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SOME RESULTS ON INVERSE SPECTRA I.

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Abstract: In this paper, we consider the following question: when a homeomorphism of limit spaces of two inverse spectra is induced by an isomorphism of cofinal subspectra? We prove two spectral theorems which generalize a number of A.V. Arhangel'skiĭ's, S.A. Pasynkov's and E.V. Ščepin's results. Some related questions are considered, too.

Key words and phrases: Isomorphism of spectra, open mapping, continuous spectra, d-open mapping, almost continuous spectra, semiopen mapping, $\varkappa$-metrizable space.

Classification: Primary 54B25, 54A25
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Introduction. In 1976 E.V. Ščepin proved the fundamental result which was named the spectral theorem for compacts (see [1], Theorem 2). Since this result was obtained, some new versions of this theorem have appeared. The most interesting of them, as we see it, were proved by A.V. Arhangel'skiĭ [2] and B.A. Pasynkov [3]. Another approach (via uniform spaces) to the Ščepin's theorem was created by W. Kulpa [8]. In the first part of the paper we present one general assertion with a clear proof which implies Arhangel'skiĭ's and Pasynkov's results mentioned above. It should be noted that the main idea of the proof of our spectral theorem was casted by the reasoning of R. Engelking (see [4], Theorem 1).

In what follows, all spaces are assumed to be completely regular if there are no other assumptions. Instead of "inverse spectrum" we write briefly "spectrum". We assume that all spectra under consideration consist of topological spaces and spectral projections (including limit ones) are continuous and onto.

§ 1. Spectral theorem for spaces similar to compacts. We shall recall some necessary notions.

Definition 1. Let $(A, \prec)$ be a directed set of indices and $S_1 = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta\in A}$, $S_2 = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta\in A}$ be spectra. For every $\alpha \in A$ let us fix a continuous mapping $\varphi_\alpha : X_\alpha \to Y_\alpha$.

I. A family $\{\varphi_\alpha : \alpha \in A\}$ is said to be a morphism of a spectrum S_1 to a spectrum S_2 if $\varphi_\alpha \circ p_\alpha^\beta = q_\alpha^\beta \circ \varphi_\beta$ for each $\alpha, \beta \in A$ such that $\alpha \prec \beta$.

II. A morphism $\{\varphi_\alpha : \alpha \in A\}$ is said to be an isomorphism if φ_α is a homeomorphism of X_α onto Y_α for every $\alpha \in A$.

Definition 2. Let ξ be an ordinal and $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\xi}$ be a well-ordered spectrum.

I. A spectrum S is said to be continuous if for every limit $\alpha^* < \xi$ a space X_{α^*} is naturally homeomorphic to a limit of a spectrum $S_{\alpha^*} = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\alpha^*}$ (the last means that a diagonal product $\Delta\{p_\alpha^{\alpha^*} : \alpha < \alpha^*\}$ is a homeomorphism of X_{α^*} onto $\varprojlim S_{\alpha^*}$).

II. A continuous spectrum S is said to be regular if ξ is a regular cardinal and $w(X_\alpha) < \xi$ for every $\alpha < \xi$.

Now Ščepin's spectral theorem for compacts can be formulated as follows: Let $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ and $T = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta<\tau}$ be regular spectra consisting of compacts with homeomorphic limits. Then there exists a closed cofinal subset $A \subseteq \tau$ such that the spectra $S_A = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta \in A}$ and $T_A = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta \in A}$ are isomorphic.

Let τ be an infinite cardinal. We write $\nabla \ell (X) \leq \tau$ if for every open cover γ of a space X there exists a subcover $\gamma' \subseteq \gamma$ such that $|\gamma'| < \tau$.

In [2], A.V. Arhangel'skiĭ proved the following theorem: Let a space X of a regular weight $\tau > \aleph_0$ be a limit of regular spectra $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ and $T = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta<\tau}$ with quotient projections and $\nabla \ell (X^n) \leq \tau$ for every $n \in \omega$. Then there exists a closed cofinal subset $A \subseteq \tau$ such that the spectra S_A and T_A are isomorphic.

