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ON APPROXIMATE DINI DERIVATES AND ONE-SIDED APPROXIMATE DERIVATIVES OF ARBITRARY FUNCTIONS

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Abstract: By the Jarník-Blumberg method we prove two theorems on approximate Dini derivates which has the following consequences: a) For an arbitrary function at all points except a σ-porous set the existence of an one-sided finite approximate derivative implies the existence of the approximate derivative. b) For an arbitrary function the set of all points at which one one-sided approximate derivative is finite and the other is infinite is countable. By the same method we prove that the finite one-sided approximate derivative is in the Baire class one.

Key words: Approximate Dini derivates, one-sided approximate derivatives, σ-porous sets, Baire class one, Jarník-Blumberg method.

Classification: 26A27

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1. Introduction. In the present article we prove some new results on approximate Dini derivates and one-sided approximate derivatives of arbitrary functions by the Jarník-Blumberg method. The main idea of the present article was used in [15] and thus the present article is in a sense a continuation of [15]. The Jarník-Blumberg method and the notion of σ-porous sets are discussed there and we shall not repeat these remarks and definitions here. We obtain results in three distinct directions:

a) The approximate analogue of the Denjoy-Young-Saks theorem for arbitrary functions ([6], cf. [2]) establishes certain relations, valid almost everywhere, which connect the approximate Dini derivates of arbitrary functions. Namely, for an arbitrary function f almost everywhere at least one from the following relations holds:

(i) There exists $f'_{ap}(x) \in R$.

(ii) $\overline{f}^{+}_{ap}(x) = \overline{f}^{-}_{ap}(x) = +\infty$; $\underline{f}^{+}_{ap}(x) = \underline{f}^{-}_{ap}(x) = -\infty$.

(iii) $\overline{f}^{+}_{ap}(x) = +\infty$, $\underline{f}^{-}_{ap}(x) = -\infty$, $\underline{f}^{+}_{ap}(x) = \overline{f}^{-}_{ap}(x) \in R$.

(iv) $\underline{f}^{+}_{ap}(x) = -\infty$, $\overline{f}^{-}_{ap}(x) = +\infty$, $\overline{f}^{+}_{ap}(x) = \underline{f}^{-}_{ap}(x) \in R$.

Note that in the case of a measurable function f the relations (iii),(iv) are almost everywhere impossible (cf. [10], p. 295). On the other hand, there exists a Lipschitz function f (see Example 3 from 5. section) for which the set of all points x at which at least one from the relations (i),(ii),(iii),(iv) holds is a first category set. Thus we can pose the following problem:

Problem P. What is the strongest relation concerning the approximate Dini derivates of arbitrary functions which holds except a first category set?

The analogical problem for Dini derivates was completely solved in [15], where we used the Jarník-Blumberg method and the Dolženko's theorem [3] on the boundary behaviour of arbitrary functions. In the present article we obtain a partial solution of Problem P using the Jarník-Blumberg method and the approximate analogue of the Dolženko's theorem proved in [13]. Namely, we prove that the set S of all points x at which $\overline{f}^{+}_{ap}(x) \neq \overline{f}^{-}_{ap}(x)$ or $\underline{f}^{+}_{ap}(x) \neq \underline{f}^{-}_{ap}(x)$ and at least one from the numbers max $(|\overline{f}^{+}_{ap}(x)|, |\underline{f}^{+}_{ap}(x)|)$, max $(|\overline{f}^{-}_{ap}(x)|, |\underline{f}^{-}_{ap}(x)|)$ is finite, is a first category set. From the approximate analogue of the Denjoy-Young-Saks theorem follows that S is also a set of measure zero. Actually we prove a little more precise result (Theorem 1) which asserts that S is a σ-porous set. Note that from this result immediately follows that for an arbitrary function f at all points except a σ-porous set the existence of an one-sided finite approximative derivative implies the existence of the approximative derivative. I was not able to solve Problem P completely. Note that H.H. Pu, J.D. Chen and H.W. Pu [9] proved that for any continuous f the relations $\overline{f}^{+}_{ap}(x) = \overline{f}^{-}_{ap}(x)$ and $\underline{f}^{+}_{ap}(x) = \underline{f}^{-}_{ap}(x)$ hold at all points x except a first category set. Examples 2, 3 of the 5. section of the present article show that this result gives the solution of Problem P for continuous functions.

