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ON THE RELATION OF THREE-VALUED LOGIC TO MODAL LOGIC

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Abstract: Three-valued logic with the third value meaning "unknown" is investigated. Each model of this three-valued logic determines a Kripke model of modal logic - the set of all two-valued completions. The aim is to characterize such Kripke models by means of modal logic. This is achieved for propositional logic and (on a weak form) for monadic predicate logic.

Key words: Three-valued logic, modal logic, Kripke models.

Classification: 03B45, 03B50

The aim of this paper is to clarify the relationship between Kleene's three-valued logic and modal logic.

Kleene's 3-valued logic was introduced (by S.L. Kleene in 1952 [1]) as a formalization for incomplete models, i.e. models where some values of predicates are missing and has found applications in mechanized hypothesis formation [2], [3]. For further investigation of this logic see [4] and [8].

In Part I we discuss the 3-valued propositional calculus, in Part II the three-valued monadic predicate calculus.

I - Propositional Calculus

1.1. Kleene's 3-valued logic uses 3-valued models in which the third value denoted by cross " $\times$ " in the present paper represents an undefined part of a predicate in an incomplete model. This approach differs from other types of 3-valued logic by the requirement that all formulas valid (or invalid) in a 3-valued model must be valid (or invalid) in every its two-valued completion, i.e. in each model that results from the given three-valued model by changing all crosses to zeros and ones in an arbitrary way. From this requirement the definitions of connectives also arise:

The language L_P of propositional calculus consists of a countable (or finite) set P of propositional variables (denoted by p, q, r,...) and the connectives $\&$, $\vee$, $\neg$. Formulas are defined by induction as usual. The set of all formulas is denoted Fla (L_P).

A valuation N of L_P is a mapping from P into $\{0,1,\times\}$. The value of a formula φ in a valuation N (denoted by $\|\varphi\|_N$) is defined by induction with the help of truth-value function: if $0 < \times < 1$ is the natural ordering on $\{0,1,\times\}$ then

$$\|p\|_N = N(p) \text{ for } p \in P$$

$$\|\varphi \,\&\, \psi\|_N = \min(\|\varphi\|_N, \|\psi\|_N),$$

$$\|\varphi \vee \psi\|_N = \max(\|\varphi\|_N, \|\psi\|_N),$$

$$\|\neg\varphi\|_N = \neg\|\varphi\|_N \text{ where } \neg 0 = 1 \ , \neg 1 = 0, \ \neg\times = \times .$$

Implication $\varphi \rightarrow \psi$ can be defined in Kleene's logic as $\neg\varphi \vee \psi$. A formula φ is <u>valid</u> in N (denoted by $N \models \varphi$) if $\|\varphi\|_N = 1$. φ is <u>3-equivalent</u> to ψ ($\varphi \equiv_3 \psi$) if for every valuation, $\|\varphi\|_N = \|\psi\|_N$.

1.2. Basic definitions and facts

a) For every formula φ there is a valuation N such that

$$\|\varphi\|_N = \times .$$

Moreover, there is a valuation N such that for every φ , $\|\varphi\|_N = \times$ ($N(p) = \times$ for all $p \in P$).

b) A valuation N is a 2-valued valuation if the range of N is included in $\{0,1\}$, φ is 2-tautology if φ is valid in every 2-valuation, φ is 2-equivalent to ψ ($\varphi \equiv_2 \psi$) iff for every 2-valuation N, $\|\varphi\|_N = \|\psi\|_N$. Fact: φ is a 2-tautology iff for every valuation N, $\|\varphi\|_N \geq \times$.

c) A 2-valuation N is called a completion of M if for every $p \in P$, if $\|p\|_M \neq \times$ then $\|p\|_N = \|p\|_M$. The set of all completions of M is denoted by $\mathcal{D}(M)$.

d) For every φ and for every N, if $\|\varphi\|_N \neq \times$ then $\|\varphi\|_M = \|\psi\|_M$ for every $M \in \mathcal{D}(N)$. Thus, each formula of Kleene's logic is monotone in the sense of [5]. On the other hand, there is a formula which is valid in every completion M of N and nevertheless is not valid in N (e.g. $p \vee \neg p$ for $\|p\|_N = \times$).

1.3. Let P be a set of propositional variables, L_P the language of propositional calculus enriched by a modality $\Box$. The formulas of $L_P^{\Box}$ are called modal formulas (denoted by Φ , Ψ,...). The set of all such formulas is denoted $Fla(L_P^{\Box})$. The modality $\Box$ is read "necessarily", a unary modality $\Diamond$, defined by $\Diamond = \neg\Box\neg$ is read "possibly". Kripke model ([6]) of modal system S5 of the language $L_P^{\Box}$ is a system $\mathcal{K} = \langle \mathfrak{M}, \mathcal{R} \rangle$ where $\mathfrak{M}$ is a non-empty set of 2-valuations of L_P and $\mathcal{R}$ is a relation of equivalence on $\mathfrak{M}$.

