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ON THE DIFFERENTIABILITY OF MULTIVALUED MAPPINGS, II

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Abstract: This paper is a continuation of our work [7]. The differentiable selections of differentiable multivalued mappings defined on an interval [a,b] and having values in the space of all bounded convex closed non-empty subsets of a Banach space are investigated.

Key words: Multivalued mappings differentiability, selections, Banach spaces.

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3. Differentiable selections. We use notions and notations introduced in [7].

Definition 1. Let F be a map of the interval $[a,b] \subseteq R$ into $\hat{X}$, then we say that F is positively (respectively negatively) conical at $t_0 \in [a,b]$ iff $F(t_0) \in \varkappa(\mathscr{C}_0(X))$ ($F(t_0) \in \in (-\varkappa(\mathscr{C}_0(X)))$ respectively) i.e. iff there exist maps A, B of the interval [a,b] into $\mathscr{C}_0(X)$ such that $F = [A,B]$ and $B(t_0) = \{0\}$ ($A(t_0) = \{0\}$ respectively). We say that F is positively (respectively negatively) conical on [a,b] if F is positively (resp. negatively) conical at every point t of [a,b]. We say that F is conical on [a,b] if at each point t of [a,b] F is either positively or negatively conical i.e.

if there exist maps A, B of interval $[a,b]$ into $\mathscr{C}_0(X)$ such that $F = [A,B]$ and A, B satisfy the following condition

(1) for each $t \in [a,b]$ either $A(t) = \{0\}$ or $B(t) = \{0\}$.

Definition 2. Let F be a map of $[a,b]$ into $\hat{X}$ and suppose that F is conical on $[a,b]$ and $F(t) = [A(t),B(t)]$ for all $t \in [a,b]$ where A, B satisfy the condition (1). Then the map f of $[a,b]$ into X is said to be a selection of F if there exist maps f_1, f_2 of $[a,b]$ into X, such that f_1, f_2 are selections of A and B, respectively, i.e. $f_1(t) \in A(t)$ and $f_2(t) \in B(t)$ for all $t \in [a,b]$, and $f = f_1 - f_2$. If f is continuous (differentiable, respectively) then f is said to be a continuous (differentiable, respectively) selection of F.

The set of all continuous selections of F is denoted by C(F). It is clear that C(F) is a convex family; i.e. if $f, g \in C(F)$, $\lambda \in [0,1]$, then $\lambda f + (1-\lambda)g \in C(F)$. It follows from definition 2 that if F is a map of $[a,b]$ into $\mathscr{C}_0(X)$, then f is a selection of F (i.e. $f(t) \in F(t)$ for all $t \in [a,b]$) if and only if f is a selection of $\hat{F}$.

Now we shall show that the definition of selections of a conical map F of $[a,b]$ into $\hat{X}$ does not depend on the choice of the maps A, B, which satisfy the condition (1). Let $F(t) = [A(t),B(t)] = [A_1(t),B_1(t)]$ where A, B and A_1, B_1 satisfy the condition (1). If $f(t) \notin \varkappa(X) = \{[\{x\},\{0\}] = [\{0\},\{-x\}] \mid x \in X\}$, then $A(t) = A_1(t)$ and $B(t) = B_1(t)$. In fact if $A(t) \neq A_1(t)$, then one of these sets is $\{0\}$, for instance let $A_1(t) = \{0\}$, then $B(t) = \{0\}$ and $A(t) +^* B_1(t) = \{0\}$, but it is impossible, as A(t) is not a singleton. If $F(t) \in \varkappa(X)$, then it is easy to see that either $A(t) = A_1(t)$, $B(t) = B_1(t)$ or

$A(t) = \{x_t\} = -B_1(t)$, $B(t) = \{0\} = -A_1(t)$ or $A(t) = \{0\} = -B_1(t)$, $B(t) = \{-x_t\} = -A_1(t)$ where $F(t) = \varkappa(\{x_t\})$. Let $f = f_1 - f_2$, where f_1, f_2 are selections of A, B respectively. Put

$$g_1(t) = \begin{cases} f_1(t) \text{ if } A(t) = A_1(t) \\ -f_2(t) \text{ if } A(t) \neq A_1(t) \end{cases} \qquad g_2(t) = \begin{cases} f_2(t) \text{ if } B(t) = B_1(t) \\ -f_1(t) \text{ if } B(t) \neq B_1(t) \end{cases}$$

then g_1, g_2 are selections of A_1, B_1 respectively and $f = g_1 - g_2$. This shows that the definition of selections does not depend on the choice of A, B.

