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ON CONGRUENCE LATTICES OF FINITE PARTIAL UNARY ALGEBRAS

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Abstract: In this note there are investigated finite partial unary algebras $\mathcal{A}$ having the property that the height of Con $\mathcal{A}$ is 2. It is shown that there are 10 types of such algebras (the classification being performed by means of properties of subalgebras) and that for each type τ the following alternative is valid: either (a) card Con $\mathcal{A} \leq 5$ for each $\mathcal{A} \in \tau$, or (b) for each positive integer $n > 5$ there exists $\mathcal{A} \in \tau$ with card Con $\mathcal{A} = n$.

Key words: Congruence lattice, unary algebra, partial unary algebra.

Classification: 08A60

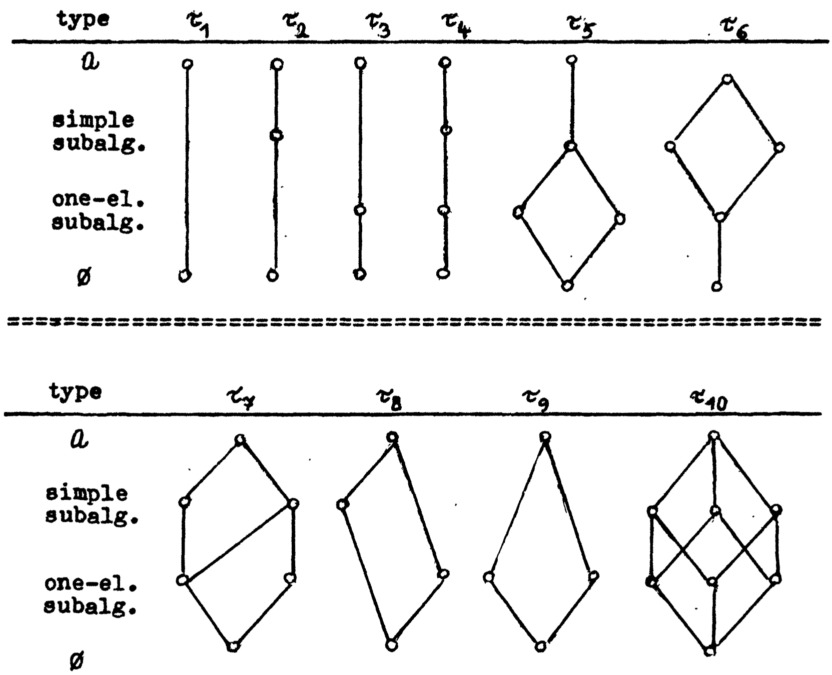
P.P. Pálfy [3] inspired by a problem proposed by P. Goralčík [1] investigated finite unary algebras $\mathcal{A}$ having the property that the height of Con $\mathcal{A}$ is 2. In this note there are studied analogous questions concerning finite partial unary algebras.

Let $\mathcal{A} = (A,F)$ be a partial unary algebra. For $f \in F$ put $B_f = \{x \in A : f(x) \text{ exists}\}$. An equivalence θ on A will be called a congruence, if the following is valid (cf. also [2], p. 177):

$$(\forall f \in F)(\forall x,y \in B_f)\ (x\,\theta\, y \Rightarrow f(x)\,\theta\, f(y)).$$

Let $A_1 \subseteq A$ and suppose that for each $x \in A_1 \cap B_f$ we have $f(x) \in A_1$; then $\mathcal{B} = (A_1,F)$ is said to be a subalgebra of $\mathcal{A}$.

Consider the types τ_i, $i \in \{1,\dots,10\}$ of finite partial unary algebras $\mathfrak{A}$ such that the lattice Sub $\mathfrak{A}$ consists of $\mathfrak{A}$, $\emptyset$ and, maybe, of some simple or one-element subalgebras, where Sub $\mathfrak{A}$ is one of the lattices listed below:



Proposition 1. Let $\mathfrak{A}$ be a finite partial unary algebra such that the height of Con $\mathfrak{A}$ is 2. Then there exists $i \in \{1,\dots,10\}$ with $\mathfrak{A} \in \tau_i$.

The proof is the same as in the proof of Proposition in [3], p. 90, and p. 93 (case 5).

Proposition 2. (Pálfy [3]) Let $\mathfrak{A}$ be a finite unary algebra and suppose that the height of Con $\mathfrak{A}$ is 2. Then either the number of nontrivial congruences is less than 4, or $\mathfrak{A}$ has no proper subalgebra.