The value of a formula in $\mathcal{K}$ is defined inductively as follows: first a value of Φ is defined in any $M \in \mathfrak{M}$ w.r.t. $\mathcal{K}$:

if $p \in P$ then $\|p\|_{\mathcal{K},M} = \|p\|_M$;
if $\Phi = \Phi_1 \& \Phi_2$ then $\|\Phi\|_{\mathcal{K},M} = \min(\|\Phi_1\|_{\mathcal{K},M}, \|\Phi_2\|_{\mathcal{K},M})$,

similarly for $\vee, \neg$;

if $\Phi = \Box \Phi_1$ then $\|\Phi\|_{\mathcal{K},M} = \min \{\|\Phi_1\|_{\mathcal{K},M};$
$N \in \mathfrak{M} \ \&\ N \mathcal{R} M\}$.

Φ is valid in M w.r.t. $\mathcal{K}$ if $\|\Phi\|_{\mathcal{K},M} = 1$.

Φ is valid in $\mathcal{K}$ if Φ is valid in every $M \in \mathfrak{M}$. In particular, if $\mathcal{R}$ contains only one equivalence class (then the corresponding S5 model is denoted by $\mathfrak{M}$) Φ is valid in $\mathcal{K}$ iff $\Box\Phi$ is valid in some (and thus in every) model $M \in \mathfrak{M}$.

Φ is S5-equivalent to Ψ if for each M and $\mathcal{K}$

$$\|\Phi\|_{\mathcal{K},M} = \|\Psi\|_{\mathcal{K},M}.$$

We say that a formula Φ is <u>boxed</u> if each occurence of a propositional variable is in the scope of a modality. For further information on modal logic see e.g. [7].

1.4. Observe that for each 3-valued valuation N, the system $\mathfrak{D}$(N) is a particular Kripke S5 model. Our question is to describe those Kripke models that are obtained as $\mathfrak{D}$(N) for some N. Clearly, we have to specify means of such description. Let us search for a description using the language of the modal logic. But by this language we cannot distinguish two modal models which satisfy the same boxed formulas. Such models will be called equivalent. Thus we shall give the necessary and sufficient condition for a Kripke S5 model to be equivalent to $\mathfrak{D}$(N) for some N.

1.5. <u>Definition</u>. A literal is a propositional variable or its negation. A conjunction of pairwise distinct literals is called a <u>fundamental conjunction</u> (FC); FC in which every propositional variable occurs no more than once is called <u>elementary conjunction</u> (EC).

1.6. Lemma. Each formula of L_P is 3-equivalent to some disjunction of fundamental conjunctions.

Proof. Easy to show.

1.7. Lemma. Let φ, $\psi \in Fla(L_P)$, let $\mathfrak{M}$ be an S5 model.

a) If φ implies ψ in the classical propositional calculus and $\mathfrak{M} \models \Diamond\varphi$ then also $\mathfrak{M} \models \Diamond\psi$. In particular, if $\mathfrak{M} \models \Diamond\varphi_i$ for at least one i = 1,...n then $\mathfrak{M} \models \Diamond \bigvee_{i=1}^{n} \varphi_i$.

b) If $\varphi \equiv_2 \psi$ then $\mathfrak{M} \models \Diamond\varphi$ iff $\mathfrak{M} \models \Diamond\psi$.

1.8. <u>Definition</u>. Let $\mathfrak{M}$, $\mathfrak{M}'$ be two S5 models. We say that they are <u>equivalent</u> (notation $\mathfrak{M} \equiv \mathfrak{M}'$) if for any $\Phi \in Fla(L_P^{\Box})$,

$$\mathfrak{M} \models \Phi \text{ iff } \mathfrak{M}' \models \Phi .$$

1.9. Lemma. Every formula $\Phi \in Fla(L_P^{\Box})$ is S5-equivalent to a disjunction of conjunctions of basic modal formulas, i.e. for every $\Phi \in Fla(L_P^{\Box})$,

$$\Phi \equiv_{S5} \bigvee_j \bigwedge_i M_{ij}\varphi_{ij}$$

where $M_{ij} \in \{\emptyset, \neg\Diamond, \Diamond\}$ and $\varphi_{ij} \in Fla(L_P)$. In particular, for every boxed formula Φ,

$$\Phi \equiv_{S5} \bigvee_j \bigwedge_i M_{ij}\varphi_{ij}$$

where $M_{ij} \in \{\Diamond, \neg\Diamond\}$ and $\varphi_{ij} \in Fla(L_P)$.