b) It is well known (see e.g. [10], p. 261) that for an arbitrary function f the set of all x for which $\overline{f}^{+}(x) < \underline{f}^{-}(x)$ or $\underline{f}^{+}(x) > \overline{f}^{-}(x)$ is countable. The approximate analogue of this theorem does not hold (it is sufficient to consider the characteristic function of an uncountable null set). On the other hand, from Theorem 1 of [14] immediately follows that the set of all points at which $\overline{f}^{+}_{ap}(x) < \underline{f}^{-}_{ap}(x)$ or $\underline{f}^{+}_{ap}(x) > \overline{f}^{-}_{ap}(x)$, and all the approximate Dini derivates are finite, is countable. Using the Jarník-Blumberg method we strengthen this result, namely we prove that it is sufficient to assume that $\overline{f}^{+}_{ap}(x)$, $\underline{f}^{+}_{ap}(x)$ or $\overline{f}^{-}_{ap}(x)$, $\underline{f}^{-}_{ap}(x)$ are finite (Theorem 2). As an interesting consequence we immediately obtain that for an arbitrary function the set of all points at which an one-sided approximate derivative is finite and the other is infinite is countable.

c) Snyder [11] first used the Jarník-Blumberg method to prove a theorem concerning approximate derivatives. He proved a theorem concerning the boundary behaviour of functions of two variables and has shown that it yields a new proof of the fact [12] that the finite approximate derivative is of Baire class one. Preiss [8] proved that the assumption of finiteness of approximate derivative can be dropped and Mišík [7] has shown that also this theorem can be proved by the Snyder's theorem. We show that the Snyder's theorem also yields that the finite one-sided approximate derivative is of Baire class one. The assumption of the finiteness is substantial.

2. Preliminaries. We denote by R the set of all real numbers and put $\overline{R} = R \cup \{-\infty, \infty\}$. The symbol μ (resp. μ_2) stands for the outer Lebesgue measure in R (resp. in R^2). The open circle of the centre $x \in R^2$ and the radius r is denoted by B(x,r). For $M \subset R$ we put $-M = \{x;\ -x \in M\}$. The Dini derivates of a function f are denoted by $\overline{f}^+(x), \underline{f}^+(x), \overline{f}^-(x), \underline{f}^-(x)$. The one-sided approximate derivatives are denoted by $f'_{ap+}(x)$ and $f'_{ap-}(x)$. The approximate Dini derivates are denoted by $\overline{f}^+_{ap}(x)$, $\underline{f}^+_{ap}(x)$, $\overline{f}^-_{ap}(x)$, $\underline{f}^-_{ap}(x)$. If $M \subset R$ is an arbitrary set, then $d^+(M,x)$ denotes the upper right outer density of M at x, the numbers $d_+(M,x)$, $d^-(M,x)$, $d_-(M,x)$ are defined similarly. The open half-plane $\{(x,y);x>y\}$ will be denoted by H. An open angle $A \subset H$ with the vertex at a point (t,t) is termed an angle at (t,t). If A is an angle at (0,0) then we denote by A_t the image of A under the translation taking (0,0) into (t,t). If $M \subset R^2$ and A is an angle at (t,t), then we define the upper outer density of M at (t,t) with respect to A as

$$d^A(M,(t,t)) = \limsup_{h\to 0_+} \mu_2(M\cap B((t,t),h)\cap A)\,(\mu_2(B((t,t),h)\cap A))^{-1}.$$

The upper lower density is defined similarly. If f is an arbitrary real function in H and A is an angle at (t,t) then we define the approximate limes superior of f at (t,t) with respect to A $\text{ap-}\limsup_{z\to(t,t),\, z\in A} f(z)$ as the upper bound of the numbers $p\in R$ such that $d^A(f^{-1}(p,\infty),(t,t))>0$. Similarly is defined $\text{ap-}\liminf_{z\to(t,t),\, z\in A} f(z)$. If $\text{ap-}\limsup_{z\to(t,t),\, z\in A} f(z) = \text{ap-}\liminf_{z\to(t,t),\, z\in A} f(z)$ then we denote the common value by $\text{ap-}\lim_{z\to(t,t),\, z\in A} f(z)$. It is easy to prove that $\text{ap-}\lim_{z\to(t,t),\, z\in A} f(z) = a$ iff there exists a measurable set $M\subset R^2$ such that $d_A(M,(t,t)) = 1$ and $\lim_{z\to(t,t),\, z\in A} f(z) = a$.

