Josef Jirásko

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NOTES ON GENERALIZED PRIME AND COPRIME MODULES II.

Josef JIRÁSKO

Abstract: The dualization of the notions generalized prime and semiprime module which are introduced in [7] and [16] is given. Generalized coprime and semicoprime modules as well as rings in which every module is generalized coprime (semicoprime) are characterized.

Key words: Coprime modules, semicoprime modules, their generalizations.

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In what follows R stands for an associative ring with unit element and R-mod denotes the category of all unitary left R-modules.

A preradical r for R-mod is a subfunctor of the identity functor i.e. r assigns to every module M its submodule r(M) such that every homomorphism $f:M \longrightarrow N$ induces a homomorphism from r(M) into r(N) by restriction.

The identity functor will be denoted by id.

A module M is r-torsion if r(M) = M and r-torsionfree if r(M) = 0. The class of all r-torsion (r-torsionfree) modules will be denoted by $\mathcal{T}_r$ ($\mathcal{F}_r$).

A preradical r is said to be

- idempotent if $r(M) \in \mathcal{T}_r$ for every module M,

- a radical if $M/r(M) \in \mathcal{F}_r$ for every module M,
- hereditary if for every module M and every monomorphism $f: A \longrightarrow r(M)$, $A \in \mathcal{T}_r$,
- superhereditary if it is hereditary and $\mathcal{T}_r$ is closed under direct products,
- cohereditary if for every module M and every epimorphism $f: M/r(M) \longrightarrow A$, $A \in \mathcal{F}_r$,
- pseudocohereditary if for every injective module M and every epimorphism $f: M/r(M) \longrightarrow A$, $A \in \mathcal{F}_r$.

The radical closure $\tilde{r}$ of a preradical r is defined by $\tilde{r}(M) = \bigcap L$, where L runs through all submodules L of M with $M/L \in \mathcal{F}_r$ and the hereditary closure $h(r)$ of a preradical r is defined by $h(r)(M) = M \cap r(E(M))$, $M \in R\text{-mod}$. $E(M)$ will be denoted an injective hull of a module M.

The superhereditary (cohereditary) preradical corresponding to a two-sided ideal I is defined by $s(M) = \{m \in M;\ Im = 0\}$ ($s(M) = IM$), $M \in R\text{-mod}$.

A submodule N of a module M is characteristic in M if there is a preradical r such that $N = r(M)$.

For a non-empty class of modules $\mathcal{A}$ $p_{\mathcal{A}}$ denotes the idempotent preradical defined by $p_{\mathcal{A}}(M) = \sum \operatorname{Im} f$, $f \in \operatorname{Hom}_R(A, M)$, $A \in \mathcal{A}$.

A module M is cofaithful if $h(p_{\{M\}}) = id$.

A submodule N of a module M is

- essential in M if $K \subseteq M$, $K \cap N = 0$ implies $K = 0$,
- small in M if $K \subseteq M$, $K + N = M$ implies $K = M$,
- d-complement in M if there is a submodule V of M such that N is minimal in the set of all submodules K of M with $K + V = M$.

A module M is cocyclic if there is a simple module S such that S is essential in M,

- hollow if every proper submodule N of M is small in M.

A ring R is

- left strongly perfect if it is isomorphic to a (finite) direct sum of full matrix rings over left perfect local rings.

Finally Soc(J) will be denoted the Socle (Jacobson radical).

The following proposition is dual to the Proposition 0.1 of [16]. We present it here without the proof.

Proposition 0.1. Let $M \in$ R-mod. Then the following are equivalent:

(i) $p_{\{M\}}$ is pseudocohereditary ($\widetilde{p_{\{M\}}}$ is pseudocohereditary)

(ii) if $0 \longrightarrow K \hookrightarrow P \longrightarrow M \longrightarrow 0$ is a projective presentation of M then $P = K + h(p_{\{M\}})(P)$ ($P = K + h(\widetilde{p_{\{M\}}})(P)$),

(iii) there is a projective presentation $0 \longrightarrow K \hookrightarrow P \longrightarrow M \longrightarrow 0$ of M such that $P = K + h(p_{\{M\}})(P)$) ($P=K+h(\widetilde{p_{\{M\}}})(P)$) .

