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Antonín Sochor; Alena Vencovská

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INDISCERNIBLES IN THE ALTERNATIVE SET THEORY

A. SOCHOR, A. VENCOVSKÁ

Abstract: In the paper we prove the existence of classes of indiscernibles in the alternative set theory. These results are used to constructions of endomorphic universes with special properties.

Key words: Alternative set theory, indiscernibles, semiset, real class, endomorphism, endomorphic universe.

Classification: Primary 03E70, 03H99
Secondary 03H15

A class of indiscernibles (of strong indiscernibles respectively) is a class of natural numbers such that there are no two finite increasing sequences of its elements which can be distinguished using a set-formula without parameters (respectively with sets of small type as parameters only). We show that no two finite increasing sequences of elements of a cofinal class of strong indiscernibles can be distinguished using a normal formula with semisets of sets of small type as parameters only.

At the beginning of the first section we deal with the existence of cofinal classes of strong indiscernibles. No set-theoretically definable class can be a cofinal class of strong indiscernibles (neither a cofinal class of indiscernibles) and hence showing the existence of cofinal class of strong indiscernibles which is a π-class, we construct such a class of the smallest possible complexity. Furthermore we show that every real (in particular analytical) cofinal class of indiscernibles is a class of strong indiscernibles.

The second section contains two applications. In the alternative set theory an important role is played by endomorphic universes. There are natural characteristics of endomorphic universes e.g. the cut of all natural numbers α such that every subset of the endomorphic universe of the cardinality α is even an element of this endomorphic universe or the cut of all natural numbers α such that every element of the endomorphic universe of the cardinality α is even a subset of the endomorphic universe in question. The necessary and sufficient conditions for a cut to be the second characteristic of an endomorphic universe are known (cf. [S-V 4]). On the other hand, the first characteristic was not yet seriously studied. In the paper we show that the first characteristic cannot be naturally described from the second one.

If A is an endomorphic universe then every set-formula with parameters in A holds in the sense of A iff it holds in the sense of V. We construct an endomorphic universe such that the above mentioned equivalence holds even for seminormal formulas with subsemisets of A as parameters.

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§ 1. **We use the usual notions and results of the book** [V]. In particular $\tau(x)$ denotes the rank of x and $\overline{P}(\alpha) = \{x; \tau(x) \leq \alpha\}$ (cf. § 1 ch. II [V]). For every class C, the symbol Sd_C denotes the system of all classes of the form $\{x; \varphi(x)\}$ where φ is a set-formula of the language FL_C (i.e. with parameters in C). Elements of Sd_V are called set-theoretically definable classes. Let $G \in Sd_0$ denote in the whole paper a one-one mapping of N onto V (such a mapping is constructed in § 1 ch. II [V]). The function G induces by the natural way an ordering of V such that every set-theoretically definable class has the G-smallest element. As usual, the symbol $P_n(X)$ denotes the class $\{x \subseteq X; x \hat{\approx} n\}$.

<u>Theorem</u>. Let $T \in Sd_{\{a\}}$ be a function with $dom(T) = P_n(S)$ & $rng(T) \subseteq \{0,1\}$. If S is a proper class then there is a proper class $R \subseteq S$ such that $R \in Sd_{\{a\}}$ and such that $T''P_n(R)$ is a singleton.

Proof. For $n > 1$ we are going to define a function F by induction (cf. § 1 ch. II [V]). For $\alpha < n-1$ we define $F(\alpha)$ as the G-smallest set of $S - F''\alpha$.

Let $F \restriction \alpha$ be defined ($\alpha \geq n-1$) and let

$$X^{\alpha} = \{x \in S; (\forall \alpha_1, \dots, \alpha_n \in \alpha)(\alpha_1 < \dots < \alpha_n \rightarrow$$
$$\rightarrow T(\{F(\alpha_1), \dots, F(\alpha_n)\}) = T(\{F(\alpha_1), \dots, F(\alpha_{n-1}), x\}))\}$$

