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ON STRICT PREPONDERANT MAXIMA

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Abstract: It is proved that the set of points of strict preponderant maxima of a real-valued function of n variables is of measure zero. The proof uses the fact that each set of positive measure contains a compact, all points of which are points of upper symmetric density greater or equal than one half.

Key words: Preponderant maximum, Density Theorem.

Classification: 26B35

It was shown in [3],[4],[1] that the set of points at which real-valued function defined on a Euclidean n-space takes on a strict density maximum, is of measure zero. We shall give a characterization of the set of points of strict preponderant maxima of a function (defined below).

The ideas of the proof of Proposition and Theorem are the same as in [1]. Lemma 2 improves a similar assertion in [1], where the property of a compact K contained in a set of positive measure is only $D^-(x,K) \geq \frac{1}{2}$ (D^- denotes ordinary upper metric density).

We assume that a Euclidean n-space R_n is fixed throughout this paper.

Definition. Let $A \subset R_n$, $x \in R_n$. Outer upper symmetric density of the set A at the point x is

$$D_s^-(x,A) = \overline{\lim_{r\to 0_+}} \frac{|B(x,r)\cap A|}{|B(x,r)|} .$$

If A is measurable, we leave out the word "outer". Similarly, we define lower density using $\underline{\lim}$.

Definition. Let f be a real-valued function on a Euclidean n-space R_n. If $x\in R_n$ has the property

$$D_s^-(x,[t;f(t)\geq f(x)])<\frac{1}{2} ,$$

we say that f attains a strict preponderant maximum at x.

We shall also need the following notation:

$$M_{\alpha}(f) = \{x\in R_n; D_s^-(x, \{t;f(t)\geq f(x)\}) < \alpha\} , \ 0<\alpha\leqq 1.$$

Especially, $M_{\frac{1}{2}}(f)$ is the set of points of strict preponderant maxima of a function f.

Lemma 1. For arbitrary $\varepsilon > 0$ there is $\beta > 0$ that

$$\frac{|B(x,\beta r)\cap B(0,r)|}{|B(x,\beta r)|} \geq \frac{1}{2} - \varepsilon$$

for any $r>0$ and $x\in B(0,r)\subset R_n$.

We leave out the proof of this lemma because of its simple geometrical interpretation.

Lemma 2. If $A\subset R_n$ is a measurable set and $|A|>a>0$, then there is a compact $K\subset A$ such that

(i) $|K|\geq a$

(ii) $D_s^-(x,K)\geq\frac{1}{2}$ for every $x\in K$.

Proof. There is no loss of generality in assuming that A is a compact subset of a unit cube $Q\subset R_n$. Let E_k be a finite $\frac{1}{k}$-net of the cube Q, $\mathfrak{D}$ denotes a family of balls

$$\mathfrak{D} = \bigcup_{k\in N}\{B(x,\tfrac{1}{k});x\in E_k\}$$

and (σ_k) - a sequence of real numbers such that $\sigma_k\searrow 0$,

$1 > \delta_1 > \delta_2 > \ldots > 0.$

We shall form an increasing sequence of compact sets

$K_m^1 = \bigcup \{S \cap A; S \in \mathcal{D}, \text{ diam } S \geq \frac{1}{m}, \frac{|S \cap A|}{|S|} > 1 - \delta_1\}, \ m = 1,2,\ldots$

If x is any point of density of A, then x belongs to all but finitely many K_m. Therefore, using Density Theorem, $|K_m| \nearrow |A|$. We can find an m_1 so that $|K_{m_1}| > a$. We relabel $K_{m_1} = K_1$ and let $\mathcal{B}_1$ denote the finite family of all $S \in \mathcal{D}$ satisfying diam $S \geq \frac{1}{m_1}$ and $\frac{|S \cap A|}{|S|} > 1 - \delta_1$. That is to say $K_1 = \bigcup_{S \in \mathcal{B}_1} S \cap A$.

Now put for $m = m_1 + 1, \ldots$

$K_m^2 = \bigcup \{S \cap K_1; S \in \mathcal{D}, \frac{1}{m_1} > \text{diam } S \geq \frac{1}{m}, \frac{|S \cap K_1|}{|S|} > 1 - \delta_2\}.$

Similarly, $|K_m^2| \nearrow |K_1|$. Hence there is an $m_2 > m_1$ such that $|K_{m_2}^2| > a$ and at the same time $\frac{|S \cap K_{m_2}^2|}{|S|} > 1 - \delta_1$ for $S \in \mathcal{B}_1$, because $\mathcal{B}_1$ is a finite family. Relabel $K_{m_2}^2 = K_2$ and let $\mathcal{B}_2$ be the finite family of all $S \in \mathcal{D}$ satisfying $\frac{1}{m_1} > \text{diam } S \geq \frac{1}{m_2}$ and $\frac{|S \cap K_1|}{|S|} > 1 - \delta_2$.