Proof. By the induction using well-known equivalences of the propositional calculus and the following evident S5-equivalences:

$$\Diamond(\varphi \vee \psi) \equiv_{S5} \Diamond\varphi \vee \Diamond\psi$$

$$\Diamond(\varphi \,\&\, \Diamond\psi) \equiv_{S5} \Diamond\varphi \,\&\, \Diamond\psi$$

$$\Diamond \Diamond \varphi \equiv_{S5} \Diamond \varphi$$

$$\Diamond \neg \Diamond \varphi \equiv_{S5} \neg \Diamond \varphi$$

1.10. Claim.

$\mathfrak{M} \equiv \mathfrak{M}'$ iff for every $\varphi \in Fla(L_p)$,

$\mathfrak{M} \models \Diamond \varphi$ iff $\mathfrak{M}' \models \Diamond \varphi$.

Proof. From the preceding lemma.

1.11. Proposition. Let N be a 3-valuation. Then $\mathcal{D}(N)$ with the trivial equivalence containing only one equivalence class ($\mathcal{D}(N) \times \mathcal{D}(N)$) is an S5-model. We call all such models T-models.

1.12. Theorem. An S5-model $\mathfrak{M}$ is equivalent to any T-model iff $\mathfrak{M}$ satisfies the following condition

(+) for every EC $\varphi \,\&\, \psi$,

if $\mathfrak{M} \models \Diamond \varphi$ and $\mathfrak{M} \models \Diamond \psi$ then $\mathfrak{M} \models \Diamond (\varphi \,\&\, \psi)$.

Proof. 1) Let $\mathfrak{M}$ be equivalent to a T-model (= $\mathcal{D}(N)$ for 3-valuation N). If $\mathfrak{M} \models \Diamond \varphi$ and $\mathfrak{M} \models \Diamond \psi$ then also $\mathcal{D}(N) \models \Diamond \varphi$ and $\mathcal{D}(N) \models \Diamond \psi$. Now, if $\varphi \,\&\, \psi$ is an elementary conjunction then no propositional variable occuring in φ occurs in ψ and vice versa. Thus if there is a completion N_1 of N such that $\| \varphi \|_{N_1} = 1$ and a completion N_2 of N such that $\| \psi \|_{N_2} = 1$ then take a completion M of N coinciding with N_1 on variables occuring in φ and with N_2 on other variables; clearly, $\| \varphi \,\&\, \psi \|_M = 1$. Thus $\mathcal{D}(N) \models \Diamond (\varphi \,\&\, \psi)$ and also $\mathfrak{M} \models \Diamond (\varphi \,\&\, \psi)$.

2) Suppose that $\mathfrak{M}$ is an S5-model satisfying (+). Let us define a 3-valuation N as follows:

$\|p\|_N = 1$ if $\mathfrak{M} \models \Box p$;

$\|p\|_N = 0$ if $\mathfrak{M} \models \Box \neg p$;

$\|p\|_N = \times$ otherwise, i.e. if $\mathfrak{M} \models \Diamond p$ and $\mathfrak{M} \models \Diamond \neg p$.

We shall show that $\mathfrak{M} \equiv \mathfrak{D}(N)$. Clearly $\mathfrak{M} \subseteq \mathfrak{D}(N)$ hence by the preceding Claim it suffices to show for every $\varphi \in \in Fla(L_P)$, if $\mathfrak{D}(N) \models \Diamond \varphi$ then also $\mathfrak{M} \models \Diamond \varphi$.

Suppose that $\mathfrak{D}(N) \models \Diamond \varphi$, i.e. that there exists $M \in \mathfrak{D}(N)$ such that $M \models \varphi$. By the Lemma 1.6, $\varphi \equiv_3 \bigvee_{i=1}^{n} K_i$ where K_i are FC for $i = 1,\dots n$. If $M \models \bigvee_{i=1}^{n} K_i$ then there exists $j \leq n$ such that $M \models K_j$. Clearly K_j is EC and $\|K_j\|_N \geq \times$. Hence if $K_j = \bigwedge_{k=1}^{m} \varepsilon_k p_k$ then $\|\varepsilon_k p_k\|_N \geq \times$ for every $k = 1,\dots m$ and thus by the definition of N, $\mathfrak{M} \models \Diamond \varepsilon_k p_k$. Since K_j is EC we obtain, repeatedly using the property (+) that

$$\mathfrak{M} \models \Diamond \bigwedge_{k=1}^{m} \varepsilon_k p_k,$$

i.e. $\mathfrak{M} \models \Diamond K_j$. By Lemma 1.7 also $\mathfrak{M} \models \Diamond \varphi$.

Q.e.d.

Thus we have answered our question. In a slight reformulation, our answer may be formulated as follows:

An S5-model $\mathfrak{M}$ is equivalent to (not S5-distinguishable from) the T-model (a set of all completions of a 3-valuation) iff $\mathfrak{M}$ has the following property: whenever $\mathfrak{M} \models \Diamond \varphi$ and $\mathfrak{M} \models \Diamond \psi$ where φ and ψ have no propositional variables in common then $\mathfrak{M} \models \Diamond (\varphi \,\&\, \psi)$.