be a proper class. Then for some $y \in X^{\alpha}$, the class

$$\{x \in X^{\alpha}; (\forall \alpha_1, \dots, \alpha_{n-1} \in \alpha)(\alpha_1 < \dots < \alpha_{n-1} \rightarrow$$
$$\rightarrow T(\{F(\alpha_1), \dots, F(\alpha_{n-1}), y\}) = T(\{F(\alpha_1), \dots, F(\alpha_{n-1}), x\}))\}$$

is a proper class since otherwise for every f with $dom(f) = P_{n-1}(\alpha)$ & $rng(f) \subseteq \{0,1\}$ there would be β so that

$$(\forall z \in X^{\alpha})(T(\{F(\alpha_1),\dots,F(\alpha_{n-1}),z\}) = f(\alpha_1,\dots,\alpha_{n-1}) \rightarrow$$
$$\rightarrow \tau(z) < \beta)$$

and defining $H(f)$ as the smallest β with the above described property, H would be a function which is an element of $Sd_{\{a,\alpha\}}$ and with $dom(H) \subseteq \{f;\ dom(f) = P_{n-1}(\alpha)\ \&\ rng(f) \subseteq \{0,1\}\}$. Hence rng(H) would be a set by the replacement schema (see § 1 ch. I [V]; more precisely, we use the formal replacement schema which is a consequence of the formal axiom of induction - cf. the axiom A4 [S1]). This would contradict the assumption that X^{α} is a proper class. Therefore we are able to define $F(\alpha)$ as the G-smallest set y such that the class in question is a proper class and $y \notin F''\alpha$.

For the function F we have defined by induction, the class rng(F) is proper since dom(F) = N and F is a one-one mapping. Moreover, in the definition of F we have used only elements of $Sd_{\{a\}}$ and thence $F \in Sd_{\{a\}}$.

Defining $\tilde{T}(\{F(\alpha_1),\dots,F(\alpha_{n-1})\}) = T(\{F(\alpha_1),\dots,F(\alpha_{n-1}),F(\alpha_{n-1}+1)\})$ for every $\alpha_1 < \dots < \alpha_{n-1}$ we get a mapping of $P_{n-1}(rng(F))$ into $\{0,1\}$ which is an element of $Sd_{\{a\}}$. Moreover, if $R \subseteq rng(F)$ is a class such that $\tilde{T}''P_{n-1}(R)$ is a singleton then $T''P_n(R)$ is a singleton, too. Therefore we can finish the proof using the obvious induction w.r.t. n (the case n = 1 being trivial).

A class $I \subseteq N$ is called a class of indiscernibles iff for every set-formula $\varphi(z_1,\dots,z_n)$ of the language FL and every two sequences $e_1 < \dots < e_n$ and $e_1' < \dots < e_n'$ of elements of I we have $\varphi(e_1,\dots,e_n) \equiv \varphi(e_1',\dots,e_n')$.

A class $I \subseteq N$ is called a class of strong indiscernibles iff for every $e \in I$, every set-formula $\varphi(z_1,\ldots,z_n)$ of the language FL_e (i.e. elements of e are admitted as parametres) and every two sequences $e < e_1 < \ldots < e_n$ and $e < e_1' < \ldots < e_n'$ of elements of I we have $\varphi(e_1,\ldots,e_n) \equiv \varphi(e_1',\ldots,e_n')$.

Theorem. There is a class of indiscernibles which is a π-class and which is no semiset.

Proof. For every set-formula $\varphi(z_1,\ldots,z_n)$ of the language FL and every sequence $x_1 < \ldots < x_n$ of elements of N we define $T_\varphi(\{x_1,\ldots,x_n\}) = 1$ iff $\varphi(x_1,\ldots,x_n)$ and $T_\varphi(\{x_1,\ldots,x_n\}) = 0$ iff $\neg\varphi(x_1,\ldots,x_n)$. Evidently $T_\varphi \in Sd_o$ and $dom(T_\varphi) = F_n(N)$ for every set-formula $\varphi \in FL$. Let $\leq$ be a fixed well-ordering of FL of type ω. By the previous theorem there is a sequence of classes $\{S_\varphi : \varphi \in FL\}$ such that for every two set-formulas $\varphi, \psi \in FL$ we have $S_\varphi \in Sd_o \& \neg Sms(S_\varphi)$, $T_\varphi "P_n(S_\varphi)$ is a singleton and $\varphi \leq \psi \rightarrow S_\psi \subseteq S_\varphi$. The intersection of all classes S_φ where φ is a set-formula of the language FL is a π-class of indiscernibles. This class is no semiset according to the last part of § 5 ch. II [V]).