In [3], B.A. Pasynkov showed that if a space X is a limit of a regular spectrum $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ with closed projections then $\nabla \ell (X) \leq \tau$. With the aid of this result Pasynkov proves that for every regular spectra $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ and $T = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta<\tau}$ with closed projections and homeomorphic limits there exists a closed cofinal subset $A \subseteq \tau$ such that the spectra S_A and T_A are isomorphic.

We shall show that it is possible to exclude the restrictions on projections of spectra in Arhangel'skiĭ's and Pasynkov's results. But we need to retain the condition $\nabla \ell (X) \leq \tau$ which is inherent to both of them. Before formulating our main result (Theorem 2) let us discuss the following question. When a space can be represented as a limit of a well-ordered spectrum consisting of spaces of smaller weight? There are many spaces which do not admit such a representation. For example, the space $T(\omega_1)$ (countable ordinals with the order topology) is "bad" in this sense. Indeed, if the space $T(\omega_1)$ is a limit of a spectrum $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\omega_1}$ consisting of spaces of a countable weight then a countable compactness of $T(\omega_1)$ implies that X_α is compact for each $\alpha < \omega_1$. However, a limit of a spectrum consisting of compacts is compact, that is a contradiction.

The following theorem shows when does the desired representation exist.

Theorem 1. Let $\tau > \aleph_0$ be a regular cardinal and $\nabla\ell(X) \leq \tau = w(X)$. Then a space X is homeomorphic to a limit of some well-ordered spectrum $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ where $w(X_\alpha) < \tau$ for every $\alpha < \tau$.

Proof. Let us assume that X is a subspace of Tychonoff cube I^τ. There exists a family $\{A_\alpha : \alpha < \tau\}$ such that 1) $A_\alpha \subseteq A_\beta$, $\alpha < \beta < \tau$; 2) $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ for every limit ordinal $\beta < \tau$; 3) $|A_\alpha| < \tau$ and 4) $\tau = \bigcup_{\alpha<\tau} A_\alpha$.

Let π_α be a natural projection of I^τ onto I^{A_α} and π_α^β be a natural projection of I^{A_β} onto I^{A_α}, $\alpha < \beta < \tau$. For every $\alpha < \tau$ put $X_\alpha = \pi_\alpha(X)$; a topology on X_α is induced from I^{A_α}. For each α, β with $\alpha < \beta$ put $p_\alpha^\beta = \pi_\alpha^\beta | X_\beta$. So we have defined a well-ordered spectrum $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ such that $w(X_\alpha) \leq |A_\alpha| \cdot \aleph_0 < \tau$ for every $\alpha < \tau$. Let φ be a diagonal product of a family of mappings $\{\pi_\alpha | X : \alpha < \tau\}$. Let $Y = \varprojlim S$ and p_α be a limit projection of Y onto X_α, $\alpha < \tau$. Then φ is a continuous mapping of X to Y. Now we shall show that φ is a homeomorphism of X onto Y.

I. φ is a monomorphism.

Indeed, let $x, y \in X$ and $x \neq y$. Then there exists an ordinal

$\alpha^* < \tau$ such that $\pi_{\alpha^*}(x) \neq \pi_{\alpha^*}(y)$. The equality $p_{\alpha^*} \circ \varphi = \pi_{\alpha^*} | X$ implies that $\varphi(x) \neq \varphi(y)$.

II. φ is an epimorphism.

Let $y \in Y$. For every $\alpha < \tau$ put $F_\alpha = X \cap \pi_\alpha^{-1}(p_\alpha(y))$. Then F_α is a non-empty closed subset of X and $F_\alpha \subseteq F_\beta$ for $\beta < \alpha < \tau$. The inequality $\nabla \ell(X) \leq \tau$ implies that $F = \bigcap_{\alpha < \tau} F_\alpha \neq \Lambda$. Let $x \in F$. Then $\pi_\alpha(x) = p_\alpha(x)$ for every $\alpha < \tau$, hence $\varphi(x) = y$. Consequently $\varphi(X) = Y$.