Corollary 0.2. Let R be a left hereditary ring and $M \in$ R-mod. Then the following are equivalent:

(i) $p_{\{M\}}$ is pseudocohereditary ($\widetilde{p_{\{M\}}}$ is pseudocohereditary),

(ii) there is an $h(p_{\{M\}})$-torsion ($h(\widetilde{p_{\{M\}}})$-torsion) projective presentation of M,

(iii) there is a projective presentation $0 \longrightarrow K \hookrightarrow P \longrightarrow M \longrightarrow 0$ of M such that $h(p_{\{M\}}) = h(p_{\{P\}})$ ($h(\widetilde{p_{\{M\}}}) = h(p_{\{P\}})$).

Corollary 0.3. Let $M \in R$-mod with a projective cover $C(M) \xrightarrow{\varphi_M} M$. Then the following are equivalent:

(i) $p_{\{M\}}$ is pseudocohereditary ($\widetilde{p_{\{M\}}}$ is pseudocohereditary)

(ii) $h(p_{\{M\}})(C(M)) = C(M)$ $(h(\widetilde{p_{\{M\}}})(C(M)) = C(M))$,

(iii) $h(p_{\{M\}}) = h(p_{\{C(M)\}})$ $(h(\widetilde{p_{\{M\}}}) = h(p_{\{C(M)\}}))$.

§ 1. Coprime and semicoprime modules

1.1. A module M is called

- coprime if $p_{\{M/N\}}(M) = M$ for every proper submodule N of M,
- pseudocoprime if $h(p_{\{M/N\}})(M) = M$ for every proper submodule N of M,
- r-coprime if $\widetilde{p_{\{M/N\}}}(M) = M$ for every proper submodule N of M,
- r-pseudocoprime if $h(\widetilde{p_{\{M/N\}}})(M) = M$ for every proper submodule N of M,
- semicoprime if $N + p_{\{M/N\}}(M) = M$ for every proper submodule N of M,
- pseudo-semicoprime if $N + h(p_{\{M/N\}})(M) = M$ for every proper submodule N of M,
- r-semicoprime if $N + \widetilde{p_{\{M/N\}}}(M) = M$ for every proper submodule N of M,
- r-pseudo-semicoprime if $N + h(\widetilde{p_{\{M/N\}}})(M) = M$ for every proper submodule N of M.

For modules M, N and their submodules $A \subseteq M$ and $B \subseteq N$ let us define $s(A,M,B,N)$ by $s(A,M,B,N) = \bigcap f^{-1}(A)$, $f \in \mathrm{Hom}_R(N,M)$, $f(B) = 0$.

Proposition 1.2. Let $M \in R$-mod. Then

(i) M is coprime if and only if $p_{\{M\}} = p_{\{M/N\}}$ for every proper submodule N of M if and only if $s(A,M,B,M) \neq M$ for all proper submodules $A,B \subsetneqq M$,

(ii) M is pseudocoprime if and only if $h(p_{\{M\}}) = h(p_{\{M/N\}})$ for every proper submodule N of M if and only if $s(A,E(M),B,M) \neq M$ for all $A \subseteq E(M)$, $M \nsubseteq A$ and $B \subsetneqq M$,

(iii) M is r-coprime if and only if $\widetilde{p_{\{M\}}} = \widetilde{p_{\{M/N\}}}$ for every proper submodule N of M if and only if $s(0,M/A,B,M) \neq M$ for all proper submodules $A,B \subseteq M$,

(iv) M is r-pseudocoprime if and only if $s(0,E(M)/A,B,M) \neq M$ for all $A \subseteq E(M)$, $M \nsubseteq A$ and $B \subsetneqq M$,

(v) M is semicoprime if and only if $s(A,M,B,M) \neq M$ for all submodules $A,B \subseteq M$ with $A + B \neq M$ if and only if $s(A,M,A,M) \neq M$ for every proper submodule A of M,

(vi) M is pseudo-semicoprime if and only if $s(A,E(M),A \cap M,M) \neq M$ for every $A \subseteq E(M)$, $M \nsubseteq A$ if and only if $s(A,E(M),B,M) \neq M$ for all $A \subseteq E(M)$, $B \subseteq M$ with $B + A \nsupseteq M$,

(vii) M is r-semicoprime if and only if $s(0,M/A,B,M) \neq M$ for all submodules $A,B \subseteq M$ with $A + B \neq M$,

(viii) M is r-pseudo-semicoprime if and only if $s(0,E(M)/A,B,M) \neq M$ for all $A \subseteq E(M)$, $B \subseteq M$ with $A + B \nsupseteq M$.

<u>Proof</u>. (i) was proved in [7].