1.13. Corollary. a) Suppose that for a given N, $card(\{p; \|p\|_N = \times\}) = n$. Then $card(\mathfrak{D}(N)) = 2^n$ and there is no modal model equivalent to $\mathfrak{D}(N)$ and different from $\mathfrak{D}(N)$.

b) Suppose now that $card(\{p; \|p\|_N = \times\}) = \aleph_0$.

Then

1) there exists a countable modal model equivalent to $\mathfrak{D}(N)$;

2) there exist $2^{2^{\aleph_0}}$ modal models equivalent to $\mathfrak{D}(N)$.

II. Monadic predicate calculus. Our aim is to investigate modal models which are sets of completions of a 3-valued model, called T-models. In the propositional calculus we have seen that by the modal language T-models are not distinguishable from those satisfying the property (+) of Theorem 1.12, thus we can only describe the set of models equivalent to T-models.

Next we want to study this problem in monadic predicate calculus. Thus we try to find those axioms which describe the set of models equivalent to T-models in some sense. But we succeed only partially: we can describe the set of modal models which are undistinguishable from T-models using formulas $\Diamond\varphi$ where φ is without modalities. But we believe that our method of proof can be useful for a complete solution in the case of monadic logic; see Remark 2.13.

2.1. The language of a monadic predicate calculus $L_{\mathcal{P}}$ consists of variables $(x,y,z,\ldots)$, a finite set of predicates $\mathcal{P} = \{P_1,P_2,\ldots P_n\}$, connectives $\&$, $\vee$, $\neg$ and quantifiers $\forall$, $\exists$. Formulas, free and bound variables, closed and open formulas are defined by induction as usual. The set of all formulas is denoted $Fla(L_{\mathcal{P}})$. A 3-valued model $\mathcal{M} = \langle M, \mathcal{P}_1,\ldots \mathcal{P}_n\rangle$ consists of a non-empty set M (called the domain of $\mathcal{M}$) and of a collection of mappings $\mathcal{P}_1,\ldots \mathcal{P}_n$ such that $\mathcal{P}_i : M \rightarrow \{0,1,\times\}$.

The valuation of formulas is defined by induction: let φ, $\psi \in Fla(L_{\mathcal{P}})$, let e be a mapping of variables to M;

i) if $\varphi = P_i(x)$ (atomic formula) then the value of φ in $\mathcal{M}$ for e is $\|P_i(x)[e]\|_{\mathcal{M}} = \mathcal{P}_i(e(x))$;

ii) $\|(\varphi \& \psi)[e]\|_{\mathcal{M}} = \min(\|\varphi[e]\|_{\mathcal{M}}, \|\psi[e]\|_{\mathcal{M}})$, similarly for $\vee, \neg$.

iii) $\|(\forall x \varphi)[e]\|_{\mathcal{M}} = \min\{\|\varphi[e_1]\|_{\mathcal{M}}\ ;\ e_1(y) = e(y)$ for each y distinct from $x\}$;

$$\|(\exists x \varphi)[e]\|_{\mathcal{M}} = \|(\neg(\forall x)\neg\varphi)[e]\|_{\mathcal{M}}$$

We say that a formula φ is <u>valid</u> in a model $\mathcal{M}$ ($\mathcal{M} \models \varphi$) if

$$\|(\forall x_1..x_k)\varphi(x_1,..x_k)\|_{\mathcal{M}} = 1 \text{ where } FV(\varphi) = \{x_1,..x_k\}.$$

Particularly, a closed formula φ is valid in $\mathcal{M}$ if $\|\varphi\|_{\mathcal{M}} = 1$.

Formulas φ and ψ are <u>3-equivalent</u> ($\varphi \equiv_3 \psi$) if for every $\mathcal{M}$ and every evaluation e,

$$\|\varphi[e]\|_{\mathcal{M}} = \|\psi[e]\|_{\mathcal{M}}.$$

2.2. For the 3-valued predicate calculus there hold the corresponding definitions and facts as for propositional calculus, especially

a) $\mathcal{M} = \langle M, \mathcal{P}_1, \dots \mathcal{P}_n \rangle$ is a 2-model if $\mathcal{P}_i : M \rightarrow \{0,1\}$. φ is a 2-tautology if φ is valid in every 2-models. φ is a 2-tautology iff for every $\mathcal{M}$, e, $\|\varphi[e]\|_{\mathcal{M}} \geq \times$.

b) A 2-model $\mathcal{M} = \langle M, \mathcal{P}_1^{\mathcal{M}}, \dots \mathcal{P}_m^{\mathcal{M}} \rangle$ is a <u>completion</u> of $\mathcal{N} = \langle N, \mathcal{P}_1^{\mathcal{N}}, \dots \mathcal{P}_n^{\mathcal{N}} \rangle$ if

i) $N = M$, $n = m$;

ii) for every $i \leq n$ and every $a \in M$,

if $\mathcal{P}_i^{\mathcal{N}}(a) \neq \times$ then $\mathcal{P}_i^{\mathcal{M}}(a) = \mathcal{P}_i^{\mathcal{N}}(a)$.