The previous statement can obviously be a little strengthened, namely for every set-theoretically definable proper class $R \subseteq N$ there is a class of indiscernibles $I \subseteq R$ which is a π-class and which is no semiset.

A class is real iff there is an indiscernibility equivalence (cf. ch. III [V]), such that the class in question is a figure in this equivalence. We shall use the following property of real classes.

Theorem. Let X be a real class, $\alpha \in N$. Then there is either a set u with $u \subseteq X$ & $u \hat{\approx} \alpha$, or for each $\gamma \in N-FN$ there is a set $u \supseteq X$ with $u \hat{\approx} \alpha \cdot \gamma$.

Proof. By the last theorem of [V 1] we can suppose that X is a figure in a totally disconnected indiscernibility equivalence $\doteq$. For the monad of the point x in this equivalence we use the notation $Mon^*(x)$. Let $\{S_n; n \in FN\}$ be a generating sequence of $\doteq$, such that S_n is an equivalence for each n (see ch. III [V]). If there is a set $x \in X$ and a set u such that $u \hat{\approx} \alpha$ and $u \subseteq Mon^*(x)$ then $u \subseteq X$ because X is a figure in $\doteq$ and therefore the first possibility holds. Suppose that $Mon^*(x)$ does not contain a subset $u \hat{\approx} \alpha$ for any $x \in X$. We have $Mon^*(x) = \bigcap \{o(x,n); n \in FN\}$, where $o(x,n) = \{y; \langle x,y \rangle \in S_n\}$. For $x \in X$ there must be $n \in FN$ such that $o(x,n) \hat{\prec} \alpha$; otherwise the classes $Y_n = \{v; v \hat{\approx} \alpha \; \& \; v \subseteq o(x,n)\}$ would form a countable descending sequence of nonempty set-theoretically definable classes with empty intersection, which is impossible. Put $u_n = \{o; o$ is an equivalence class in the equivalence S_n and $o \hat{\succeq} \alpha\}$. Each u_n is finite because $\doteq$ is compact and thus for every $n \in FN$ there is $k \in FN$ so that $\bigcup u_n \hat{\preceq} k \cdot \alpha$ Furthermore, for each $x \in X$ there is $n \in FN$ and $o \in u_n$ such that $x \in o$, i.e. $X \subseteq \bigcup \{\bigcup u_n; n \in FN\}$. By the axiom of prolongation, for each $\gamma \in N-FN$ there is a set $u \hat{\preceq} \gamma \cdot \alpha$ such that $X \subseteq u$.

Consequence. If X is a real class which is no semiset then for every α there is a set u with $u \hat{\approx} \alpha$ & $u \subseteq X$.

Theorem. If I is a real class of indiscernibles which is no semiset then I is a class of strong indiscernibles.

Proof. Let $\varphi(y_1,\dots,y_m,z_1,\dots,z_n)$ be a set-formula of the language FL and let $e\in I$ be given. By the last statement there is a set $x\subseteq I-e$ so that $(2^{e^m}+1)n\mathrel{\hat{\precsim}}x$. Hence there is a set $y\subseteq\{\langle e_1,\dots,e_n\rangle\in x^n;\ e_1<\dots<e_n\}$ such that $((\langle e_1,\dots,e_n\rangle\in y\ \&\ \langle e_1',\dots,e_n'\rangle\in y\ \&\ \langle e_1,\dots,e_n\rangle\neq$ $\neq\langle e_1',\dots,e_n'\rangle)\rightarrow(e_n<e_1'\vee e_n'<e_1))\ \&\ y\mathrel{\hat{\approx}}2^{e^m}+1$. Therefore there are $\langle\varepsilon_1,\dots,\varepsilon_n\rangle$ and $\langle\varepsilon_1',\dots,\varepsilon_n'\rangle$ elements of y such that $\varepsilon_n<\varepsilon_1'$ and such that the formula $(\forall q_1,\dots,q_m\in e)(\varphi(q_1,\dots,q_m,\varepsilon_1,\dots,\varepsilon_n)\equiv$ $\equiv\varphi(q_1,\dots,q_m,\varepsilon_1',\dots,\varepsilon_n'))$ holds.