(viii). If M is r-pseudo-semicoprime, $A \subseteq E(M)$, $B \subseteq M$, $M \nsubseteq A + B$ and $s(0,E(M)/A,B,M) = M$ then $\widetilde{p_{\{M/B\}}}(E(M)/A) = 0$ and hence $M = B + h(\widetilde{p_{\{M/B\}}})(M) \subseteq B + A \cap M \subseteq B + A$, a contradiction. Conversely suppose $N \subsetneqq M$ and $N + h(\widetilde{p_{\{M/N\}}})(M) \neq M$. Put $A = \widetilde{p_{\{M/N\}}}(E(M))$ and $B = N$. Then $M \nsubseteq A + B$ and hence $s(0,E(M)/A,B,M) \neq M$. Thus $\widetilde{p_{\{M/B\}}}(E(M)/A) \neq 0$, a contradiction.

The rest can be proved similarly as in (viii).

Remark 1.3. In Proposition 1.2 N and B can be replaced by N and B with M/N and M/B cocyclic and E(M) by Q, where $M \subseteq Q$, Q injective.

Proposition 1.4. Let $M \in$ R-mod.

If M is injective then

(i) M is coprime if and only if M is pseudocoprime,

(ii) M is r-coprime if and only if M is r-pseudocoprime,

(iii) M is semicoprime if and only if M is pseudosemicoprime,

(iv) M is r-semicoprime if and only if M is r-pseudo-semicoprime.

If M is hollow then

(v) M is coprime if and only if M is semicoprime,

(vi) M is r-coprime if and only if M is r-semicoprime,

(vii) M is pseudocoprime if and only if M is pseudo-semicoprime,

(viii) M is r-pseudocoprime if and only if M is r-pseudo-semicoprime

Proof. Obvious.

Proposition 1.5. Every completely reducible module is semicoprime.

Proof. Obvious.

Remark 1.6. The classes of all coprime, pseudocoprime, r-coprime, r-pseudocoprime, semicoprime, pseudo-semicoprime, r-semicoprime and r-pseudo-semicoprime modules are closed under factormodules.

Proposition 1.7. Let $N \subseteq M \subseteq Q$, where Q is injective. Then

(i) N is pseudocoprime if and only if $N \nsubseteq s(A,Q,B,M)$ whenever $N \nsubseteq A$ and $N \nsubseteq B$,

(ii) N is pseudo-semicoprime if and only if $N \nsubseteq s(A,Q,B,M)$ whenever $N \nsubseteq A + B$;

if M is injective then

(iii) N is coprime implies $N \nsubseteq s(A,M,B,M)$ whenever $N \nsubseteq A$ and $N \nsubseteq B$,

(iv) N is semicoprime implies $N \nsubseteq s(A,M,B,M)$ whenever $N \nsubseteq A + B$;

if N is a characteristic submodule of M then

(v) if $N \nsubseteq s(A,M,B,M)$ whenever $N \nsubseteq A$ and $N \nsubseteq B$ then N is coprime,

(vi) if $N \nsubseteq s(A,M,B,M)$ whenever $N \nsubseteq A + B$ then N is semicoprime.

Proof. (iii) was proved in [7].

(ii). If $A \subseteq Q$, $B \subseteq M$, $N \nsubseteq A + B$ and N is pseudo-semicoprime then $s(A,Q,B \cap N,N) \neq N$. Hence there is a homomorphism $f: N \longrightarrow Q$, $f(B \cap N) = 0$ such that $\mathrm{Im}\, f \nsubseteq A$. Now f can be extended to a homomorphism $g: M \longrightarrow Q$ with $g(B) = 0$. Thus $g(N) \nsubseteq A$ and consequently $N \nsubseteq s(A,Q,B,M)$.

Conversely if $A \subseteq Q$, $B \subseteq N$ and $N \nsubseteq A + B$ then $N \nsubseteq s(A,Q,B,M)$ by assumption and hence there is a homomorphism $f: M \longrightarrow Q$, $f(B) = 0$ such that $f(N) \nsubseteq A$. Now it suffices to restrict f to N. We have $s(A,Q,B,N) \neq N$.

The rest can be proved similarly.

Proposition 1.8. Every d-complement of a pseudocoprime module is pseudocoprime.