The results of this note are as follows:

Proposition 3. Let $\mathcal{A}$ be a finite partial unary algebra such that the height of Con $\mathcal{A}$ is 2. Let $\mathcal{A} \in \tau_i$, $i \in \{9,10\}$. Then the number of nontrivial congruences is less than 4.

Proposition 4. Let $i \in \{1,\ldots,8\}$ and let $n>3$ be a positive integer. There exists a finite partial unary algebra $\mathcal{A} \in \tau_i$ such that the height of Con $\mathcal{A}$ is 2 and that the number of nontrivial congruences of $\mathcal{A}$ is n.

The proof of Proposition 3 is analogous to that of Proposition in [3] (the cases 6 and 11), and therefore it will be omitted.

Before proving Proposition 4 let us introduce some denotations. Let $\mathcal{A} = (A,F)$ be a partial unary algebra. Put $N(\mathcal{A}) =$ $=$ card (Con $\mathcal{A} - \{\iota, \omega\}$). For $x,y \in A$ we shall denote $\theta(x,y)$ the smallest congruence θ on $\mathcal{A}$ such that $x\theta y$. If x, y are distinct elements of A having the property that for each $z,u \in$ $\in A$ with $z\,\theta(x,y)u$ we have either $z = u$ or $\{z,u\} = \{x,y\}$, then we shall write $\theta(x,y) = [\{x,y\}]$.

If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $\theta \in$ Con $\mathcal{B}$, then we define θ^* in the following way: for $x,y \in A$ we put $x\,\theta^* y$ if and only if either $x = y$ or $x\theta y$. It is obvious that $\theta^* \in$ Con $\mathcal{A}$.

Let k be a positive integer and $Z_k = \{0,1,\ldots,k-1\}$. Further let $G_k = \{f_\alpha : \alpha \in \Gamma\}$ be the set of all unary operations f_α on Z_k such that (Z_k, f_α) is a cycle. Put $\mathcal{C}_k = (Z_k, G_k)$. It is obvious that Con $\mathcal{C}_k = \{\iota, \omega\}$.

Proof of Proposition 4. Assume that $n>3$ is a positive integer. Let $i = 1$. Let us define a partial unary algebra

$\mathcal{A} = (A,F)$ such that $\mathcal{C}_{n-1}$ is a subalgebra of (A,G_{n-1}), $A = Z_{n-1} \cup \{a\}$, $a \notin Z_{n-1}$. Put $F = G_{n-1} \cup \{f,h\}$, $f(a) = 0$, $h(0) = a$, where $B_f = \{a\}$, $B_h = \{0\}$, $a \notin B_g$ for each $g \in G_{n-1}$. Then $\mathcal{A}$ has no proper subalgebra. If $k \in Z_{n-1}$, then $\theta(a,k) = [\{a,k\}]$ according to the relations $a \notin B_g \cup B_h$ for each $g \in G_{n-1}$, $k \notin B_f$. Suppose that θ is a nontrivial congruence, $\theta \notin \{\theta_k = \theta(a,k) : k \in Z_{n-1}\}$. Then there are $m,j \in Z_{n-1}$, $m \neq j$, with $m\,\theta\,j$, and from the definition of $\mathcal{C}_{n-1}$ we obtain that $\theta = \iota^*_{Z_{n-1}}$. Hence $\operatorname{Con}\mathcal{A} = \{\iota, \omega, \iota^*_{Z_{n-1}}\} \cup \{\theta_k : k \in Z_{n-1}\}$. The height of $\operatorname{Con}\mathcal{A}$ is 2 and $N(\mathcal{A}) = n$.

Let $i = 2$. Let us define a partial unary algebra $\mathcal{A} = (A,F)$ in such a way that $\mathcal{C}_{n-1}$ is a sublagebra of (A,G_{n-1}), $A = Z_{n-1} \cup \{a\}$, $a \notin Z_{n-1}$. Put $F = G_{n-1} \cup \{f\}$, $f(a) = 0$, where $B_f = \{a\}$, $a \notin B_g$ for each $g \in G_{n-1}$. Then (Z_{n-1},F) is the only proper subalgebra of $\mathcal{A}$ and it is simple. If $k \in Z_{n-1}$, then $\theta(a,k) = [\{a,k\}]$ (since $k \notin B_f$ and $a \notin B_g$ for each $g \in G_{n-1}$). If θ is a nontrivial congruence such that $\theta \notin \{\theta_k = \theta(a,k) : k \in Z_{n-1}\}$, then there are $m,j \in Z_{n-1}$, $m \neq j$, with $m\,\theta\,j$. From the definition of $\mathcal{C}_{n-1}$ it follows that $m\,\theta\,k$ for each $k \in Z_{n-1}$, and hence $\theta = \iota^*_{Z_{n-1}}$. Thus $\operatorname{Con}\mathcal{A} = \{\iota, \omega, \iota^*_{Z_{n-1}}\} \cup \{\theta_k : k \in Z_{n-1}\}$. The height of $\operatorname{Con}\mathcal{A}$ is 2 and $N(\mathcal{A}) = n$.