Now we shall formulate three theorems concerning the boundary behaviour of functions of two variables which we shall use in the 4. section.

Theorem A. Let f be an arbitrary function in H. Then the set of all $t\in R$ for which there exist angles A_1, A_2 at (t,t) such that $\text{ap-}\limsup_{z\to(t,t),\, z\in A_1} f(z) \neq \text{ap-}\limsup_{z\to(t,t),\, z\in A_2} f(z)$ is σ-porous.

Proof. The theorem is an easy consequence of Theorem 12 from [13].

Theorem B. Let f be an arbitrary function in H. Then the set of all $t\in R$ for which there exist angles A_1, A_2 at (t,t) such that $\text{ap-}\limsup_{z\to(t,t),\, z\in A_1} f(z) < \text{ap-}\liminf_{z\to(t,t),\, z\in A_2} f(z)$ is countable.

Proof. Use Theorem 13 from [13] or look in [1].

Theorem C. Let f be an arbitrary function in H and A an angle at (0,0). If for each $t\in R$ there exists finite or infinite $\underset{z\to(t,t),\ z\in A_t}{\text{ap-lim}} f(z) := g(t)$, then the function g is of Baire class one.

Proof. The theorem is an easy consequence of Theorem 1 from [11].

3. Lemmas

Lemma 1. Let $v>0$, $t\in R$ and let $M,N\subset R$ be such that $d_+(M,t) = 1$ and $d^-(N,t)>0$. Define the angle A at (t,t) as $A = \{(x,y);y<t<x,\ (t-y)>v(x-t)\}$. Then $d^A(M\times N,(t,t))>0$.

Proof. Put $\alpha = \operatorname{arctg} v$ and $T_r = \{(x,y)\in A;\ r\sin\alpha > t-y\}$ for $r>0$. Further put $S_r = A\cap B((t,t),r)$. Then T_r is an open triangle and $T_r\subset S_r$. Let $C\supset M\times N$ be a measurable set such that $\mu_2(C\cap W) = \mu_2((M\times N)\cap W)$ for an arbitrary measurable set W. Since $d^-(N,t)>0$ there exist $b>C$ and a sequence $p_n\searrow 0$ such that

(1) $(1/p_n)\ \mu((t-p_n,t)\cap N)>b$ for all n.

Put $r_n = p_n/\sin\alpha$. Since $d_+(M,t) = 1$ we have for sufficiently large n

(2) $(1/h)\ \mu((t,t+h)\cap M)>1/2$ whenever $0<h<p_n \operatorname{cotg}\alpha = p_n/v$.

For these n we have by the Fubini theorem

(3) $\mu_2(C\cap T_{r_n}) = \int_{t-p_n}^{t} \mu\{x;(x,y)\in C\cap T_{r_n}\}\,dy$

and by (2)

(4) $\mu\{x;(x,y)\in C\cap T_{r_n}\}>(1/2)(t-y)v^{-1}$ whenever $y\in(t-p_n,t)\cap N$.

From (1) follows $\mu((t-p_n,t-b\,p_n/2)\cap N)>b\,p_n/2$. For $y\in(t-p_n,t-b\,p_n/2)\cap N$ we have by (4)

$\mu(\{x;(x,y)\in C\cap T_{r_n}\})>b\,p_n/4v$. Therefore by (3) we obtain $\mu_2(C\cap T_{r_n})>(b\,p_n/2)(b\,p_n/4v) = K\,r_n^2$ and

$\mu_2(C\cap S_{r_n})/\ \mu_2 S_{r_n} > K\ r_n^2/\ \mu_2 S_{r_n} = L$, where K, L do not depend on n. Consequently $d^A(M\times N,(t,t)) = d^A(C,(t,t)) \geqq L > 0$.

Lemma 2. Let $v > 0$, $t \in R$ and let $M, N \subset R$ be measurable sets such that $d_+(M,t) = 1$, $d_-(N,t) = 1$. Put $A = \{(x,y); y < t < x, (t-y) > v(x-t)\}$. Then $d_A(M\times N,(t,t)) = 1$.