In the remainder we always suppose that X is a Banach space and F is a map of some nrighborhood of interval [a,b] into $\mathcal{C}_0(X)$.

If F is continuous on [a,b], then $\int_a^b \hat{F}(t)dt \in \varkappa(\mathcal{C}_0(X))$ (see [4]) and we put:

$$\int_a^b F(t)dt = \varkappa^{-1}\left(\int_a^b \hat{F}(t)dt\right)$$

<u>Lemma 1</u>. Let F be a continuous map on [a,b], then

$$\int_a^b F(t)dt = \lim_n \frac{(b-a)}{n} \left(\overline{\sum_{i=0}^{n-1} F\left(a + \frac{i(b-a)}{n}\right)}\right)$$

<u>Proof</u>. From definition of integral $\int_a^b F(t)dt$ it immediately implies our assertion.

Let F be continuous on [a,b], then by theorem 3.2" [8] it follows that $C(F) \neq \emptyset$ and $F(t) = \{f(t) \; f \in C(F)\}$. We define

$${}^0\!\int_a^b F(t)dt = \overline{\left\{\int_a^b f(t)dt \,\middle|\, f \in C(F)\right\}}$$

<u>Lemma 2</u>. If F is continuous on [a,b], then

$$\int_a^b F(t)dt = {}^{o}\!\int_a^b F(t)dt$$

Proof. 1) Of course for each $f \in C(F)$ we have that $\int_a^b f(t)dt \in \int_a^b F(t)dt$. This means ${}^{o}\!\int_a^b F(t)dt \subseteq \int_a^b F(t)dt$.

2) We shall prove that $\int_a^b F(t)dt) \subseteq {}^{o}\!\int_a^b \hat{F}(t)dt$, and the proof will be complete. Let $z \in \int_a^b F(t)dt)$ and let $\varepsilon > 0$, $\varepsilon < b-a$ be given; then from the continuity of F on $[a,b]$ and by the Lemma 1 there exists a positive integer n and $x_i \in F(a_i)$ where $a_i = a + i\Delta_n$; $\Delta_n = \frac{1}{n}(b-a)$ for $i = 0,1,\ldots,n-1$ such that

(1) $\| z - \Delta_n \sum x_i \| < \frac{\varepsilon}{3}$

(2) $d(F(t),F(t') < \varepsilon_1 = \frac{\varepsilon}{3(b-a)}$ for all $t,t' \in [a,b]$, $|t-t'| \leqq \Delta_n$.

Let $M = \operatorname{Sup}\{\|x\| \mid x \in F(t),\ t \in [a,b]\} = \max\{\|\hat{F}(t)\| \mid t \in [a,b]\}$. Put $b_i = a_{i+1} - \frac{\varepsilon}{6nM}$ for $i = 0,1,\ldots,n-2$ and $b_{n-1} = a_n = b$. But (2) implies that $F(t) \cap S^o(x_i, \varepsilon_1) \neq$ for all $t \in [q_i,b_i]$ and $i = 0,1,\ldots,n-1$, where $S^o(x_i, \varepsilon_1) = \{x \in X : \|x - x_i\| < \varepsilon_1\}$. It is clear that the map G defined by

$$G(t) = \begin{cases} \overline{F(t) \cap S^o(x_i, \varepsilon_1)} & \text{for } t \in [a_i,b_i],\ i = 0,1,\ldots,n-1 \\ F(t) & \text{for } t \in U(b_i,a_{i+1}) \end{cases}$$

is lower semi-continuous on $[a,b]$. By Theorem 3.2" [8] there exists a continuous selection f of G (so as of F) on $[a,b]$. Then, of course, $\| f(t) - x_i \| \leqq \varepsilon_1$ for all $t \in [a_i,b_i]$ $i = 0, 1,\ldots,n-1$ and $\| f(t) - x_i \| \leqq 2M$ for $t \in U(b_i,a_{i+1})$, whence

$$\| z - \int_a^b f(t)dt \| \leqq \| z - \Delta_n \sum x_i \| + \| \sum_0^{n-1} \int_{a_i}^{b_i} (f(t) - x_i)dt + \sum_0^{n-2} \int_{b_i}^{a_{i+1}} (f(t) - x_i)dt \| < \frac{\varepsilon}{3} + \sum \int_{a_i}^{b_i} \varepsilon_1 dt +$$

$$+ \sum \int_{b_i}^{a_{i+1}} 2M\, dt < \frac{\varepsilon}{3} + (b-a)\,\varepsilon_1 + 2M(n-1)\cdot \frac{\varepsilon}{6nM} < \varepsilon$$