III. The mapping φ^{-1} is continuous.

Let $y \in Y$, $\varphi(x) = y$ and $\mathcal{O}$ be an open neighbourhood of x in X. Then there exist an ordinal $\alpha < \tau$ and an open subset $V \subseteq X_\alpha$ such that $x \in X \cap \pi_\alpha^{-1} V \subseteq \mathcal{O}$. Put $W = p_\alpha^{-1} V$. The equality $\varphi(x) = y$ implies that $y \in W$ and $\varphi^{-1}(W) = X \cap \pi_\alpha^{-1} V \subseteq \mathcal{O}$. Thus the theorem is proved.

It should be noted that a mapping φ is a homeomorphism of X onto a dense subset of Y if the condition $\nabla \ell(X) \leq \tau$ is not assumed. In connection with this fact it seems to be natural to introduce the following definition (see also [6], p. 1059).

Definition 3. a) We shall say that a spectrum $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta < \xi}$ is almost continuous if for every limit $\alpha^* < \xi$ a space X_{α^*} is naturally homeomorphic to a subspace of $\varprojlim S_{\alpha^*}$ where $S_{\alpha^*} = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta < \alpha^*}$ (then this subspace is dense in $\varprojlim S_{\alpha^*}$ because projections of a spectrum S are assumed to be onto).

b) We shall say that an almost continuous spectrum S is almost regular if ξ is a regular cardinal and $w(X_\alpha) < \xi$ for every $\alpha < \xi$.

An almost continuity of a spectrum S means that for every limit ordinal $\alpha^* < \xi$ the family $\{(p_\alpha^{\alpha^*})^{-1}\mathcal{O} : \alpha < \alpha^*$ and $\mathcal{O}$ is open in $X_\alpha\}$ forms a base of a space X_{α^*}.

In some sense the notion of an almost continuous spectrum is better than the notion of a continuous one. It is confirmed by the following facts:

a) each completely regular space X of regular weight $\tau > \aleph_0$ with $\nabla\ell(X) \le \tau$ is a limit of an almost continuous spectrum consisting of completely regular spaces of smaller weights, but

b) not every such a space X can be represented as a limit of a regular spectrum.

Indeed, the spectrum constructed in the proof of Theorem 1 is almost continuous. Namely, from the condition $A_{\alpha^*} = \bigcup_{\alpha<\alpha^*} A_\alpha$ which holds for every limit $\alpha^* < \tau$, we obtain that the family $\{(p_\alpha^{\alpha^*})^{-1}\mathcal{O} : \alpha < \alpha^*$ and $\mathcal{O}$ is open in $X_\alpha\}$ forms a base for X_{α^*}.

However, the σ-product of $\aleph_1$ many of the discrete dubletons is an example of Lindelöf space of weight $\aleph_1$ which is not representable as a limit of a continuous spectrum consisting of spaces of a countable weight. This fact easily follows from Theorem 2 which is the main result of the first part of the paper.

<u>Theorem 2</u>. Let a space X of a regular weight $\tau > \aleph_0$ with $\nabla\ell(X) \le \tau$ be a limit of each of two almost regular spectra $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ and $T = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta<\tau}$. Then there exists a closed cofinal subset $A \subseteq \tau$ such that the spectra $S_A = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta\in A}$ and $T_A = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta\in A}$ are isomorphic.

The general idea of the proof of this theorem is a factorization of continuous functions on X.