Let $e<e_1<\dots<e_n$ and $e<e_1'<\dots<e_n'$ be two sequences of elements of I. Then there is a sequence $e_1''<\dots<e_n''$ of elements of I such that $e_n<e_1''\ \&\ e_n'<e_1''$. Since I is a class of indiscernibles, we have $(\forall q_1,\dots,q_m\in e)(\varphi(q_1,\dots,q_m,$ $e_1,\dots,e_n)\equiv\varphi(q_1,\dots,q_m,e_1'',\dots,e_n''))$ and also $(\forall q_1,\dots,q_m\in e)(\varphi(q_1,\dots,q_m,e_1',\dots,e_n')\equiv$ $\equiv\varphi(q_1,\dots,q_m,e_1'',\dots,e_n''))$ from which the formula $(\forall q_1,\dots,q_m\in e)(\varphi(q_1,\dots,q_m,e_1,\dots,e_n)\equiv$ $\equiv\varphi(q_1,\dots,q_m,e_1',\dots,e_n'))$ follows.

Let us note that every π-class is a real class and therefore there is a class of strong indiscernibles which is no semiset.

Lemma. Let I be a class of strong indiscernibles, which is no semiset and let $e<e_0$ be two elements of I. Then for every $n\in FN$ we have $\overline{P}(e+n)\subseteq Def_{e_0}$.

Proof. We have $Def_{\{e\}}\cap N\subseteq e_0$, because otherwise there would be a set-formula $\varphi(z_1,z_2)$ of the language FL such that the formula $(\exists!\alpha)\varphi(\alpha,e)$ holds and such that the number β satisfying $\varphi(\beta,e)$ would be greater or equal to e_0. Let $e' > \beta$ be an element of I. We would have $(\exists\alpha)(\varphi(\alpha,e)\,\&$ $\&\ \alpha > e_0)$ and consequently $(\exists\alpha)(\varphi(\alpha,e)\ \&\ \alpha > e')$ which is a contradiction.

Consider the number $\alpha_n = \max\{G^{-1}(x);\ \tau(x) \leq e+n\}$, where G is the mapping mentioned at the beginning. Obviously $\alpha_n \in \mathrm{Def}_{\{e\}}$, it means that $\alpha_n < e_0$ and thus $\overline{P}(e+n) \subseteq G''e_0 \subseteq \mathrm{Def}_{e_0}$.

The lemma has important consequences, e.g. the following statement where Id denotes the identity mapping. (For notions of automorphism and similarity see ch. V [V].)

<u>Theorem</u>. Let I be a class of strong indiscernibles, which is no semiset and let $e < e_0 < e_1 < \dots < e_n$ and $e < e_0 < e_1' < \dots < e_n'$ be two sequences of elements of I. Then there is an automorphism F such that $F \upharpoonright \overline{P}(e)$ equals to $\mathrm{Id} \upharpoonright \overline{P}(e)$ and $F(\langle e_1,\dots,e_n\rangle) = \langle e_1',\dots,e_n'\rangle$.

Proof. In [Ve 2] was proved a statement concerning prolongation of similarities to automorphisms, which says: for a set u with $P(u) \supseteq u$ and a pair $\langle x',x\rangle$ there exists an automorphism F such that $F \upharpoonright u = \mathrm{Id} \upharpoonright u$ and $F(x) = x'$ iff the mapping $(\mathrm{Id} \upharpoonright \bigcup\{P^n(u);n \in FN\}) \cup \{\langle x',x\rangle\}$ is a similarity (the symbol $P^n(u)$ denotes the set obtained by application of the operation of power-set n-times, starting from u). In our case we know that $(\mathrm{Id} \upharpoonright e_0) \cup \{\langle e_1',\dots,e_n'\rangle, \langle e_1,\dots,e_n\rangle\}$ is a similarity according to the fact that I is a class of strong indiscernibles. Thus the lemma implies that $(\mathrm{Id} \upharpoonright \bigcup\{P^n(\overline{P}(e));n \in FN\}) \cup \{\langle e_1',\dots,e_n'\rangle, \langle e_1,\dots,e_n\rangle\}$ is a similarity, too (since $P^n(\overline{P}(e)) = \overline{P}(e+n)$).