Let $i = 3$. Let us define $\mathcal{A} = (A,F)$ such that $\mathcal{C}_n$ is a subalgebra of (A,G_n), $A = Z_n \cup \{o\}$, $o \notin Z_n$. Put $F = G_n \cup \{f\}$, where $f(1) = 0$, $f(0) = o$, $B_f = \{1,0\}$, $o \notin B_g$ for each $g \in G_n$. Then $(\{o\},F)$ is the only proper subalgebra of $\mathcal{A}$. Further $\theta(o,k) = [\{o,k\}]$ for each $k \in Z_n$. If θ is a nontrivial congruence, $\theta \notin \{\theta(o,k) : k \in Z_n\}$, then there are $m,j \in Z_n$, $m \neq j$, such that $m\,\theta\,j$. From the properties of $\mathcal{C}_n$ we get that $m\,\theta\,k$

for each $k \in Z_n$, hence $0 \theta 1$, which implies $f(0) \theta f(1)$, i.e., $o \theta 0$, and therefore $\theta = \iota$. Hence $N(\mathfrak{A}) = n$. The height of $\mathrm{Con}\,\mathfrak{A}$ is 2.

Let i = 4. Let us define a partial unary algebra $\mathfrak{A} = (A,F)$ such that $\mathcal{C}_{n-1}$ is a subalgebra of (A,G_{n-1}), $A = Z_{n-1} \cup \{o,a\}$, $o \neq a, o,a \notin Z_{n-1}$. Put $F = G_{n-1} \cup \{f\}$, $f(a) = f(0) = o$, $f(1) = 0$, $g(o) = o$ for each $g \in G_{n-1}$, $B_f = \{a,0,1\}$, $B_g = Z_{n-1} \cup \{o\}$ for each $g \in G_{n-1}$. Then $(\{o\},F)$ and $\mathcal{S} = (Z_{n-1} \cup \{o\},F)$ are the only proper subalgebras of $\mathfrak{A}$. Let $\theta \in \mathrm{Con}\,\mathcal{S}$. If there are $m,j \in Z_{n-1}$, $m \neq j$, with $m \theta j$, then from the definition of $\mathcal{C}_{n-1}$ we get $m \theta k$ for each $k \in Z_{n-1}$. Then $0 \theta 1$, $f(0) \theta f(1)$, i.e., $o \theta 0$, and hence $\theta = \iota_{Z_{n-1} \cup \{o\}}$. If there is $m \in Z_{n-1}$ with $m \theta o$, then $g(m) \theta g(o)$, i.e., $g(m) \theta o$ for each $g \in G_{n-1}$, and therefore $\theta = \iota_{Z_{n-1} \cup \{o\}}$. Thus $\mathcal{S}$ is simple. If $k \in Z_{n-1}$, $k \neq 1$, then $\theta(k,a) = [\{k,a\}]$. Further, $\theta(1,a) = \iota$, since $f(1)\, \theta(1,a)\, f(a)$, i.e., $0 \ \theta(1,a) o$, and $\mathcal{S}$ is simple. Let θ be a nontrivial congruence, $\theta \notin \{\theta(k,a) : k \in Z_{n-1}\}$, $\theta \neq \theta(o,a) = [\{o,a\}]$. Then either there are $m,j \in Z_{n-1}$, $m \neq j$, with $m \theta j$, or $m \theta o$ for some $m \in Z_{n-1}$. Since $\mathcal{S}$ is simple, $\theta = \iota^*_{Z_{n-1} \cup \{o\}}$. Hence $\mathrm{Con}\,\mathfrak{A} = \{\iota, \omega, \iota^*_{\{o,a\}}, \iota^*_{Z_{n-2} \cup \{o\}}\} \cup \{\theta(k,a) : k \in Z_{n-1}, k \neq 1\}$, $N(\mathfrak{A}) = n$ and the height of $\mathrm{Con}\,\mathfrak{A}$ is 2.