Lemma 1. Let a space X of a regular weight $\tau > \aleph_0$ with $\nabla \ell(X) \leq \tau$ be a limit of a well-ordered spectrum $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ and f be a continuous function on X. Then there exist an ordinal $\alpha < \tau$ and a continuous function f_α on X_α such that $f = f_\alpha \circ p_\alpha$, where p_α is a limit projection of X onto X_α.

Proof. For every $i \in \omega$ let γ_i be a countable open cover of R by intervals of length $< 1/i$. Fix $i \in \omega$. Since f is continuous, for every $x \in X$ there exist an ordinal $\alpha(x) < \tau$ and an open subset $\mathcal{O}_x \subseteq X_{\alpha(x)}$ such that the set $f(p_{\alpha(x)}^{-1}\mathcal{O}_x)$ is contained in some member of γ_i. The inequality $\nabla \ell(X) \leq \tau$ implies that there exists a subset $K_i \subseteq X$ with $|K_i| < \tau$ such that $\mu_i = \{p_{\alpha(x)}^{-1}\mathcal{O}_x : x \in K_i\}$ is a cover of X. Put $A_i = \{\alpha(x) : x \in K_i\}$.

Now put $A = \bigcup_{i\in\omega} A_i$. Then $|A| < \tau$ because τ is a regular cardinal and $|A_i| \leq |K_i| < \tau$ for each $i \in \omega$. Consequently there exists an ordinal $\alpha < \tau$ such that $\beta < \alpha$ for every $\beta \in A$. We claim that $x, y \in X$ and $p_\alpha(x) = p_\alpha(y)$ implies $f(x) = f(y)$.

Indeed, let $p_\alpha(x) = p_\alpha(y)$ and $i \in \omega$. As $p_\beta(x) = p_\beta(y)$ for every $\beta \in A_i$ and μ_i is a cover of X, there is a point $z \in K_i$ such that $x, y \in p_{\alpha(z)}^{-1}\mathcal{O}_z$ and the set $f(p_{\alpha(z)}^{-1}\mathcal{O}_z)$ is contained in some member of γ_i. Consequently $|f(x) - f(y)| < 1/i$. The last inequality is valid for each $i \in \omega$ hence $f(x) = f(y)$.

Now we shall define a function f_α on X_α. Let $x_\alpha \in X_\alpha$, $x \in X$ and $p_\alpha(x) = x_\alpha$. Put $f_\alpha(x_\alpha) = f(x)$. This definition is correct because a value $f_\alpha(x_\alpha)$ does not depend on a choice of a point $x \in p_\alpha^{-1} x_\alpha$. From the definition of the function f_α it follows that $f = f_\alpha \circ p_\alpha$. It remains to show that f_α is continuous. Let $x_\alpha \in X_\alpha$, $x \in X$ and $p_\alpha(x) = x_\alpha$. Let $\mathcal{U}$ be an open neighbourhood of a point $f_\alpha(x_\alpha)(= f(x))$. Then there exists a number $i \in \omega$ such that $St_{\gamma_i}(f(x)) \subseteq \mathcal{U}$. Moreover, there exists a point $z \in K_i$ such that $x \in p_{\alpha(z)}^{-1} \mathcal{O}_z$ and a set $f(p_{\alpha(z)}^{-1} \mathcal{O}_z)$ is contained in some member of a cover γ_i. It is obvious that $f(p_{\alpha(z)}^{-1} \mathcal{O}_z) \subseteq St_{\gamma_i}(f(x)) \subseteq \mathcal{U}$. As $\alpha(z) \in A_i$, we conclude that $\alpha(z) < \alpha$. Put $V = (p_{\alpha(z)}^{\alpha})^{-1} \mathcal{O}_z$. Then $y \in V$ and the equality $f = f_\alpha \circ p_\alpha$ implies that $f_\alpha(V) = f(p_\alpha^{-1} V) = f(p_{\alpha(z)}^{-1} \mathcal{O}_z) \subseteq \mathcal{U}$. Thus the lemma is proved.