This means that $d(z, {}^{c}\!\int_a^b F(t)dt) < \varepsilon$ for all $\varepsilon > 0$ and hence $z \in {}^{o}\!\int_a^b F(t)dt$, or $\int_a^b F(t) \subseteq {}^{o}\!\int_a^b F(t)dt$. This completes the proof.

Theorem 1. Suppose that F is continuously positively or negatively conically differentiable on $[a,b]$. Then there exists a convex family $\mathcal{F}$ of differentiable selections of the map F on $[a,b]$ such that

1) f' is a continuous selection of the map $\hat{F}'$, where $\hat{F}'(t) = D\hat{F}(t)(1)$ for all $t \in [a,b]$ and all $f \in \mathcal{F}$;

2) $A(t) = \{f'(t) \mid f \in \mathcal{F}\}$ if $\hat{F}'(t) = [A(t), \{0\}]$

$B(t) = \{-f'(t) \mid f \in \mathcal{F}\}$ if $\hat{F}'(t) = [\{0\}, B(t)]$

3) $F(t) = \overline{\{f(t) \mid f \in \mathcal{F}\}}$ for all $t \in [a,b]$.

Proof: 1) Let F be positively conically differentiable on $[a,b]$, $\hat{F}'(t) = D\hat{F}(t)(1) = [A(t), \{0\}]$. From the continuity of $\hat{F}'$ it follows that $A(t)$ is continuous on $[a,b]$. Of course, we have

$$\hat{F}(t) = \hat{F}(a) + \int_a^t \hat{F}'(\tau)\, d\tau = \hat{F}(a) + [\int_a^t A(\tau)d\tau, \{0\}]$$

for all $t \in [a,b]$.

Hence $F(t) = F(a) +^{*} \int_a^t A(\tau)d\tau$. By the Lemma 2 it follows that

$$F(t) = F(a) +^{*} {}^{o}\!\int_a^t A(\tau)d\tau .$$

Put

$$\mathcal{F} = \{f_{x,g} : f_{x,g}(t) = x + \int_a^t g(\tau)d\tau \mid x \in F(a),\ g \in C(\hat{F}')\};$$

then it is clear that $\mathcal{F}$ has the properties 1) 2) and 3).

2) Let F be negatively conically differentiable on $[a,b]$, $\hat{F}'(t) = D\hat{F}(t)(1) = [\{0\}, B(t)]$, then

$$\hat{F}(b) = \hat{F}(t) + \int_t^b \hat{F}'(t)dt = \hat{F}(t) + [\{0\}, \int_t^b B(\tau)d\tau]$$

hence

$$F(t) = F(b) +^{*} {}^{o}\!\int_t^b B(\tau)d\tau \quad .$$

Put

$$\mathcal{F} = \{f_{x,g} : f_{x,g}(t) = x + \int_t^b (-g(\tau))d\tau \mid x \in F(b),\ g \in C(\hat{F}') = - C(B)\};$$

then $\mathcal{F}$ satisfies the conditions 1) 2) and 3) and the proof is complete.

Remark 1. If int $F(a) \neq \emptyset$ (where int $F(a)$ denotes the interior of $F(a)$) and if F is continuously positively differentiable on $[a,b]$, then we put $\mathcal{F}' = \{f_{x,g} : f_{x,g}(t) = x + \int_a^t g(\tau)d\tau \mid x \in \text{int } F(a), g \in C(\hat{F}')\}$. Then for each $t \in [a,b]$, $\{f(t) \mid f \in \mathcal{F}'\}$ is a convex open subset and $F(t)$ is its closed hull. It means that int $F(t) = \{f(t) \mid f \in \mathcal{F}'\}$.