Let i = 5. Let us define a partial unary algebra $\mathfrak{A} = (A,F)$ such that $\mathcal{C}_{n-1}$ is a subalgebra of (A,G_{n-1}), $A = Z_{n-1} \cup \{o_1,o_2\}$, $o_1 \neq o_2, o_1, o_2 \notin Z_{n-1}$. Put $F = G_{n-1} \cup \{f,h\}$, where $f(0) = o_2$, $f(1) = 0$, $h(0) = o_1$, $h(o_2) = o_2 = g(o_2)$ for each $g \in G_{n-1}$, $B_f = \{0,1\}$, $B_h = \{0,o_2\}$ and $B_g = Z_{n-1} \cup \{o_2\}$

for each $g\in G_{n-1}$. Then $(\{o_1\},F)$, $(\{o_2\},F)$ and $(\{o_1,o_2\},F)$ are the only proper subalgebras of $\mathcal{A}$. We have $\Theta(k,o_1) = [\{k,o_1\}]$ for each $k\in Z_{n-1}$. Let Θ be a nontrivial congruence, $\Theta\notin\{\Theta(k,o_1):k\in Z_{n-1}\}$, $\Theta\neq\Theta(o_1,o_2)=[\{o_1,o_2\}]$. Suppose that there are $m,j\in Z_{k-1}$, $m\neq j$, such that $m\Theta j$. Then $m\Theta k$ for each $k\in Z_{n-1}$ (from the definition of $\mathcal{C}_{n-1}$), hence $0\Theta 1$ and $f(0)\Theta f(1)$, i.e., $o_2\Theta 0$, and therefore $h(o_2)\Theta h(0)$, i.e., $o_2\Theta o_1$, which implies $\Theta=\iota$. Now suppose that there is $m\in Z_{n-1}$ with $m\Theta o_2$. Then $g(m)\Theta g(o_2)$, i.e., $g(m)\Theta o_2$ for each $g\in G_{n-1}$, hence $m\Theta k$ for each $k\in Z_{n-1}$, and this is the above case, therefore $\Theta=\iota$. Thus $\mathrm{Con}\,\mathcal{A}=\{\iota,\omega,\iota^*_{\{o_1,o_2\}}\}\cup\{\Theta(k,o_1):k\in Z_{n-1}\}$, $N(\mathcal{A})=n$ and the height of $\mathrm{Con}\,\mathcal{A}$ is 2.

Let i = 6. Let us define a partial unary algebra $\mathcal{A}=(A,F)$ such that $\mathcal{C}_{n-2}$ is a subalgebra of (A,G_{n-2}), $A=Z_{n-2}\cup\{a,o\}$, $a\neq o$, $a,o\notin Z_{n-2}$. Put $F=G_{n-2}\cup\{f,h\}$, $f(0)=o$, $f(1)=0$, $h(a)=o$, $g(o)=o$ for each $g\in G_{n-2}$, $B_f=\{0,1\}$, $B_h=\{a\}$, $B_g=Z_{n-1}\cup\{o\}$ for each $g\in G_{n-2}$. Then $(\{o\},F)$, $(\{o,a\},F)$ and $\mathcal{S}=(Z_{n-1}\cup\{o\},F)$ are the only proper subalgebras of $\mathcal{A}$, $(\{o,a\},F)$ is simple. Let $\Theta\in\mathrm{Con}\,\mathcal{S}$. If there are $m,j\in Z_{n-2}$, $m\neq j$, with $m\Theta j$, then from the definition of $\mathcal{C}_{n-2}$ it follows that $m\Theta k$ for each $k\in Z_{n-2}$. Hence $0\Theta 1$, $f(0)\Theta f(1)$, i.e., $o\Theta 1$, therefore $\Theta=\iota_{Z_{n-2}\cup\{o\}}$. If $m\Theta o$ for some $m\in Z_{n-2}$, then $g(m)\Theta g(o)$, i.e., $g(m)\Theta o$ for each $g\in G_{n-2}$, hence $k\Theta o$ for each $k\in Z_{n-2}$, i.e. $\Theta=\iota_{Z_{n-2}\cup\{o\}}$. Thus $\mathcal{S}$ is simple. Obviously $\Theta(k,a)=[\{k,a\}]$ for each $k\in Z_{n-2}$. Let Θ be a nontrivial congruence, $\Theta\notin\{\Theta(k,a):k\in Z_{n-2}\}$. Then either there are $m,j\in Z_{n-2}$, $m\neq j$, with $m\Theta j$, or there is $m\in Z_{n-2}$ with $m\Theta o$.