Corollary 1. Let X be Lindelöf subspace of a product $\prod_{\alpha \in A} X_\alpha$ and f be a continuous function on X. Then there exist a countable subset $B \subseteq A$ and a continuous function f_B on $\pi_B(X)$ such that $f = f_B \circ (\pi_B X)$.

Remark. Let X be Lindelöf subspace of a product $\prod_{\alpha \in A} X_\alpha$ and f be a continuous mapping of X to a space Y with a G_δ-diagonal. Then there exist a countable subset $B \subseteq A$ and a mapping $f_B: \pi_B(X) \to Y$ such that $f = f_B \circ (\pi_B | X)$. This was noted by M. Hušek (see [7], Theorem 10). But f_B is not necessarily continuous in this case.

Lemma 2. Let a space X of a regular weight $\tau > \aleph_0$ with $\nabla \ell(X) \leq \tau$ be a limit of a well-ordered spectrum $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha, \beta < \tau}$. Let also f be a continuous mapping of X to a space Y of weight $< \tau$. Then there exist an ordinal $\alpha^* < \tau$ and a continuous mapping $f^*: X_{\alpha^*} \to Y$ such that $f = f^* \circ p_{\alpha^*}$.

Proof. Put $\lambda = w(Y)$. A space X is completely regular, hence there exists a family $\{\varphi_\alpha : \alpha < \lambda\}$ of continuous functions on Y which separates points and closed sets of Y. For every $\alpha < \lambda$ put $\psi_\alpha = \varphi_\alpha \circ f$. Then ψ_α is continuous for every $\alpha < \lambda$. According to Lemma 1 for every $\alpha < \lambda$ there exist an ordinal $\beta(\alpha) < \tau$ and a continuous function g_α on $X_{\beta(\alpha)}$ such that $\psi_\alpha = g_\alpha \circ p_{\beta(\alpha)}$. Put $A = \{\beta(\alpha) : \alpha < \lambda\}$. Then $|A| \leq \lambda$, hence there exists an ordinal $\alpha^* < \tau$ such that $A \subseteq \alpha^*$. For each $\alpha < \lambda$ put $f_\alpha = g_\alpha \circ p^{\alpha^*}_{\beta(\alpha)}$. Let $\tilde{f} = \Delta\{f_\alpha : \alpha < \lambda\}$ be a diagonal product of a family of functions $\{f_\alpha : \alpha < \lambda\}$, $\varphi = \Delta\{\varphi_\alpha : \alpha < \lambda\}$ and $f^* = \varphi^{-1} \circ \tilde{f}$. Then the mappings $\tilde{f}$ and φ are continuous and φ is a homeomorphism of Y onto $\varphi(Y)$ because of a choice of a family $\{\varphi_\alpha : \alpha < \lambda\}$. Hence a mapping $f^* : X_{\alpha^*} \longrightarrow Y$ is continuous. It remains to show that $f^* \circ p_{\alpha^*} = f$, or equivalently, $\tilde{f} \circ p_{\alpha^*} = \varphi \circ f$. But the last equality follows immediately from the fact that $f_\alpha \circ p_{\alpha^*} = \psi_\alpha = \varphi_\alpha \circ f$ for each $\alpha < \lambda$. Thus the lemma is proved.

We will say that a spectrum $S = \{X_\alpha, p^\beta_\alpha\}_{\alpha,\beta<\tau}$ with a limit X has the factorization property, shortly FP, if for each continuous mapping $f: X \to Y$ to a space Y of weight $< \tau$ there exist an ordinal $\alpha < \tau$ and a continuous mapping $f_\alpha : X_\alpha \to Y$ such that $f = f_\alpha \circ p_\alpha$, where p_α is a limit projection of X onto X_α.

So, Lemma 2 states that a spectrum $S = \{X_\alpha, p^\beta_\alpha\}_{\alpha,\beta<\tau}$ of a regular length $\tau > \aleph_0$ with a limit X satisfies FP if $\nabla \mathcal{L}(X) \leq \tau$.