<u>Theorem</u>. Let I be a class of strong indiscernibles which is no semiset. Then for every two sequences $e < e_0 < e_1 < \ldots < e_n$ and $e < e_0' < e_1' < \ldots < e_n'$ of elements of I and every formula $\varphi(z_1,\ldots,z_m,Z_1,\ldots,Z_m)$ of the language $FL_{\overline{P}(e)}$, the formula $\varphi(e_1,\ldots,e_n,X_1,\ldots,X_m) \equiv$ $\equiv \varphi(e_1',\ldots,e_n',X_1,\ldots,X_m)$ holds whenever $X_1,\ldots,X_m$ are subclasses of $\overline{P}(e)$.

Proof. By the previous theorem there is an automorphism F with $F \restriction \overline{P}(e) = Id \restriction \overline{P}(e)$ & $F''X_1 = X_1$ & ... & $F''X_m =$ $= X_m$ & $F(e_1) = e_1'$ & ... & $F(e_n) = e_n'$. Further according to the second theorem of § 1 ch. V [V] we have
$\varphi(e_1',\ldots,e_n',X_1,\ldots,X_m) \equiv \varphi(F(e_1),\ldots,F(e_n),F''X_1,\ldots,F''X_m) \equiv$
$\equiv \varphi(e_1,\ldots,e_n,X_1,\ldots,X_m)$.

<u>Theorem</u>. a) If I is an infinite class of indiscernibles then for every $e \in I$ and every infinite $x \subseteq I$ we have $e \notin Def_{I-\{e\}}$ and $x \notin Def_I$.

b) If I is a class of strong indiscernibles and if e_0 is an element of I such that $I-e_0$ is infinite then for every $e_0 < e \in I$ and every infinite $x \subseteq I-e_0$ we have $e \notin Def_{(I-\{e\}) \cup e_0}$ and $x \notin Def_{I \cup e_0}$.

Proof. We are going to prove the statement b), the first statement can be proved quite analogically. Let $e_0 < e$ and let $\varphi(z,z_1,\ldots,z_n)$ be a set-formula of the language FL_{e_0} such that there is a sequence $e_0 < e_1 < \ldots < e_n$ of elements of I so that $\varphi(e,e_1,\ldots,e_n)$ & $(\exists ! x)\varphi(x,e_1,\ldots,e_n)$ & $e \notin \{e_1,\ldots,e_n\}$ & $e > e_0$. Since $I-e_n$ is infinite, there is a sequence $e_0 < e_1' < \dots < e_n'$ of elements of I and $e' \in I$ so that $e \neq e' > e_0$ & $(0 < i \leq n \rightarrow (e_i < e \equiv e_i' < e \equiv e_i' < e'))$. Since I is a class of strong indiscernibles, we get $\varphi(e, e_1', \dots, e_n')$ & $\varphi(e', e_1', \dots, e_n')$ & $(\exists ! x)\, \varphi(x, e_1', \dots, e_n')$ which is a contradiction.

If an infinite $x \subseteq I - e_0$ would be an element of $Def_{I \cup e_0}$ then there would be finite $y \subseteq I \cup e_0$ so that $x \in Def_y$ and thus the G-smallest element of x-y would be an element of Def_y which is impossible as we have previously proved.

Consequence. If I is a real class of indiscernibles which is no semiset then $P(I) - Def_I \neq 0$.

Proof. Since I is real, there is an infinite $x \subseteq I$; moreover, I is a class of strong indiscernibles and therefore it is sufficient to use the last statement.