Since $\mathcal{S}$ is simple, we obtain that $\theta = \iota^*_{Z_{n-2}\cup\{o\}}$. Hence $\mathrm{Con}\,\mathcal{A} = \{\iota, \omega, \iota^*_{\{o,a\}}, \iota^*_{Z_{n-2}\cup\{o\}}\} \cup \{\theta(k,a) : k \in Z_{n-2}\}$, $N(\mathcal{A}) = n$ and the height of $\mathrm{Con}\,\mathcal{A}$ is 2.

Let i = 7. Let us define a partial unary algebra $\mathcal{A} = (A,F)$ such that $\mathcal{C}_{n-2}$ is a subalgebra of (A,G_{n-2}), $A = Z_{n-2} \cup \{o_1,o_2\}$, $o_1 \neq o_2$, $o_1,o_2 \notin Z_{n-2}$. Put $F = G_{n-2} \cup \{f\}$, $f(1) = 0$, $f(0) = o_1$, $g(o_1) = o_1$ for each $g \in G_{n-2}$, $B_f = \{0,1\}$, $B_g = Z_{n-2} \cup \{o_1\}$ for each $g \in G_{n-2}$. Then $(\{o_1\},F)$, $(\{o_2\},F)$, $(\{o_1,o_2\},F)$ and $\mathcal{S} = (Z_{n-2} \cup \{o_1\},F)$ are the only proper subalgebras of $\mathcal{A}$. Let $\theta \in \mathrm{Con}\,\mathcal{S}$. If there are $m,j \in Z_{n-2}$, $m \neq j$, such that $m\,\theta\,j$, then $m\,\theta\,k$ for each $k \in Z_{n-2}$. Hence $0\,\theta\,1$, $f(0)\,\theta\,f(1)$, i.e., $o_1\,\theta\,0$, and therefore $\theta = \iota_{Z_{n-2}\cup\{o_1\}}$. If $m\,\theta\,o_1$ for some $m \in Z_{n-2}$, then $g(m)\,\theta\,g(o_1)$, i.e., $g(m)\,\theta\,o_1$ for each $g \in G_{n-2}$, hence $k\,\theta\,o_1$ for each $k \in Z_{n-2}$ and $\theta = \iota_{Z_{n-2}\cup\{o_1\}}$. Therefore $\mathcal{S}$ is simple. Obviously $\theta(k,o_2) = [\{k,o_2\}]$ for each $k \in Z_{n-2}$. Let θ be a nontrivial congruence, $\theta \notin \{\theta(k,o_2) : k \in Z_{n-2}\}$, $\theta \neq \theta(o_1,o_2) = [\{o_1,o_2\}]$. Then either there are $m,j \in Z_{n-2}$, $m \neq j$, with $m\,\theta\,j$, or there is $m \in Z_{n-2}$ with $m\,\theta\,o_1$. From the fact that $\mathcal{S}$ is simple we get that $\theta = \iota^*_{Z_{n-2}\cup\{o_1\}}$. Hence $\mathrm{Con}\,\mathcal{A} = \{\iota, \omega, \iota^*_{\{o_1,o_2\}}, \iota^*_{Z_{n-2}\cup\{o_1\}}\} \cup \{\theta(k,o_2) : k \in Z_{n-2}\}$, $N(\mathcal{A}) = n$ and the height of $\mathrm{Con}\,\mathcal{A}$ is 2.

Let i = 8. Let $\mathcal{A} = (A,F)$ be a partial unary algebra such that $\mathcal{C}_{n-1}$ is a subalgebra of (A,G_{n-1}), $A = Z_{n-1} \cup \{o\}$, $o \notin Z_{n-1}$, $F = G_{n-1}$, $B_g = Z_{n-1}$ for each $g \in G_{n-1}$. Then $\mathcal{C}_{n-1}$ and $(\{o\},F)$ are the only proper subalgebras of $\mathcal{A}$, $\mathcal{C}_{n-1}$ is simple. Obviously $\theta(k,o) = [\{k,o\}]$ for each $k \in Z_{n-1}$. If θ is a nontrivial congruence of $\mathcal{A}$, $\theta \notin \{\theta(k,o) : k \in Z_{n-1}\}$, then according to the fact that $\mathcal{C}_{n-1}$ is simple we obtain $\theta =$ $= \iota^*_{Z_{n-1}}$. Hence Con $\mathcal{A} = \{\iota, \omega, \iota^*_{Z_{n-1}}\} \cup \{\theta(k,o) : k \in Z_{n-1}\}$, $N(\mathcal{A}) = n$ and the height of Con $\mathcal{A}$ is 2.

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