Similarly, if int $F(b) \neq \emptyset$ and F is continuously negatively differentiable on $[a,b]$, then there exists a convex subfamily $\mathcal{F}'$ of the family $\mathcal{F}$ such that int $F(t) = \{f(t) \mid f \in \mathcal{F}'\}$ for all $t \in [a,b]$.

For each $A \in \mathcal{B}(X)$, put $\delta(A) = \sup\{\|x-y\| \mid x \in A, y \in A\}$. If G is a conical map of $[a,b]$ into $\hat{X}$ and $G(t) = [A(t),B(t)]$ where A, B satisfy the condition (1), we put:

$$\delta(G(t)) = \delta(A(t)) + \delta(B(t)).$$

In the remainder of this section we always suppose that F is continuously conically differentiable on $[a,b]$ and $\hat{F}'(t) = [A(t),B(t)]$ where A, B satisfy the condition (1).

Lemma 3. 1) There exists a finite or countable family of disjoint open intervals $(a_n,b_n) \subseteq (a,b)$ such that F is positively or negatively conically differentiable on $[a_n,b_n]$ for all n, and $\hat{F}'(t) \in \varkappa(X)$ for all $t \in (a,b) \setminus \setminus \bigcup_n (a_n,b_n)$.

2) If f_n is a continuous selection of $\hat{F}'$ on $[a_n,b_n]$ for n = 1,2,... then the map defined by

$$f(t) = \begin{cases} f_n(t) & \text{for } t \in [a_n.b_n] \text{ for } n = 1,2,\dots \\ x_t \text{ where } \hat{F}'(t) = \varkappa(\{x_t\}) & \text{for } t \in [a,b] \setminus \bigcup_n [g_n,b_n] \end{cases}$$

is a continuous selection of $\hat{F}'$ on $[a,b]$.

<u>Proof</u>. Let $[A,\{0\}] \in \varkappa(\mathscr{C}_0(X)) \setminus \varkappa(X)$; then $\sigma(A) > 0$ and for each $[\{0\},B] \in (-\varkappa(\mathscr{C}_0(X)))$ we have

(3): $\|[A,\{0\}] - [\{0\},B]\| = \sup\{\|x+y\| \mid x \in A, y \in B\} \geq \geq \frac{1}{2}\sigma(A)$.

This means that $\varkappa(\mathscr{C}_0(X)) \setminus \varkappa(X)$ is open in $\varkappa(\mathscr{C}_0(X)) \cup \cup (-\varkappa(\mathscr{C}_0(X)))$. Similarly $(-\varkappa(\mathscr{C}_0(X))) \setminus \varkappa(X)$ is open in $\varkappa(\mathscr{C}_0(X)) \cup (-\varkappa(\mathscr{C}_0(X)))$. Then $\{t \in (a,b) : \hat{F}'(t) \in \varkappa(\mathscr{C}_0(X)) \setminus \setminus \varkappa(X)\}$ and $\{t \in (a,b) \mid \hat{F}'(t) \in (-\varkappa(\mathscr{C}_0(X))) \setminus \varkappa(X)\}$ are disjoint open subsets of (a,b). It follows that there exists a finite or countable family of disjoint open intervals $(a_n,b_n) \subseteq (a,b)$ such that for each n either $\hat{F}'(t) \in \varkappa(\mathscr{C}_0(X))$ for all $t \in (a_n,b_n)$, so as for all $t \in [a_n,b_n]$ (since $\varkappa(\mathscr{C}_0(X))$ is closed and $\hat{F}'$ is continuous), or $\hat{F}'(t) \in \in (-\varkappa(\mathscr{C}_0(X)))$ for all $t \in (a_n,b_n)$, so as for all $t \in [a_n,b_n]$; $\hat{F}'(t) \in \varkappa(X)$ for all $t \in (a,b) \setminus \bigcup(a_n,b_n)$. This completes the proof of the first part of the Lemma.

It is clear that the map f defined in 2) is continuous at every point t, for which $\hat{F}'(t) \notin \varkappa(X)$. If $\hat{F}'(t) = = \varkappa(\{x_t\}) \in \varkappa(X)$ then for each $t' \in [a,b]$ we have $\|f(t) - f(t')\| = = \|x_t - f(t')\| \leq \|\hat{F}'(t) - \hat{F}'(t')\|$. It shows that f is continuous at t and this completes the proof.