Inductively, we proceed in the above fashion to obtain a sequence (K_j) of compact sets and a sequence of finite families $(\mathcal{B}_j)$ of the balls of $\mathcal{D}$ such that

(1) $K_j \subset A, \ K_{j-1} \supset K_j$

(2) $|K_j| > a$

(3) for $S \in \mathcal{B}_k$ $\frac{|K_j \cap S|}{|S|} > 1 - \delta_k, \ j \geq k$

(4) diam $S \leq \frac{1}{m_{k-1}}$ for $S \in \mathcal{B}_k$ $(m_o = 1)$

(5) $K_j \subset \bigcup_{S \in \mathcal{B}_j} S$

If we put $K = \bigcap_{j=1}^{\infty} K_j$, it is clear that $K \subset A$ and $|K| \geq a$. Let $x \in K$. For every k there is $S_k \in \mathcal{B}_k$ such that $x \in S_k$ and $\frac{|K_j \cap S_k|}{|S_k|} > 1 - \delta_k$ for $k \leq j$. Therefore

$$\frac{|K \cap S_k|}{|S_k|} \geq 1 - \delta_k \text{ for every } k.$$

Hence there is a sequence of balls $(S_k) \subset \mathcal{D}$ such that $x \in S_k$ for every k and diam $S_k \to 0$. Let us choose $\varepsilon > 0$. We can find k_0 that $k \geq k_0$ implies

$$\frac{|K \cap S_k|}{|S_k|} \geq 1 - \varepsilon .$$

Let us choose another $\varepsilon_1 > 0$. By Lemma 1, there exists $\beta > 0$ such that if we denote by $\tilde{S}_k$ the ball with the center at x and diam $\tilde{S}_k = \beta \cdot$ diam S_k, then

$$\frac{|S_k \cap \tilde{S}_k|}{|S_k|} \geq \frac{1}{2} - \varepsilon_1.$$

Now

$$\frac{|K \cap \tilde{S}_k|}{|\tilde{S}_k|} \geq \frac{|K \cap \tilde{S}_k \cap S_k|}{|\tilde{S}_k|} \geq \frac{|\tilde{S}_k \cap S_k|}{|\tilde{S}_k|} - \varepsilon \frac{|S_k|}{|\tilde{S}_k|} \geq \frac{1}{2} - \varepsilon_1 - \varepsilon_2 \frac{|S_k|}{|\tilde{S}_k|} =$$

$$= \frac{1}{2} - \varepsilon_1 - \varepsilon \beta^n \text{ for } k \geq k_0.$$

Since $\varepsilon_1 > 0$ is arbitrary small, we have $D_s^-(x,K) \geq \frac{1}{2} - \varepsilon \beta^n$. But also $\varepsilon > 0$ is arbitrary and this completes the proof.

<u>**Lemma 3**</u>. If f is a measurable function, then $M_\alpha(f)$ is a measurable set.

<u>Proof</u>. It suffices to show that if r is fixed, then the function $\chi_r(x) = |B(x,r) \cap \{t; f(t) \geq f(x)\}|$ is measurable.

Let $E = \{x; \chi_r(x) < \lambda\}$ for $\lambda > 0$. Let us introduce an auxiliary function $\phi(x)$ given by

$$\phi(x) = \sup\{c \in R;\ |\{t; f(t) \geq c\} \cap B(x,r)| \geq \lambda\}.$$

Clearly, $E = \{x; f(x) > \phi(x)\}$. We will show $\phi(x)$ is an upper semicontinuous function, thereby establishing the lemma.

Let x_0 be fixed, $c > \phi(x_0)$. By definition of $\phi(x)$ we have

$$|\{t; f(t) \geq c\} \cap B(x_0,r) = \lambda - d,\ d > 0.$$

By choosing $\delta > 0$ small enough, we obtain

$$|B(x,r) \setminus B(x_0,r)| < d \text{ for } x \in B(x_0,\delta),$$

hence $|\{t; f(t) \geq c\} \cap B(x,r)| < \lambda + d - d = \lambda$ i.e.

$$\phi(x) \leq c \text{ for } x \in B(x_0,\delta).$$

Proposition. If $f: R_n \to R$ is measurable, then $|M_{\frac{1}{2}}(f)| = 0$.

