a,b)$, i.e. V is PTT and the proof is complete.

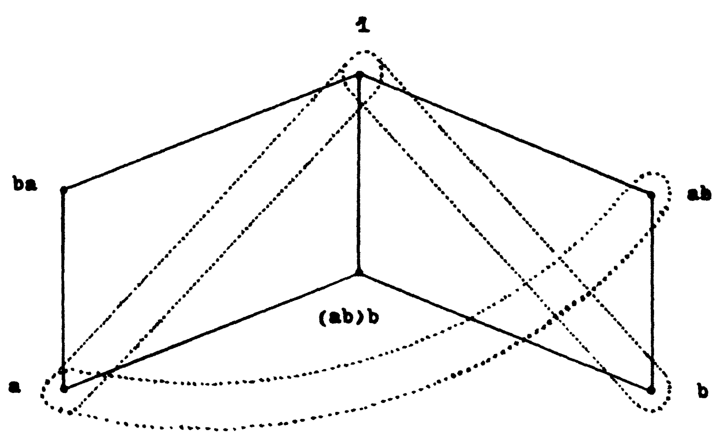
Remark 3. The Principal Tolerance Triviality and the n-permutability $(n \geq 3)$ are independent conditions:

(1) As was noted above, the variety $\mathbb{D}$ of distributive lattices is PTT; however, $\mathbb{D}$ is not n-permutable for any $n \geq 2$; see, e.g. [13, p. 79].

(2) The variety $\mathbb{I}$ of implication algebras, see [1], is 3-permutable; this is shown in [4], [11] or [3; p. 356]. It remains to prove that $\mathbb{I}$ is not PTT: Take the free algebra $F_2(a,b) \in \mathbb{I}$ with two free generators a, b. Let us recall, see [1], that this algebra is the grupoid $\langle \{1,a,b,ab,ba,(ab)b\}, \cdot \rangle$ with the following operational table:

| $\cdot$ | 1 | a | b | ab | ba | (ab)b |
|---|---|---|---|---|---|---|
| 1 | 1 | a | b | ab | ba | (ab)b |
| a | 1 | 1 | ab | ab | 1 | 1 |
| b | 1 | ba | 1 | 1 | ba | 1 |
| ab | 1 | a | (ab)b | 1 | ba | (ab)b |
| ba | 1 | (ab)b | b | ab | 1 | (ab)b |
| (ab)b | 1 | ba | ab | ab | ba | 1 . |

Further, it is well-known, see [1], [11], that any implication algebra $\langle I, \cdot\rangle \in \mathbb{I}$ may be expressed as a join semilattice $\langle I, \vee\rangle$ where $a \vee b := (ab)b$ and, conversely, $ab = (a \vee b)'_b$ (= the complement of $a \vee b$ in the principal filter $[b)$ of $\langle I, \vee\rangle$ ). In particular, the following diagram corresponds to the above mentioned implication algebra $F_2(a,b)$:



Now, consider the tolerance $T(a,ab)$ on $F_2(a,b)$. We have

$\langle a,1\rangle \in T(a,ab)$ since $\langle a,1\rangle = \langle ab,a\rangle \ \langle a,a\rangle$;

$\langle 1,b\rangle \in T(a,ab)$ since $\langle ab,ba\rangle = \langle (ab)b,(ab)b\rangle \ \langle ab,a\rangle$,

$$\langle (ab)b,b\rangle = \langle ab,ba\rangle \ \langle b,b\rangle ,$$

$$\langle 1,b\rangle = \langle a,1\rangle \ \langle (ab)b,b\rangle .$$

Suppose $T(a,ab) = \Theta(a,ab)$. Then $\langle a,1\rangle, \langle 1,b\rangle \in T(a,ab) = \Theta(a,ab)$ implies $\langle a,b\rangle \in \Theta(a,ab)$, i.e. we get $\langle a,b\rangle \in T(a,ab)$, a contradiction.

3. <u>Some characterizations of congruence permutability</u>. As was noted above, the relational equality $\Theta(a,b) = T(a,b)$, i.e. the PTT property, is a weaker condition than the permutability of congruences in a variety of algebras. Nevertheless, for two (and more) generating pairs of elements the following Theorem holds:

<u>Theorem 4</u>. For a variety V, the following conditions are equivalent:

(1) V has permutable congruences;

(2) $\Theta(\langle a,b\rangle, \langle b,c\rangle) = T(\langle a,b\rangle, \langle b,c\rangle)$ for each $\mathfrak{A} \in V$ and every a,b,c of $\mathfrak{A}$;

(3) $Q(\langle a,b\rangle, \langle b,c\rangle) = R(\langle a,b\rangle, \langle b,c\rangle)$ for each $\mathfrak{A} \in V$ and every a,b,c of $\mathfrak{A}$.

<u>Proof</u>. (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) follow directly from H. Werner's Theorem, see [12].

(2) $\Longrightarrow$ (1). Consider the equality $\Theta(\langle x,y\rangle, \langle y,z\rangle) = T(\langle x,y\rangle, \langle y,z\rangle)$ on the free algebra $F_3(x,y,z)$ in V. By the transitivity of congruences, we get $\langle x,z\rangle \in T(\langle x,y\rangle, \langle y,z\rangle)$

and thus, by Lemma 1(b), there is a 4-ary algebraic function $\sigma$ such that $\langle x,z\rangle = (\sigma \times \sigma)(\langle x,y\rangle, \langle y,z\rangle, \langle y,x\rangle, \langle z,y\rangle)$, i.e. $x = \sigma(x,y,y,z)$ and $z = \sigma(y,z,x,y)$. Since $\sigma$ is an algebraic function over the free algebra $F_3(x,y,z)$, we get a 7-ary polynomial s of V with

$$x = s(x,y,y,z,x,y,z)$$

$$z = s(y,z,x,y,x,y,z).$$

But $p(x,y,z) := s(x,z,y,y,x,y,z)$ is the well-known Mal´cev polynomial ($x = p(x,z,z)$, $z = p(x,x,z)$, see [10]), proving the permutability of congruences.

(3) $\Rightarrow$ (1). Analogously, the equality $Q(\langle x,y\rangle, \langle y,z\rangle) = R(\langle x,y\rangle, \langle y,z\rangle)$ on the free algebra $F_3(x,y,z)$ yields $\langle x,z\rangle \in R(\langle x,y\rangle, \langle y,z\rangle)$, and so $\langle x,z\rangle = (\tau \times \tau)(\langle x,y\rangle, \langle y,z\rangle)$ for some binary algebraic function $\tau$ over $F_3(x,y,z)$. So we have a 5-ary polynomial t of V with

$$x = t(x,y,x,y,z)$$

$$z = t(y,z,x,y,z).$$

Putting $p(x,y,z) := t(x,z,x,y,z)$, we again obtain the Mal´cev polynomial and (1) is thus proved.

Remark 4. The original strong Mal´cev condition characterizing permutable varieties, see [10], is very simple and useful for proving purposes if a given variety is permutable. However, if we proceed to prove the contrary, this condition is not too convenient. More suitable conditions for such a case are those of the foregoing Theorem 4.

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