Remark 2: By theorem 3.2" [8] there exists a continuous selection of F (i.e. of A or of B) on $[a_n,b_n]$ for each n, and then by the Lemma 3 there exists a continuous selection of F on $[a,b]$.

Lemma 4. For each $\varepsilon > 0$ there exists a convex family $\mathcal{F}_\varepsilon(F)$ of continuously differentiable maps of $[a,b]$ into X such that:

1) f' is a continuous selection of $\hat{F}'$ on $[a,b]$ for all $f \in \mathcal{F}_\varepsilon(F)$.

2) $d(F(t),\{f(t) \mid f \in \mathcal{F}_\varepsilon(F)\}) \leq \varepsilon(b-a)$ for all $t \in [a,b]$

3) If $t \in [a,b]$ and $\sigma(\hat{F}'(t)) > \varepsilon$ then either

$A(t) = \{f'(t) \mid f \in \mathcal{F}_\varepsilon(F)\}$ or $B(t) = \{-f'(t) \mid f \in \mathcal{F}_\varepsilon(F)\}$.

Proof: Let $\Delta > 0$ be such that for all $t,t' \in [a,b]$,

(4) $|t-t'| < \Delta$, we have $\| \hat{F}'(t) - \hat{F}'(t') \| < \frac{\varepsilon}{2}$.

Let (a_n,b_n), n = 1,2,... be the intervals constructed in Lemma 3. If there exists a $t \in (a_n,b_n)$ such that $\sigma(\hat{F}'(t)) \geq \varepsilon$, then by (3) and (4) we get that $b_n - a_n \geq \Delta$. That is, there is only a finite number of intervals (a_n,b_n), which contain a point t such that $\sigma(\hat{F}'(t)) \geq \varepsilon$. This means that there exists a finite number of points $s_0 = a < s_1 < \ldots < s_{n-1} < s_n = b$ such that $\hat{F}'(s_i) \in \mathcal{K}(X)$ for i = 1,2,...,n-1, and for each i = 0,1,...,n-1 either F is positively (or negatively) conically differentiable on $[s_i,s_{i+1}]$ or $\sigma(\hat{F}'(t)) < \varepsilon$ for all $t \in [s_i,s_{i+1}]$. Let φ be a continuous selection of $\hat{F}'$ on $[a,b]$ (the existence of φ is guaranteed by Lemma 3), then it is clear that $\|\hat{F}'(t) - [\{\varphi(t)\},\{0\}]\| \leq \sigma(\hat{F}'(t))$. Let I be the set of all k, $t \leq k \leq n$, such that there exists a convex family of differentiable maps of $[a,s_k]$ into X, which satisfies the conditions 1) 2) and 3) of our Lemma on $[a,s_k]$. To prove the assertion it is sufficient to prove that $1 \in I$ and if $k \in I$, $k < n$, then $k+1 \in I$.

1) a) If F is positively (or negatively) conically differentiable on $[s_0,s_1]$, then take the family $\mathcal{F}$ constructed in the Theorem 1 for F on $[s_0,s_1]$. It is clear that $\mathcal{F}$ satisfies the conditions 1) 2) and 3).

b) If $\delta(\hat{F}'(t)) < \varepsilon$ for all $t \in [s_0,s_1]$ then we put $\mathcal{F} = \{f_x : f_x(t) = x + \int_a^t \varphi(\tau)d\tau \mid x \in F(a),\ t \in [s_0,s_1]\}$. It is clear that $\mathcal{F}$ satisfies the conditions 1) and 3) and for each $t \in [s_0,s_1]$ $G(t) = \{f(t) \mid f \in \mathcal{F}\} = F(a) + \int_a^t \varphi(\tau)d\tau \in \mathscr{C}_0(X)$ and $\|(\hat{F} - \hat{G})'(t)\| = \|\hat{F}'(t) - [\{\varphi(t)\},\{0\}]\| < \varepsilon$. It follows that $d(F(t),G(t)) = \|\hat{F}(t) - \hat{G}(t)\| < \varepsilon(t-a) \leq$
$\leq \varepsilon(s_1 - a)$.

This means that $1 \in I$.