The set of all completions of $\mathcal{N}$ is denoted by $\mathfrak{D}(\mathcal{N})$.

c) For every φ, e, $\mathcal{N}$ if $\mathcal{M} \in \mathfrak{D}(\mathcal{N})$ and $\|\varphi[e]\|_{\mathcal{N}} \neq \times$ then $\|\varphi[e]\|_{\mathcal{M}} = \|\varphi[e]\|_{\mathcal{N}}$. Thus each formula of Kleene's monadic predicate calculus is <u>monotone</u>.

2.3. Let $L^{\Box}_{\mathcal{P}}$ be the monadic predicate language with n predicate symbols and the modality $\Box$. Formulas of $L^{\Box}_{\mathcal{P}}$ are called modal formulas (denoted by Φ, Ψ,...), the modality $\Box$ is read "necessarily", the modality $\Diamond = \neg\Box\neg$ is read "possibly". A Kripke' modal S5-model is a system $\mathcal{K} = \langle \mathfrak{M}, \mathcal{R} \rangle$ where $\mathfrak{M}$ is a non-empty set of 2-valued models of $L_{\mathcal{P}}$ on the same domain and $\mathcal{R}$ is an equivalence relation on $\mathfrak{M}$. The valuation of formulas in $\mathcal{K}$ is defined as follows:
first we define the value of Φ in any $\mathcal{M} \in \mathfrak{M}$ for an evaluation e w.r.t. $\mathcal{K}$:

i) if $\Phi = P_i(x)$ then $\|\Phi[e]\|_{\mathcal{M},\mathcal{K}} = \mathcal{P}_i^{\mathcal{M}}(e(x))$;

ii) if $\Phi = \Phi_1 \& \Phi_2$ ($\Phi_1 \vee \Phi_2$, $\neg\Phi_1$, $\forall x \Phi_1$, $\exists x \Phi_1$) then the truth value of Φ is determined from the values of Φ_1 (and Φ_2 resp.) as usual;

iii) if $\Phi = \Box\Phi_1$ then $\|\Phi[e]\|_{\mathcal{M},\mathcal{K}} = \min\{\|\Phi_1[e]\|_{\mathcal{N},\mathcal{K}};\ \mathcal{N} \in \mathfrak{M}\ \&\ \mathcal{M}\mathcal{R}\mathcal{N}\}$. We write $(\mathcal{K},\mathcal{M}) \models \Phi[e]$ instead of $\|\Phi[e]\|_{\mathcal{M},\mathcal{K}} = 1$.

2.4. <u>Conventions and definitions</u>. In the sequel, we restrict ourselves to modal models with the equivalence relation having only one equivalence class, i.e. models of the form $\mathcal{K} = \langle \mathfrak{M}, \mathfrak{M} \times \mathfrak{M} \rangle$. Such a model is denoted simply by $\mathfrak{M}$. Thus from now on, a modal model is simply a non-empty set of 2-valued models with the same domain. Observe that the truth value of a formula $\Diamond\varphi$ in a model $\mathfrak{M}$ does not depend on the choice of a particular $\mathcal{M} \in \mathfrak{M}$, thus we write $\mathfrak{M} \models \Diamond\varphi[e]$

instead of $(\mathfrak{M}, \mathcal{M}) \models \Diamond \varphi [e]$. Similarly for each formula Φ in which each atomic formula occurs in the scope of a modality ($\Box$ or $\Diamond$). Call such formulas <u>boxed</u>. A boxed formula Φ is <u>valid</u> in $\mathfrak{M}$ if $\mathfrak{M} \models \Phi [e]$ for each e.

2.5. <u>Definition</u>. Two modal models $\mathfrak{M}$, $\mathfrak{N}$ with the same domain are <u>weakly equivalent</u> if for each formula φ without modalities and each e we have

$$\mathfrak{M} \models \Diamond \varphi [e] \text{ iff } \mathfrak{N} \models \Diamond \varphi [e].$$

2.6. <u>Definition</u>. Let $\mathcal{M}$ be a 3-valued model. The <u>T-model</u> associated with $\mathcal{M}$ is $\mathfrak{D}(\mathcal{M})$ - the set of all completions of $\mathcal{M}$. (Clearly, $\mathfrak{D}(\mathcal{M})$ is a modal model.)

2.7. <u>Remark</u>. We may now make precise our aim: to characterize modal models weakly equivalent to T-models.