Proof. Let $\varepsilon > 0$. Then for sufficiently small $r > 0$ we have $\mu(M \cap (t,t+r)) > (1-\varepsilon)r$ and $\mu(N \cap (t-r,t)) > (1-\varepsilon)\ r$. Put $C_r = (t,t+r) \times (t-r,t)$ and $S_r = A \cap B((t,t),r)$. By the Fubini theorem we have for sufficiently small r $\mu_2((M\times N) \cap C_r) > (1-\varepsilon)^2\ r^2$. Therefore we have $\lim_{r\to 0_+} \mu_2((M\times N)\cap C_r)/\ \mu_2(C_r) = 1$. Obviously $S_r \subset C_r$ and $\mu_2(C_r)/\ \mu_2(S_r)$ does not depend on r. Consequently $d_A(M\times N,(t,t)) = 1$.

Lemma 3. Let $v > 0$, $t \in R$ and let $M \subset R$ be a measurable set such that $d_+(M,t) = 1$. Put $A = \{(x,y); x > y > t, (x-t) > v(y-t)\}$. Then $d_A(M\times M,(t,t)) = 1$.

Proof. The proof is quite similar to the proof of Lemma 2.

In the rest of the present section f is an arbitrary real function on R and $g(x,y) = (f(x) - f(y))(x-y)^{-1}$.

Lemma 4. Let $\overline{f}^+_{ap}(t)$, $\underline{f}^+_{ap}(t)$ be finite and $\overline{f}^+_{ap}(t) < T$, $T \in R$. Then there exists an angle A at (t,t) such that

$\operatorname{ap-lim\,sup}_{z\to(t,t),\, z\in A} g(z) < T$.

Proof. Choose real numbers b, B such that $b < \underline{f}^+_{ap}(t) \leq \overline{f}^+_{ap}(t) < B < T$. By the definition of the approximate derivates there exists a measurable set $M \subset R$ such that $d_+(M,t) = 1$ and

(5) $\qquad b < g(x,t) < B$ for $x \in M$.

Let $v > 1$. Put $A^v = \{(x,y); x > y > t, (x-t) > v(y-t)\}$. By Lemma 3 we have $d_{A^v}(M\times M,(t,t)) = 1$. For $(x,y) \in A^v$ obviously $(y-t)/(x-y) <$

$< 1/(v-1)$. Therefore for $(x,y) \in (M\times M) \cap A^v$ we have by (5)

$$g(x,y) = \frac{(f(x)-f(t))+(f(t)-f(y))}{x-y} \leq \frac{(x-t)\,B+(t-y)\,b}{x-y} =$$

$= B + (b-B)(t-y)/(x-y) \leq B + (|b| + |B|)/(v-1)$.

Consequently there exists $v > 1$ such that $\text{ap-lim}\sup_{z\to(t,t),\, z\in A} g(z) < T$.

Lemma 5. Let $\underline{f}^+_{ap}(t) > -\infty$ and $\overline{f}^-_{ap}(t) > a \in R$. Then there exists an angle A at (t,t) such that $\text{ap-lim}\sup_{z\to(t,t),\, z\in A} g(z) > a$.

Proof. Choose real numbers q, b such that $\underline{f}^+_{ap}(t) > q$ and $\overline{f}^-_{ap}(t) > b > a$. Since $\underline{f}^+_{ap}(t) > q$ there exists a measurable set $M \subset R$ such that $d_+(M,t) = 1$ and $g(x,t) > q$ whenever $x \in M$. Since $\overline{f}^-_{ap}(t) > b$ there exists a set $N \subset R$ such that $d^-(N,t) > 0$ and $g(y,t) > b$ whenever $y \in N$. For $v > 0$ put $A_v = \{(x,y); y < t < x, (t-y) > v(x-t)\}$. By Lemma 1 $d^{A_v}(M\times N,(t,t)) > 0$ and for $(x,y) \in A_v$ obviously $(x-t)/(x-y) < 1/v$. Therefore for $(x,y) \in A_v \cap (M\times N)$ we have

$$g(x,y) = \frac{(f(x)-f(t))+(f(t)-f(y))}{x-y} > \frac{q(x-t)+b(t-y)}{x-y} =$$

$= b + (q-b)(x-t)/(x-y) > b - (|q| + |b|)/v$.