2) Let $k \in I$, $k < n$ and let $\overline{\mathcal{F}}_k$ be a convex family of differentiable maps of $[a,s_k]$ into X satisfying the conditions 1) 2) and 3).

a) Let F be positively (or negatively) conically differentiable on $[s_k,s_{k+1}]$ and let $\mathcal{F}_k$ be the convex family of differentiable selections of F on $[s_k,s_{k+1}]$ constructed in Theorem 1. For each $g \in \overline{\mathcal{F}}_k$, $h \in \mathcal{F}_k$ put

$$f_{g,h}(t) = \begin{cases} g(t) & \text{for } t \in [a,s_k] \\ h(t) + (g(s_k) - h(s_k)) & \text{for } t \in [s_k,s_{k+1}] \end{cases}$$

Then, of course, $f_{g,h}$ is continuously differentiable on $[a,s_{k+1}]$ and $f'_{g,h}$ is a continuous selection of $\hat{F}'$ on $[a,s_{k+1}]$. We write $g \Delta h$ for $g \in \overline{\mathcal{F}}_k$, $h \in \mathcal{F}_k$ if $\|g(s_k) - h(s_k)\| \leq$

$\leq \varepsilon(s_k-a) + \frac{1}{2}\varepsilon(s_{k+1}-s_k)$. Put $\mathcal{F} = \{f_{g,h} | f \in \overline{\mathcal{F}}_k, h \in \mathcal{F}_k, g \Delta h\}$. Then $\mathcal{F}$ is a convex family, since $\overline{\mathcal{F}}_k$, $\mathcal{F}_k$ are convex. On the other hand, $d(F(s_k), \{g(s_k) | g \in \overline{\mathcal{F}}_k\}) =$
$= d(\{h(s_k) | h \in \mathcal{F}_k\}, \{g(s_k) | g \in \overline{\mathcal{F}}_k\}) \leqq \varepsilon(s_k-a) < \varepsilon(s_k-a) +$
$+ \frac{1}{2}\varepsilon(s_{k+1}-s_k)$. For each $g \in \overline{\mathcal{F}}_k$ (resp. $h \in \mathcal{F}_k$) there exists an $h \in \mathcal{F}_k$ (resp. $g \in \overline{\mathcal{F}}_k$) such that $g \Delta h$. Then $\mathcal{F}$ satisfies the conditions 1) and 3) on $[a, s_{k+1}]$ and the condition 2) on $[a, s_k]$. Let $x \in \{f(t) | f \in \mathcal{F}\}$ for $t \in [s_k, s_{k+1}]$, then there exists $g \in \overline{\mathcal{F}}_k$, $h \in \mathcal{F}_k$ such that $x = f_{g,h}(t)$ and

$$\| x - h(t) \| = \| f_{g,h}(t) - h(t) \| = \| g(s_k) - h(s_k) \| < \varepsilon(s_{k+1}-a)$$

If $x \in F(t)$ for $t \in [s_k, s_{k+1}]$, then there exists $h \in \mathcal{F}_k$ such that $\| x - h(t) \| < \frac{\varepsilon}{2}(s_{k+1}-s_k)$. Let $g \in \overline{\mathcal{F}}_{k+1}$ be such that $g \Delta h$, then $\| x - f_{g,h}(t) \| \leq \| x - h(t) \| + \| h(t) - f_{g,h}(t) \| <$
$< \varepsilon(s_{k+1} - a)$. This shows that $d(F(t), \{f(t) | f \in \mathcal{F}\}) \leqq$
$\leqq \varepsilon(s_{k+1} - a)$ for all $t \in [s_k, s_{k+1}]$. This means that $\mathcal{F}$ satisfies the conditions 1),2),3).

b) If $\sigma(\hat{F}'(t)) < \varepsilon$ for all $t \in [s_k, s_{k+1}]$ then for each $g \in \overline{\mathcal{F}}_k$ put

$$f_g(t) = \begin{cases} g(t) & \text{for } t \in [a, s_k] \\ g(s_k) + \int_{s_k}^{t} \varphi(\tau)\,d\tau & \text{for } t \in [s_k, s_{k+1}]. \end{cases}$$

Denote $\mathcal{F} = \{f_g : g \in \overline{\mathcal{F}}_k\}$. Then it is clear that $\mathcal{F}$ satisfies the conditions 1),3) on $[a, s_{k+1}]$ and the condition 2) on $[a, s_k]$. For $t \in [s_k, s_{k+1}]$ set

$$G(t) = \overline{\{f(t) | f \in \mathcal{F}\}} = \overline{\{g(s_k) | g \in \overline{\mathcal{F}}_k\}} + \int_{s_k}^{t} \varphi(\tau)\,d\tau \in \mathcal{C}_0(X).$$

Then $\| (\hat{F} - \hat{G})'(t) \| = \| \hat{F}'(t) - [\{\varphi(t)\}, \{0\}] \| < \varepsilon$.