The assumption that I is real is essential in the last theorem as follows. Let A be an endomorphic universe which is no semiset and such that A has only finite subsets (every endomorphic universe which has a standard extension fulfils these properties; see [S-V 1]). Further let I be a subclass of A such that in the sense of A, the class I is a class of strong indiscernibles which is no semiset. Then I is no semiset and $P(I) = \{x; x \subseteq I \ \& \ Fin(x)\} \subseteq Def_I$. Let us recall that the formula φ^A is obtained from the formula φ by restricting set-quantifiers to elements of A and class quantifiers to subclasses of A. Furthermore let us remind that if A is an endomorphic universe then the equivalence $\varphi^A \equiv \varphi$ holds for every set-formula of the language FL_A. Thus to prove that I is a class of strong indiscernibles

(in the sense of V) it is sufficient to realize that the equivalence $((\forall q_1,\dots,q_m \in e_0)(\varphi(q_1,\dots,q_m,e_1,\dots,e_n) \equiv$
$\equiv \varphi(q_1,\dots,q_m,e_1',\dots,e_n')))^{A} \equiv$
$\equiv (\forall q_1,\dots,q_m \in e_0)(\varphi(q_1,\dots,q_m,e_1,\dots,e_n) \equiv$
$\equiv \varphi(q_1,\dots,q_m,e_1',\dots,e_n'))$ holds for every $e_0,e_1,\dots,e_n,e_1',\dots,e_n' \in I$.

§ 2. Let us consider two characteristics of endomorphic universes $PE(A) = \{\alpha ; (\forall x \mathrel{\hat{\approx}} \alpha)(x \subseteq A \rightarrow x \in A)$ & & $(\exists x \mathrel{\hat{\approx}} \alpha)(x \subseteq A)\}$ and $EP(A) = \{\alpha ; \alpha \subseteq A\}$. We are going to show that there is no normal formula $\psi(z,Z)$ of the language FL so that $PE(A) = \{x; \psi(x,EP(A))\}$ (even if we suppose that there is a set d with $A[d] = V$; the class $A[d]$ was defined in [S-V 1]). For this purpose it is sufficient to construct two endomorphic universes A, B and sets d, d_1 with $EP(A) = EP(B)$ & $PE(A) \neq PE(B)$ & $A[d] = B[d_1] = V$.

Let $B \neq V$ be a fully revealed endomorphic universe such that there is a set d_1 with $B[d_1] = V$ (by [S-V 1] there is an endomorphic universe $B' \neq V$ so that $B'[d_1] = V$, every revealment of B' fulfils our requirements and by [S-V 2] a revealment of B' exists). The class $PE(B)$ is fully revealed and hence we can choose $\alpha \in PE(B)-FN$. According to the first section we are able to construct in the sense of B a π-class I of strong indiscernibles. Let $EP(B) \subseteq \beta \in B$ and let us choose $e,e_0,d \in B$ such that in the sense of B we have $\{e,e_0\} \cup d \subseteq I - \beta$ & $d \mathrel{\hat{\approx}} \alpha$ & $(\forall e' \in d)(e < e' < e_0)$; such a choice is possible since I is in the sense of B a real class which is no semiset. Thus the formula $d \subseteq B$ follows from the assumption $\alpha \in$ PE(B) $\subseteq$ EP(B). Furthermore, by the first section we have $d \notin \mathrm{Def}_{I \cup e}$ and hence $d \notin \mathrm{Def}_{\beta \cup d \cup \{e_0\}}$ since $\mathrm{Def}_{\beta \cup d \cup \{e_0\}} \subseteq \mathrm{Def}_{e \cup I}$. By [Ve 2] the following theorem holds: If x, d' are sets such that $d' \in \bigcup \mathrm{Def}_x - \mathrm{Def}_x$ & $x \in \mathrm{Def}_{x \cup \{d'\}}$ then there is an endomorphic universe A with $A[d'] = V$ & & $d' \notin A$ & $x \subseteq A$. Putting $d' = d$ and $x = \beta \cup d \cup \{e_0\}$ we see that the assumptions hold since $x = (\beta \cup d \cup \{e_0\}) \in$ $\in \mathrm{Def}_{\{\beta - 1, e_0, d\}} \subseteq \mathrm{Def}_{x \cup \{d\}}$. Hence there exists an endomorphic universe $A \subset B$ so that $A[d] = B$ & $\beta \cap B \subseteq A$ & $d \subseteq A$. The class A is an endomorphic universe (in the sense of V) and moreover $A[\{d, d_1\}] = (A[d])[d_1] = B[d_1] = V$ (cf. [S-V 1]). It is EP(A) = EP(B) because $EP(B) = \{\gamma ; \gamma \subseteq B\} \subseteq \{\gamma ; \gamma \subseteq \beta \cap B\} \subseteq$ $\subseteq \{\gamma ; \gamma \subseteq A\} = EP(A) \subseteq \{\gamma ; \gamma \subseteq B\} = EP(B)$. On the other hand, $d \subseteq A$ & $d \notin A$ & $d \mathrel{\hat{\approx}} \alpha$ and thence $\alpha \notin PE(A)$ and therefore $PE(A) \neq PE(B)$ and we are done.