Consequently there exists $v > 0$ such that $\text{ap-lim}\sup_{z\to(t,t),\, z\in A_v} g(z) > a$.

Lemma 6. Let $\underline{f}^+_{ap}(t) > -\infty$ and $\underline{f}^-_{ap}(t) > \overline{f}^+_{ap}(t)$. Then there exists an angle A at (t,t) such that $\text{ap-lim}\inf_{z\to(t,t),\, z\in A} g(z) > \overline{f}^+_{ap}(t)$.

Proof. Let b, q be such real numbers that $\underline{f}^-_{ap}(t) > b > \overline{f}^+_{ap}(t)$ and $\underline{f}^+_{ap}(t) > q$. By the definition of approximate derivates there exist measurable sets $M, N \subset R$ such that $d_+(M,t) = 1$, $d_-(N,t) = 1$, $g(x,t) > q$ for $x \in M$ and $g(y,t) > b$ for $y \in N$. Let the symbol A_v have the same meaning as in Lemma 5. By Lemma 2 we have $d_{A_v}(M\times N,(t,t)) = 1$. For $(x,y) \in (M\times N) \cap A_v$ we obtain by the same way as in the proof of Lemma 5 $g(x,y) > b - (|q| + |b|)/v$. Consequently there exists $v > 0$ such that $\text{ap-}\lim_{z \to (t,t),\, z \in A_v} \inf g(z) > \bar{f}^{+}_{ap}(t)$.

4. Theorems

Theorem 1. Let f be an arbitrary function on R. Then there exists a σ-porous set P such that for any $t \in R - P$

(i) $\underline{f}^{+}_{ap}(t) = \underline{f}^{-}_{ap}(t)$, $\bar{f}^{+}_{ap}(t) = \bar{f}^{-}_{ap}(t)$ or

(ii) $\max(|\underline{f}^{+}_{ap}(t)|, |\bar{f}^{+}_{ap}(t)|) = \max(|\underline{f}^{-}_{ap}(t)|, |\bar{f}^{-}_{ap}(t)|) = +\infty$.

Proof. For an arbitrary function f on R denote by S(f) the set of all points t at which $-\infty < \underline{f}^{+}_{ap}(t) \leq \bar{f}^{+}_{ap}(t) < \bar{f}^{-}_{ap}(t)$. By Lemma 5 for any $t \in S(f)$ there exists an angle A at (t,t) such that $\text{ap-}\lim_{z \to (t,t),\, t \in A} \sup g(z) > \bar{f}^{+}_{ap}(t)$. Therefore by Lemma 4 there exists an angle A^* at (t,t) such that $\text{ap-}\lim_{z \to (t,t),\, t \in A} \sup g(z) > \text{ap-}\lim_{z \to (t,t),\, t \in A^*} \sup g(z)$. Thus by Theorem A the set S(f) is σ-porous for any function f. Let P be the set of all points at which no from the relations (i),(ii) holds. Then it is easy to prove that

$$P \subset S(f(x)) \cup S(-f(x)) \cup (-S(f(-x))) \cup (-S(-f(-x))).$$

Therefore P is σ-porous.

Corollary. For an arbitrary function f the set of all points at which an one-sided approximate derivative of f exists and is finite but the approximate derivative does not exist is σ-porous.

Theorem 2. Let f be an arbitrary function on R. Then there exists a countable set C such that for any $x \in R - C$ at least one from the following relations holds:

(i) $\underline{f}^-_{ap}(x) \leqq \overline{f}^+_{ap}(x)$ and $\underline{f}^+_{ap}(x) \leqq \overline{f}^-_{ap}(x)$

(ii) $\underline{f}^-_{ap}(x) = -\infty$ and $\overline{f}^+_{ap}(x) = +\infty$

(iii) $\overline{f}^-_{ap}(x) = +\infty$ and $\underline{f}^+_{ap}(x) = -\infty$.