Proof. Let N be a d-complement of a pseudocoprime module M and $V \subsetneq M$ such that N is minimal in the set of all submodules D of M with the property D + V = M. Suppose $A \subseteq E(M)$, $B \subseteq M$, $N \nsubseteq A$, $N \nsubseteq B$ and $N \subseteq s(A,E(M),B,M)$. Then $B \cap N \subsetneq N$ and hence $(B \cap N) + V \subsetneq M$. Further $s(A,E(M),((B \cap N) + V),M) \supseteq s(A,E(M),(B \cap N),M) \supseteq s(A,E(M),B,M) \cap s(A,E(M),N,M) \supseteq N$. Thus $s(A,E(M),((B \cap N) + V),M) \supseteq N + V = M$. Hence $M \subseteq A$ since $(B \cap N) + V \subsetneq M$ and M is pseudocoprime, a contradiction. Therefore N is pseudocoprime by Proposition 1.7.

Proposition 1.9. Let I be a two-sided ideal in R, s be the superhereditary and r the cohereditary preradical corresponding to I. Then

(i) M is pseudocoprime implies r(M) = 0 if $r(M) \neq M$,

(ii) M is pseudo-semicoprime implies s(M) + r(M) = M.

Moreover, if I is idempotent then

(iii) M is r-pseudocoprime implies r(M) = 0 if $r(M) \neq M$,

(iv) M is r-pseudo-semicoprime implies s(M) + r(M) = M.

Proof. (iv). As it is easy to see $p_{\{M/r(M)\}}(M) \subseteq s(M)$. Hence $h(\widehat{p_{\{M/r(M)\}}})(M) \subseteq s(M)$ since I is idempotent. Now if $r(M) \neq M$ then $M = r(M) + h(\widehat{p_{\{M/r(M)\}}})(M) \subseteq r(M) + s(M)$. The remaining assertions can be proved similarly.

Corollary 1.10. Let M be a pseudocoprime module such that $Soc(R/(0:M)) \neq 0$. Then M is completely reducible.

Proof. It follows from Proposition 1.9 (i).

Proposition 1.11. Let $M \in$ R-mod. Then

(i) if M is pseudocoprime and $J(M) \neq M$ then M is completely reducible,

(ii) if M is r-pseudocoprime and $J(M) \neq M$ then $\widetilde{Soc}(M) =$
$= M$,

(iii) if M is pseudo-semicoprime then $J(M/Soc(M)) =$
$= M/Soc(M)$,

(iv) if M is r-pseudo-semicoprime then $J(M/\widetilde{Soc}(M)) =$
$= M/\widetilde{Soc}(M)$,

(v) if M is finitely generated pseudo-semicoprime then M is completely reducible,

(vi) if M is finitely generated r-pseudo-semicoprime then $\widetilde{Soc}(M) = M$.

Proof. (v) and (vi) follow immediately from (iii) and (iv).

(ii). Let N be a maximal submodule of M. Then $M =$
$= h(\widetilde{p_{\{M/N\}}})(M) \subseteq \widetilde{Soc}(M)$ since M is r-pseudocoprime.

(i) can be proved similarly as (ii).

(iv). Let $\widetilde{Soc}(M) \neq M$. If $J(M/\widetilde{Soc}(M)) \neq M/\widetilde{Soc}(M)$ then there is a maximal submodule N of M with $\widetilde{Soc}(M) \subseteq N$. Hence $M = N +$
$+ h(\widetilde{p_{\{M/N\}}})(M) \subseteq N + \widetilde{Soc}(M) = N$, a contradiction. Thus
$J(M/\widetilde{Soc}(M)) = M/\widetilde{Soc}(M)$.

(iii) can be proved similarly as (iv).

Proposition 1.12.

(i) Every module is coprime iff every module is pseudocoprime iff R is coprime iff R is pseudocoprime iff every nonzero module is a generator iff every nonzero module is cofaithful iff R is isomorphic to a matrix ring over a skew-field.

(ii) Every module is semicoprime iff every module is pseudo-semicoprime iff R is semicoprime iff R is pseudo-semicoprime iff $p_{\{M\}}$ is cohereditary for every module M iff $p_{\{M\}}$ is pseudo-cohereditary for every module M iff every idempotent preradical is cohereditary iff every idempotent preradical is pseudocohereditary iff R is a completely reducible ring.

(iii) Every module is r-coprime iff $\widetilde{p_{\{M\}}} = id$ for every nonzero module M iff R is isomorphic to a matrix ring over local left and right perfect ring.