2.8. <u>Definition</u>. Let $\varphi(x)$ be an open formula with one free variable. φ is said to be a <u>fundamental disjunction</u> (FD) if φ is a disjunction containing only atomic formulas or their negations. Similarly, we define FC, CFD, DFC (fundamental conjunction, conjunction of fundamental disjunctions etc.). φ is an <u>elementary disjunction</u> if φ is FD and contains every atomic formula at most once.

2.9. <u>Definition</u>. A <u>canonical sentence</u> is a formula of the form $(\forall x)\varphi(x)$ where φ is a FD.

2.10. <u>Theorem</u>. Every formula is 3-equivalent to a Boolean combination of canonical sentences and open formulas (Boolean combination means by the help of connectives $\&$, $\vee$, $\neg$). Particularly, every closed formula is 3-equivalent to a Boolean combination of canonical sentences.

Proof. Similarly as in the 2-valued case. (See e.g. [3].)

Corollary. Every formula is 3-equivalent to a disjunction of conjunctions of canonical sentences, their negations, atomic formulas and their negations (disjunctive normal form).

2.11. Definition. Let $\mathcal{M}$ be a 3-model. We extend the language L_n by constants $\{\underline{a}; a \in M\}$ and interpret each $\underline{a}$ by a. L_n^* means the extended language. A closed quantifier free formula in L_n^* is called an instance. Every instance is a Boolean combination of atomic formulas of the form $P(\underline{a})$.

An elementary conjunction is a conjunction φ of distinct atomic instances in which each atom occurs at most once, i.e. there is no atomic instance occuring in φ both in positive and in negated form.

2.12. Theorem. A modal model $\mathfrak{M}$ is weakly equivalent to a T-model iff $\mathfrak{M}$ satisfies the following two conditions:

(1) for each instance $\varphi \,\&\, \psi$ which is EC,

if $\mathfrak{M} \models \Diamond\varphi$ and $\mathfrak{M} \models \Diamond\psi$ then also $\mathfrak{M} \models \Diamond(\varphi \,\&\, \psi)$;

(2) for any open L_n-formulas $\varphi, \varphi_1, \ldots \varphi_k$ with one free variable and for each sequence $a_1, \ldots a_k$ of elements of $\mathfrak{M}$,

if $\mathfrak{M} \models (\forall x)\Diamond(\varphi(x) \,\&\, \bigwedge_{i \leq k} \varphi_i(\underline{a}_i))$ then

$$\mathfrak{M} \models \Diamond(\forall x)(\varphi(x) \,\&\, \bigwedge_{i \leq k} \varphi_i(\underline{a}_i)).$$

Proof. I. Let $\mathfrak{M}$ be weakly equivalent to $\mathfrak{D}(\mathcal{M})$ for a 3-model $\mathcal{M}$ with the domain M. We will show that $\mathfrak{M}$ satisfies both conditions.

(1) Let the instance $\varphi \,\&\, \psi$ be an EC, let $\mathfrak{M} \models \Diamond\varphi$ and $\mathfrak{M} \models \Diamond\psi$. Then also $\mathfrak{D}(\mathcal{M}) \models \Diamond\varphi$ and $\mathfrak{D}(\mathcal{M}) \models \Diamond\psi$; i.e.

there exist $\mathcal{N}_1, \mathcal{N}_2 \in \mathfrak{D}(\mathcal{M})$ such that $\mathcal{N}_1 \models \varphi$ and $\mathcal{N}_2 \models \psi$. Define a completion $\bar{\mathcal{N}}$ in the following way: for each atom $P_i(\underline{a})$ occuring in φ (positively or negatively), define

$$\mathcal{P}_i^{\bar{\mathcal{N}}}(a) = \mathcal{P}_i^{\mathcal{N}_1}(a);$$

similarly, if $P_i(\underline{a})$ occurs in ψ, put

$$\mathcal{P}_i^{\bar{\mathcal{N}}}(a) = \mathcal{P}_i^{\mathcal{N}_2}(a).$$

(Recall that $\varphi \,\&\, \psi$ is an EC!) The rest is completed arbitrary. Clearly, $\bar{\mathcal{N}} \in \mathfrak{D}(\mathcal{M})$ and $\bar{\mathcal{N}} \models \varphi \,\&\, \psi$, thus $\mathfrak{D}(\mathcal{M}) \models$ $\models \Diamond(\varphi \,\&\, \psi)$ and also from the assumption, $\mathfrak{M} \models \Diamond(\varphi \,\&\, \psi)$.