Proof. For an arbitrary function f on R denote by Q(f) the set of all points at which $-\infty < \underline{f}^+_{ap}(t) \leqq \overline{f}^+_{ap}(t) < \underline{f}^-_{ap}(t)$. Let $t \in Q(f)$. Then by Lemma 6 there exists an angle A at (t,t) such that $\text{ap-lim}_{z\to(t,t),\, z\in A} \inf g(z) > \overline{f}^+_{ap}(t)$. By Lemma 4 there exists an angle A^* at (t,t) such that $\text{ap-lim}_{z\to(t,t),\, z\in A^*} \sup g(z) <$ $< \text{ap-lim}_{z\to(t,t),\, z\in A} \inf g(z)$. Therefore by Theorem B the set Q(f) is countable for any function f. Let C be the set of all points at which no from the relations (i),(ii),(iii) holds. Then

$$C \subset Q(f(x)) \cup Q(-f(x)) \cup (-Q(f(-x))) \cup (-Q(-f(-x)))$$

and therefore C is countable.

Corollary. For an arbitrary function f on R the set of all points at which the one-sided approximate derivatives of f exist, are not equal and one from them is finite, is countable.

Theorem 3. Let f be a function on R for which at each $t \in R$ $f'_{ap+}(t) \in R$. Then the function $a(t) := f'_{ap+}(t)$ is in the Baire class one.

Proof. By Theorem C it is sufficient to prove that for any $t \in R$ $a(t) = \text{ap-lim}_{z\to(t,t),\, z\in A_t} g(z)$, where $A_t = \{(x,y); t < y < x, (x-t) > 2(y-t)\}$. Let $t \in R$. By the definition of $f'_{ap+}(t)$ there exists a measurable set M such that $d_+(M,t) = 1$ and $\lim_{x\in t,\, x\in M} g(x,t) = a(t)$. By Lemma 3 $d_{A_t}(M \times M, (t,t)) = 1$ and therefore it is sufficient to prove

$$\lim_{z\to(t,t),\, z\in A_t \cap (M\times M)} g(z) = a(t). \tag{6}$$

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that a(t)-

$-\varepsilon < g(x,t) < a(t) + \varepsilon$ whenever $|x-t| < \delta$ and $x \in M$. Therefore for $(x,y) \in (M\times M) \cap A_t \cap B((t,t),\delta)$ we have

$$g(x,y) = \frac{(f(x)-f(t))+(f(t)-f(y))}{x-y} \leqq \frac{(a(t)+\varepsilon)(x-t)+(t-y)(a(t)-\varepsilon)}{x-y} =$$

$= a(t) + \varepsilon((x-t) + (y-t))/((x-y) \leqq a(t) + 3\varepsilon$, and analogically we obtain $g(x,y) \geqq a(t) - 3\varepsilon$. Thus (6) is proved and the proof is complete.

Note. Example 1 of the following section shows that the assumption of the finiteness of $a(t)$ is substantial.

5. Examples

Example 1. Let f be the well known Dirichlet function. Then obviously $f'_{ap+}(x) = 0$ for irrational x and $f'_{ap+}(x) = -\infty$ for rational x. Therefore f'_{ap+} is not in the Baire class one.

Example 2. Let f be the well known Weierstrass function (see e.g. [5], p. 141). Then at all points except a first category set $\overline{f}^+(x) = \overline{f}^-(x) = +\infty$ and $\underline{f}^+(x) = \underline{f}^-(x) = -\infty$ ([5], p. 142). Since for any continuous function g at all points of a residual set $\overline{g}^+(x) = \overline{g}^-(x) = \overline{g}^+_{ap}(x) = \overline{g}^-_{ap}(x)$ and $\underline{g}^+(x) = \underline{g}^-(x) = \underline{g}^+_{ap}(x) = \underline{g}^-_{ap}(x)$ (see [9] and [4]) we obtain that $\overline{f}^+_{ap}(x) = \overline{f}^-_{ap}(x) = +\infty$ and $\underline{f}^+_{ap}(x) = \underline{f}^-_{ap}(x) = -\infty$ at all points except a first category set.

Example 3. Let the real numbers $a \leqq b$ be given. Let $g = f_1$ and $h = f_2$, where f_1, f_2 are the functions from the Examples 1, 2 from [15]. The functions g, h are continuous. Using the same theorem as in the Example 2 we obtain that $\overline{g}^+_{ap}(x) = \overline{g}^-_{ap}(x) = b$, $\underline{g}^+_{ap}(x) = \underline{g}^-_{ap}(x) = a$, $\overline{h}^+_{ap}(x) = \overline{h}^-_{ap}(x) =$

$= +\infty$, $\underline{h}^{+}_{ap}(x) = \underline{h}^{-}_{ap}(x) = a$ at all points except a first category set.

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