Lemma 3. Let a space X of a regular weight $\tau > \aleph_0$ be a limit of each of two almost regular spectra $S = \{X_\alpha, p^\beta_\alpha\}_{\alpha,\beta<\tau}$ and $T = \{Y_\alpha, q^\beta_\alpha\}_{\alpha,\beta<\tau}$ having FP. Then for every $\alpha < \tau$ there exist an ordinal $\alpha^* < \tau$ with $\alpha \leq \alpha^*$ and a homeomorphism φ^* of X_{α^*} onto Y_{α^*} such that $\varphi^* \circ p_{\alpha^*} = q_{\alpha^*}$.

Proof. Let us fix an ordinal $\alpha < \tau$. Put $\beta_0 = \alpha$. Since q_{β_0} is a continuous mapping of X onto Y_{β_0} and the weight of Y_{β_0} is less than τ, Lemma 2 implies that there exist an ordinal $\gamma < \tau$ and a continuous mapping $\varphi : X_\gamma \longrightarrow Y_{\beta_0}$ such that $q_{\beta_0} = \varphi \circ p_\gamma$. Put $\alpha_0 = \max\{\beta_0, \gamma\}$ and $\varphi_0 = \varphi \circ p_\gamma^{\alpha(0)}$. It is obvious that $q_{\beta_0} = \varphi_0 \circ p_{\alpha_0}$. Applying Lemma 2 ω-times we construct increasing sequences of ordinals $\{\alpha_i : i \in \omega\}$ and $\{\beta_i : i \in \omega\}$ where $\beta_i \leq \alpha_i \leq \beta_{i+1} < \tau$ for each $i \in \omega$, and sequences of continuous mappings $\{\varphi_i : i \in \omega\}$, $\{\psi_i : i \in \omega\}$, where $\varphi_i : X_{\alpha_i} \longrightarrow Y_{\beta_i}$, $\psi_i : Y_{\beta_{i+1}} \longrightarrow X_{\alpha_i}$ and $\varphi_i \circ p_{\alpha_i} = q_{\beta_i}$, $\psi_i \circ q_{\beta_{i+1}} = p_{\alpha_i}$. Put $\alpha^* = \sup\{\alpha_i : i \in \omega\}$(= $= \sup\{\beta_i : i \in \omega\}$). Since spectra S and T are almost continuous, without any loss of generality one can assume that X_α (Y_α) is a subspace of $\varprojlim S_\alpha$ ($\varprojlim T_\alpha$ resp.) for every limit ordinal $\alpha < \tau$, where $S_\alpha = \{X_\beta, p_\beta^\gamma\}_{\beta,\gamma<\alpha}$ and $T_\alpha = \{Y_\beta, q_\beta^\gamma\}_{\beta,\gamma<\alpha}$. For every $i \in \omega$ put $\tilde{\varphi}_i = \varphi_i \circ p_{\alpha_i}^{\alpha^*}$ and $\tilde{\psi}_i = \psi_i \circ q_{\beta_i}^{\alpha^*}$. Put also $\varphi^* = \triangle\{\tilde{\varphi}_i : i \in \omega\}$ and $\psi^* = \triangle\{\tilde{\psi}_i : i \in \omega \setminus \{0\}\}$. Then φ^* is a continuous mapping of X_{α^*} to $\varprojlim T_{\alpha^*}$ and ψ^* is a continuous mapping of Y_{α^*} to $\varprojlim S_{\alpha^*}$. We claim that φ^* is a homeomorphism of X_{α^*} onto Y_{α^*}.