(2) Let $\mathfrak{M} \models (\forall x)\Diamond(\varphi(x) \,\&\, \bigwedge_i \varphi_i(\underline{a}_i))$. $\mathfrak{M}$ and $\mathfrak{D}(\mathcal{M})$ have the same domain, thus we have (from the assumption)

$$\mathfrak{D}(\mathcal{M}) \models (\forall x)\Diamond(\varphi(x) \,\&\, \bigwedge_i \varphi_i(\underline{a}_i)).$$

Put $\varphi(x) \,\&_{i \leqq k} \bigwedge \varphi_i(\underline{a}_i) = \bar{\varphi}(x)$. For every $a \in M$, $\| \bar{\varphi}(a) \|_{\mathcal{M}} \geqq \times$ and there exists $\mathcal{N}^a \in \mathfrak{D}(\mathcal{M})$ such that $\mathcal{N}^a \models \bar{\varphi}[a]$. We will construct the completion $\bar{\mathcal{N}}$ by putting

$$\mathcal{P}^{\bar{\mathcal{N}}}(a) = \mathcal{P}^{\mathcal{N}^a}(a) \text{ for } a \in M.$$

Clearly, $\bar{\mathcal{N}} \in \mathfrak{D}(\mathcal{M})$. We claim that $\bar{\mathcal{N}} \models (\forall x)\bar{\varphi}(x)$: $\bar{\mathcal{N}} \models \varphi_i[\underline{a}_i]$ because $\mathcal{N}^{a_i} \models \varphi_i[\underline{a}_i]$ and the validity of an instance $\varphi_i(\underline{a}_i)$ depends only on valuations of $P_k(\underline{a}_i)$ $(k \leqq n)$ which are the same as in $\mathcal{N}^{a_i}$. Similarly, for $a \in M$, $\bar{\mathcal{N}} \models \varphi[a]$. Thus $\bar{\mathcal{N}} \models (\forall x)\varphi(x)$. By the same reasons, $\bar{\mathcal{N}} \models {}_{i \leqq k}\bigwedge \varphi_i(\underline{a}_i)$ thus $\bar{\mathcal{N}} \models (\forall x)\bar{\varphi}(x)$ from which it follows that

$$\mathfrak{D}(\mathcal{M}) \models \Diamond(\forall x)\bar{\varphi}(x)$$

and thus also $\mathfrak{M} \models \Diamond(\forall x)(\varphi(x) \,\&_{i \leqq k} \bigwedge \varphi_i(\underline{a}_i))$.

II. On the other hand, let $\mathfrak{M}$ with the domain M be a modal model satisfying the conditions (1) and (2). Define a

3-model $\mathcal{M}$ with the domain M: for $a \in M$, $i \leq n$,

$$\mathcal{P}_i^{\mathcal{M}}(a) = 1 \text{ if } \mathfrak{M} \models \Box P_i(\underline{a}),$$

$$\mathcal{P}_i^{\mathcal{M}}(a) = 0 \text{ if } \mathfrak{M} \models \Box \neg P_i(\underline{a}),$$

$$\mathcal{P}_i^{\mathcal{M}}(a) = \times \quad \text{otherwise.}$$

We claim that $\mathfrak{M}$ is weakly equivalent to $\mathfrak{D}(\mathcal{M})$. Clearly, $\mathfrak{M} \subseteq \mathfrak{D}(\mathcal{M})$. We are going to prove that for each closed L_n^*-formula φ, if $\mathfrak{D}(\mathcal{M}) \models \Diamond \varphi$ then also $\mathfrak{M} \models \Diamond \varphi$.

1) First let us prove this for φ being an instance. The proof is the same as in the propositional calculus (Theorem 1.12) because $\{\mathcal{P}_i^{\mathcal{M}}(a);\ i \leq n \,\&\, a \in M\}$ is a valuation for a language $L_{\mathcal{P}}$, where $\mathcal{P} = \{P_i^{\mathcal{M}}(\underline{a});\ i \leq n \,\&\, a \in M\}$.

2) Now let φ be closed formula of a language L_n^*, let $\mathfrak{D}(\mathcal{M}) \models \Diamond \varphi$. By Corollary of Theorem 2.10,

$$\varphi \equiv_3 \bigvee_{i \leq k} K_i \quad \text{where}$$

$$K_i = \bigwedge_{j \in J} (\forall x)\, \varphi_{ij}(x) \;\&\; \bigwedge_{k=0}^{m} (\exists x)\, \overline{\varphi}_{ik}(x) \;\&\; \bigwedge_{k=m+1}^{q} \overline{\varphi}_{ik}(\underline{a}_k).$$

From the monotonicity of formulas it follows that $\|\varphi\|_{\mathcal{M}} \geq \times$ and that there exists $i \leq k$ such that $\|K_i\|_{\mathcal{M}} \geq \times$. For this i there exists $\mathcal{N} \in \mathfrak{D}(\mathcal{M})$ for which $\mathcal{N} \models K_i$, i.e.

$$\mathcal{N} \models \bigwedge_{j \in J} (\forall x)\, \varphi_{ij}(x) \;\&\; \bigwedge_{k=0}^{m} (\exists x)\, \overline{\varphi}_{ik}(x) \;\&\; \bigwedge_{k=m+1}^{q} \overline{\varphi}_{ik}(\underline{a}_k).$$