A formula is called seminormal iff it contains only quantifiers of the form $(\exists Z\ \mathrm{Sms}(Z))$. Let us note that every normal formula is (equivalent to) a seminormal one since every set is a semiset.

<u>Theorem</u>. Let I be a class of strong indiscernibles which is no semiset and let J be a subclass of I such that $\bigcup J$ is no σ-class. Then the class $A \doteq \{x ; (\exists e \in J)\ \tau(x) \leq e\}$ is an endomorphic universe such that for any seminormal formula $\varphi(Z_1, \dots, Z_n)$ of the language FL_A and any $X_1, \dots, X_n \subseteq A$ semisets in the sense of A we have $\varphi^A(X_1, \dots, X_n) \equiv \varphi(X, \dots, X_n)$.

Proof. A is a revealed class since $\bigcup J$ is no σ-class and moreover $\mathrm{Def}_A = A$ by the first section. Hence A is an endomorphic universe according to § 1 [S-V 1].

Our statement concerning seminormal formulas can be proved by induction on their length. The only nontrivial case is when φ has the form $(\exists Z\ Sms(Z))\ \psi(Z,Z_1,\dots,Z_n)$. Let us suppose that for any $X,X_1,\dots,X_n$ semisets in the sense of A we have $\psi^A(X,X_1,\dots,X_n) \equiv \psi(X,X_1,\dots,X_n)$. If $X_1,\dots,X_n$ semisets in the sense of A are given then $\varphi^A(X_1,\dots,X_n) \rightarrow$ $\rightarrow \varphi(X_1,\dots,X_n)$ because of $Sms^A(Z) \rightarrow Sms(Z)$. Let us choose $e \in J$ so that $X_1,\dots,X_n \subseteq \overline{P}(e)$ and so that all parameters occurring in ψ belong to $\overline{P}(e)$, further let X be a class with $Sms(X)\ \&\ \psi(X,X_1,\dots,X_n)$. Since $\cup J$ is no set, we are able to fix $e_0, e_1' \in J$ such that $e < e_0 < e_1'$ and according to the facts that I is no semiset and that X is a semiset, we can fix even $e_1 \in I$ such that $e_0 < e_1\ \&\ X \subseteq \overline{P}(e_1)$. By the first section we can find an automorphism F such that F is identical on $\overline{P}(e)$ and such that $F(e_1) = e_1'$. Thus we have $\psi(F''X,X_1,\dots,X_n)\ \&\ F''X \subseteq \overline{P}(e_1')$ from which $\varphi^A(X_1,\dots,X_n)$ follows.

Let us remind that if I is a π-class of strong indiscernibles which is no semiset, then we are able to find a descending sequence $\{e_n; n \in FN\}$ of elements of I and to define $J = I \cap \bigcap \{e_n; n \in FN\}$. Then $\cup J$ is a π-class and hence it is no σ-class according to the last statement of § 5 ch. II [V]. Thus we can fix in the alternative set theory classes having properties desired in the last theorem.

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Matematický ústav ČSAV, Žitná 25, Praha 1, Československo

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