Proof. Suppose not. There is a positive number $a > 0$ such that $|M_{\frac{1}{2}}(f)| > a$. Since f is measurable, by Lusin's Theorem and Lemma 2, there is a compact $K \subset M_{\frac{1}{2}}(f)$ satisfying (i),(ii) of Lemma 2 and on which f is continuous. Then there is an $x_0 \in K$ that $f(t) \geq f(x_0)$ for every $t \in K$. This implies $K \subset \{t; f(t) \geq f(x_0)\}$ hence

$$D_s^-(x_0, \{t; f(t) \geq f(x_0)\}) \geq \frac{1}{2}$$

contradicting the fact that $x_0 \in M_{\frac{1}{2}}(f)$.

Lemma 4. For each $f: R_n \to R$, $c \in R$, $r > 0$ the function

$$\Theta(x) = |B(x,r) \cap \{t; f(t) \geq c\}|$$

is upper semicontinuous.

Proof. Let $x_0 \in R_n$, $\varepsilon > 0$. Then we can choose $\delta > 0$ such that $|B(x,r) \setminus B(x_0,r)| < \varepsilon$ for $x \in B(x_0,\delta)$. Now

$$\Theta(x) = |B(x,r) \cap \{t; f(t) \geq c\}| \leq |B(x_0,r) \cap \{t; f(t) \geq c\}| +$$

$$+ |(B(x,r) \setminus B(x_0,r)) \cap \{t; f(t) \geq c\}| < \Theta(x_0) + \varepsilon .$$

Corollary. The function $x \to D_s^-(x, \{t; f(t) \geq c\})$ is measurable for each $f: R_n \to R$ and $c \in R$.

Proof. We can write

$$D_s^-(x, \{t; f(t) \geq c\}) = \lim_{m \to \infty} \sup_{\substack{0 < r < \frac{1}{m} \\ r\text{-rational}}} \frac{|\{t; f(t) \geq c\} \cap B(x,r)|}{|B(x,r)|}$$

Theorem. If f is arbitrary, then $|M_{\frac{1}{2}}(f)| = 0$.

Proof. Let us define the function

$$u(x) = \inf\{ c \in Q; D_s^-(x, \{t; f(t) \geq c\}) < \tfrac{1}{2}\}$$

where Q denotes rational numbers. Considering the corollary we get the measurability of u. Further, it is easy to find out that $f(x) \leq u(x)$ almost everywhere, because each set $\{x; f(x) > q > u(x)\}$ where $q \in Q$, has measure zero.

If $x \in M_{\frac{1}{2}}(f)$, then we see from the definition of u that $u(x) \leq f(x)$, i.e. $u(x) = f(x)$ almost everywhere in $M_{\frac{1}{2}}(f)$. Let us denote by M the set of all points of $M_{\frac{1}{2}}(f)$ which are points of outer density of $M_{\frac{1}{2}}(f)$ and $f(x) = u(x)$. Clearly, $|M \cap A| = |M_{\frac{1}{2}}(f) \cap A|$ for each measurable set A. If $x \in M$, then $D_s^-(x, \{t; f(t) \geq f(x)\}) < \frac{1}{2}$ and also

$$D_s^-(x, \{t \in M; f(t) \geq f(x)\}) < \tfrac{1}{2}.$$

Because $u(x) = f(x)$ for $x \in M$, we have

$$D_s^-(x, \{t \in M; u(t) \geq u(x)\}) < \tfrac{1}{2}.$$

This is equivalent to the fact $D_s^-(x, \{t \in M_{\frac{1}{2}}(f); u(t) \geq u(x)\}) < \frac{1}{2}$. Considering that x is a point of outer density of $M_{\frac{1}{2}}(f)$, it is easy to prove that $D_s^-(x, \{t; u(t) \geq u(x)\}) < \frac{1}{2}$, i.e. $x \in M_{\frac{1}{2}}(u)$.

Therefore $M \subset M_{\frac{1}{2}}(u)$, which is of measure zero by Proposition and we have $|M_{\frac{1}{2}}(f)| = 0$.

J. Foran [2] showed that for an arbitrary function f of n variables $|M_{2^{-n}}(f)| = 0$ holds. He formulated a problem (P 1019) there, if the number 2^{-n} can be improved. We can see now that 2^{-n} eas improved to $\frac{1}{2}$ for every n. A further improving is not possible because even the set

$$\{x; D_s^-(x, \{t; f(t) \geq f(x)\}) \leq \tfrac{1}{2}\}$$

equals R_n for $f(x_1, \ldots, x_n) = x_1$.

This problem was also solved in a different way by L. Zajíček [5].

As we said in the introduction, the characterization of the set of points of preponderant maxima is to be of measure zero. It follows from Theorem and from the simple fact that a characteristic function of a set E of measure zero attains its preponderant maxima exactly at the points of E.

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