Put $\psi_i(x) = \bigwedge_{j \in J} \varphi_{ij}(x)$. Clearly, $\mathcal{N} \models (\forall x)\, \psi_i(x)$. From the validity of the second part of the conjunction it follows that there exist elements $a_1, \ldots a_m \in M$ such that

$$\mathcal{N} \models \bigwedge_{k=1}^{m} \overline{\varphi}_{ik}(\underline{a}_k)$$

so that we have

$$\mathcal{N} \models \bigwedge_{k=1}^{q} \overline{\varphi}_{ik}(\underline{a}_k).$$

Since $\mathcal{N} \in \mathfrak{D}(\mathcal{M})$, we have also

$\mathfrak{D}(\mathcal{M}) \models \Diamond\,(\psi_i(a)\ \&_{k=1}\bigwedge^{q}\ \overline{\varphi}_{ik}(\underline{a}_k))$ for every $a \in M$.

$\psi_i(a)\ \&_{k=1}\bigwedge^{q}\ \overline{\varphi}_{ik}(\underline{a}_k)$ is an instance, thus by the preceding part of the proof also for every $a \in M$,

$$\mathfrak{M} \models \Diamond\,(\psi_i(a)\ \&_{k \leq q}\bigwedge\ \overline{\varphi}_{ik}(\underline{a}_k)).$$

M is the domain of $\mathfrak{M}$, thus

$$\mathfrak{M} \models (\forall x)\,\Diamond\,(\psi_i(x)\ \&_{k \leq q}\bigwedge\ \overline{\varphi}_{ik}(\underline{a}_k)).$$

By the property (2),

$$\mathfrak{M} \models \Diamond(\forall x)(\psi_i(x)\ \&_{k \leq q}\bigwedge\ \overline{\varphi}_{ik}(\underline{a}_k)),$$

i.e. there exists $\mathcal{N} \in \mathfrak{M}$ such that

$$\mathcal{N} \models (\forall x)(\psi_i(x)\ \&_{k \leq q}\bigwedge\ \overline{\varphi}_{ik}(\underline{a}_k)).$$

$\mathcal{N}$ is a 2-model, thus

$$\mathcal{N} \models (\forall x)(\psi_i(x)\ \&_{k \leq m}\bigwedge (\exists x)\,\overline{\varphi}_{ik}(x)\ \&_{k=m+1}\bigwedge^{q}\ \overline{\varphi}_{ik}(\underline{a}_k)),$$

i.e. $\mathcal{N} \models K_i$; thus $\mathcal{N} \models \bigvee_{i \leq k} K_i$, which means

$$\mathfrak{M} \models \Diamond\varphi\ .$$

This completes the proof.

Q.e.d.

2.13. Remark. We say that $\mathfrak{M}$, $\mathfrak{N}$ are strongly equivalent (notation: $\mathfrak{M} \equiv \mathfrak{N}$) if for each boxed Φ and each valuation e,

$$\mathfrak{M} \models \Phi\,[e] \text{ iff } \mathfrak{N} \models \Phi\,[e].$$

The two conditions of the Theorem 2.12 are necessary for strong equivalence of a model to a T-model because models which are strongly equivalent are weakly equivalent. The open problem is to characterize models strongly equivalent to T-models. We know that the property

(+) if $\mathfrak{M} \models (\forall x) \Diamond (\varphi(x,\underline{y}) \ \& \bigwedge_{i \leq n} \varphi_i(\underline{a}_i))$ then

$$\mathfrak{M} \models \Diamond (\forall x)(\varphi(x,\underline{y}) \ \& \bigwedge_{i \leq n} \varphi_i(\underline{a}_i))$$

where every occurence of $\underline{y}$ in φ is in the scope of a box is necessary for equivalence but we do not know if it is sufficient.

2.14. Remark. For non-monadic logic presented method of proof fails. Let

$\varphi(x,y) = (x \neq y \ \& (R(x,y) \ \& \ P(y)) \vee (\neg R(y,x) \ \& \ Q(y)))$

and $\mathcal{M} = \langle \{a,b\}; \mathcal{P}^{\mathcal{M}}, \mathcal{Q}^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}} \rangle$ be a 3-model of the type $\langle 1,1,2 \rangle$, such that $\mathcal{Q}^{\mathcal{M}}(b) = 1$, $\mathcal{P}^{\mathcal{M}}(a) = 1$, $\mathcal{R}^{\mathcal{M}}(b,a) = \times$, other values are zero, = is a 2-valued identity.

Then

$$\mathfrak{D}(\mathcal{M}) \models (\forall x) \Diamond (\exists y) \varphi(x,y)$$

but

$$\mathfrak{D}(\mathcal{M}) \not\models \Diamond (\forall x)(\exists y) \varphi(x,y).$$

We do not see how to describe modal models equivalent to T-models using the modal language without the possibility of exchange of $\Diamond$ and $\forall$.

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