(iv) Every module is r-pseudo-coprime iff $h(\widetilde{p_{\{M\}}}) = id$ for every nonzero module M. If every module is r-pseudo-coprime then R is isomorphic to a matrix ring over local right perfect ring. Moreover if R is left hereditary then the converse is true.

(v) Every module is r-semicoprime iff $\widetilde{p_{\{M\}}}$ is cohereditary for every module M iff every idempotent radical is cohereditary iff R is left and right strongly perfect ring.

(vi) Every module is r-pseudo-semicoprime iff $\widetilde{p_{\{M\}}}$ is pseudocohereditary for every module M iff every idempotent radical is pseudocohereditary. If every module is r-pseudo-semicoprime then R is a right strongly perfect ring. Moreover if R is left hereditary then the converse is true.

<u>Proof</u>. The equivalence of the first and last condition of (i) was proved in [7]. Further every module is coprime (pseudocoprime) iff $p_{\{M\}} = id$ $(h(p_{\{M\}}) = id)$ for every nonzero module M iff R has no nontrivial idempotent (hereditary) preradicals. The rest follows from Proposition 1.11 (i) or it is clear.

(ii). It follows from Propositions 1.5, 1.11 (v) or it is clear.

(iii). As it is easy to see every module is r-coprime iff $\widetilde{p_{\{M\}}}$ = id for every nonzero module M iff R has no nontrivial idempotent radicals. The rest follows from [15], Proposition VI.1.24.

(iv). Every module is r-pseudo-coprime iff $h(\widetilde{p_{\{M\}}})$ = id for every nonzero module M. If $h(\widetilde{p_{\{M\}}})$ = id for every nonzero module M then R has no nontrivial hereditary radicals. If R is left hereditary then the converse is true. Now it suffices to use [15], Proposition VI.1.20.

(v). Every module is r-semicoprime iff $\widetilde{p_{\{M\}}}$ is cohereditary for every module M iff every idempotent radical is cohereditary and it suffices to use [15], Proposition VI.1.25.

(vi). Every module is r-pseudo-semicoprime iff $\widetilde{p_{\{M\}}}$ is pseudocohereditary for every module M iff every idempotent radical is pseudocohereditary. In this case R is right strongly perfect by [15], Proposition VI.1.21 since every hereditary radical is cohereditary. Now if R is left hereditary then h(r) is a hereditary radical for a radical r hence h(r) is cohereditary in a right strongly perfect ring if r is a radical and consequently every radical is pseudocohereditary in this case.

Let $\mathcal{A}$ be the class of all pseudocoprime modules. Put $\mathcal{R}_1 = p_{\mathcal{A}}$

<u>Proposition 1.13</u>. Every module M with $\mathcal{R}_1(M) = M$ is pseudosemicoprime.

<u>Proof</u>. If $\mathcal{R}_1(M) = M$, $A \subseteq E(M)$, $M \nsubseteq A$ then $\mathcal{R}_1(M) \nsubseteq A$. Hence there is a pseudocoprime module N and a homomorphism $f: N \longrightarrow M$ such that $f(N) \nsubseteq A$. Further there is a homomorphism

$h:E(N) \longrightarrow E(M)$ such that $h \circ i_N = i_M \cdot f$, where $i_X : X \hookrightarrow E(X)$ is the inclusion. Now N is pseudocoprime hence $s(h^{-1}(A), E(N), N \cap h^{-1}(A), N) \neq N$. Thus there is a homomorphism $k:N \longrightarrow E(N)$ with $k(N \cap h^{-1}(A)) = 0$ such that $\operatorname{Im} k \nsubseteq h^{-1}(A)$. Consider the following diagram

$$\begin{array}{ccc} N/(N \cap h^{-1}(A)) & \xrightarrow{\ \bar{h}\ } & M/(A \cap M) \\ {\scriptstyle \bar{k}} \downarrow & & \\ E(N) \ , & & \end{array}$$

where $\bar{h}$, $\bar{k}$ are induced by h, k respectively. Now $\bar{h}$ is a monomorphism hence there is a homomorphism $p:M/(A \cap M) \longrightarrow E(N)$ which makes this diagram commutative. Put $q = hp\pi$, where $\pi : M \longrightarrow$ $\longrightarrow M/(A \cap M)$ is the natural epimorphism. As it is easy to see $\operatorname{Im} q \nsubseteq A$. Hence $s(A, E(M), A \cap M, M) \neq M$ and consequently M is pseudo-semicoprime by Proposition 1.2 (vi).

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Bělohorská 137, 16900 Praha 6, Československo

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