(1) Let $x \in X$. Put $x^* = p_{\alpha^*}(x)$ and $y^* = q_{\alpha^*}(x)$. Let us show that $\varphi^*(x^*) = y^*$ and $\psi^*(y^*) = x^*$. We have: $x^* \in X_{\alpha^*}$ and $y^* \in Y_{\alpha^*}$ so $\tilde{\varphi}_i(x^*) = \tilde{\varphi}_i p_{\alpha^*}(x) = \varphi_i p_{\alpha_i}(x) = q_{\beta_i}(x) = q_{\beta_i}^{\alpha^*}(y^*)$ for each $i \in \omega$. Hence an almost continuity of a spectrum T implies that $\varphi^*(x^*) = y^*$. The same arguments

imply the equality $\psi^*(y^*) = x^*$. Thus $\varphi^*(X_{\alpha^*}) \subseteq Y_{\alpha^*}$ and $\psi^*(Y_{\alpha^*}) \subseteq X_{\alpha^*}$.

(2) $\psi^* \circ \varphi^* = id_{X_{\alpha^*}}$ and $\varphi^* \circ \psi^* = id_{Y_{\alpha^*}}$.

Indeed, let $x^* \in X_{\alpha^*}$. Choose a point $x \in X$ such that $p_{\alpha^*}(x) = x^*$. Put $y^* = q_{\alpha^*}(x)$. The item (1) implies that $\varphi^*(x^*) = y^*$ and $\psi^*(y^*) = x^*$. Consequently $\psi^* \circ \varphi^* = id_{X_{\alpha^*}}$. The same reasoning shows that $\varphi^* \circ \psi^* = id_{Y_{\alpha^*}}$.

The items (1) and (2) imply that φ^* is a homeomorphism of X_{α^*} onto Y_{α^*} and $\varphi^* \circ p_{\alpha^*} = q_{\alpha^*}$. This completes our proof.

Let all suppositions of Lemma 3 be satisfied. Put $A = \{\alpha < \tau$: there exists a homeomorphism φ of X_α onto Y_α such that $\varphi \circ p_\alpha = q_\alpha\}$. Then Lemma 3 implies the following.

Lemma 4. The set A is a closed cofinal subset of τ.

The conclusion of Theorem 2 immediately follows from Lemma 4.

In connection with Corollary 1 the following question naturally arises. Let X be a subspace of the Tychonoff product $\prod_{\alpha \in A} X_\alpha$ and X contains a dense Lindelöf subspace. Is it true that every continuous function on X depends at most on countably many coordinates? The answer is negative even in a case of a separable space X.

Example I. Let ω be the set of all finite ordinals with the discrete topology. In the space ω^{ω_1} with the usual Tychonoff topology we define two non-empty disjoint sets F_0 and F_1 by the following way. Put $F_0 = \{x \in \omega^{\omega_1} : x(\alpha) \neq 0$ for every $\alpha \in \omega_1$ and there exists $\beta \in \omega_1$ such that $x(\beta) = 1\}$ and $F_1 = \{x \in \omega^{\omega_1} : x(\alpha) \neq 1$ for every $\alpha \in \omega_1$ and there exists

$\beta \in \omega_1$ such that $x(\beta) = 0\}$. From the definition it follows that $[F_0] \cap F_1 = \Lambda$ and $F_0 \cap [F_1] = \Lambda$. Consequently, disjoint sets F_0 and F_1 are clopen in the space $X = F_0 \cup F_1$. Let f be a function of X which equals to zero on F_0 and one on F_1. Obviously, f is continuous. For every subset $T \subseteq \omega_1$ let π_T be the natural projection of ω^{ω_1} onto ω^T. Then $\pi_T(F_0) =$ $= (\omega \setminus \{0\})^T$ and $\pi_T(F_1) = (\omega \setminus \{1\})^T$ for each countable $T \subseteq$ $\subseteq \omega_1$.

Hence $\pi_T(F_0) \cap \pi_T(F_1) = (\omega \setminus \{0,1\})^T$. Thus the function f depends on uncountably many coordinates. One can easily prove that the sets F_0 and F_1 are separable. Hence the space X is separable, too. So X contains a dense Lindelöf subspace.

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