Hence $d(F(t), G(t)) = d(F(t), \{f(t) | f \in \mathcal{F}\}) = \| \hat{F}(t) - \hat{G}(t) \|$

$$\leqq \| \hat{F}(s_k) - \hat{G}(s_k) \| + \varepsilon(t - s_k)$$

$$\leqq \varepsilon(s_k - a) + \varepsilon(t - a_k) \leqq \varepsilon(s_{k+1} - s_k).$$

This shows that k + 1 I and this completes the proof.

Theorem 2. Let $\varepsilon > 0$ and let $F_{\varepsilon}(t) = F(t) +^{*} \varepsilon S_1$, where $S_1 = \{x \in X \mid \|x\| \leq 1\}$. Then $\hat{F}'(t) = \hat{F}'(t)$ for all $t \in [a,b]$ and there exists a family $\mathcal{F}$ of differentiable selections of F_{ε} such that

1) f' is a continuous selection of $\hat{F}'$ (so as of $\hat{F}'_{\varepsilon}$) for all $f \in \mathcal{F}$;

2) $F_{\varepsilon}(t) = \overline{\{f(t) \mid f \in \mathcal{F}\}}$ for all $t \in [a,b]$.

3) For each $t \in [a,b]$ either $A(t) = \{f'(t) \mid f \in \mathcal{F}\}$ or $B(t) = \{-f'(t) \mid f \in \mathcal{F}\}$.

Proof. Without loss of generality we can suppose that $b = a + 1$. It is clear that $\hat{F}'_{\varepsilon}(t) = \hat{F}'(t)$ for all $t \in [a,b]$ and $\varepsilon > 0$. Let $r_n > 0$ be such that $\varepsilon = \sum_{1}^{\infty} r_n$. Put $\varepsilon_n = \sum^{n} r_i$. By Lemma 4 there exists a convex family $\mathcal{F}_{r_{n+1}}(F_{\varepsilon_n})$ of differentiable maps of [a,b] into X such that:

1) f' is a continuous selection of $\hat{F}'_{\varepsilon_n}$ (so as of $\hat{F}'$) for $f \in \mathcal{F}_{r_{n+1}}(F_{\varepsilon_n})$.

2) $d(F_{\varepsilon_n}(t), \{f(t) \mid f \in \mathcal{F}_{r_{n+1}}(F_{\varepsilon_n})\}) \leq r_{n+1}$

3) If $t \in [a,b]$, $\delta(\hat{F}'(t)) \geq r_{n+1}$, then either $A(t) = \{f'(t) \mid f \in \mathcal{F}_{r_{n+1}}(F_{\varepsilon_n})\}$ or $B(t) = \{-f'(t) \mid f \in \mathcal{F}_{r_{n+1}}(F_{\varepsilon_n})\}$

Put $\mathcal{F} = \bigcup_{0}^{\infty} \mathcal{F}_{r_{n+1}}(F_{\varepsilon_n})$. Then $\mathcal{F}$ is a convex family of differentiable selections of F_{ε} satisfying the conditions 1) 3) and for $t \in [a,b]$ we have $\{f(t) \mid f \in \mathcal{F}\} \subseteq F_{\varepsilon}(t)$, and

$$d(F_{\varepsilon}(t), \{f(t) \mid f \in \mathcal{F}\}) \leq d(F_{\varepsilon}(t), \{f(t) \mid f \in \mathcal{F}_{r_{n+1}}(F_{\varepsilon_n})\}) \leq$$
$$\leq d(F_{\varepsilon}(t), F_{\varepsilon_n}(t)) + d(F_{\varepsilon_n}(t), \{f(t) \mid f \in \mathcal{F}_{r_{n+1}}(F_{\varepsilon_n})\}) \leq$$
$$\leq (\varepsilon - \varepsilon_n) + r_{n+1}$$

for all n. Hence $d(F_{\varepsilon}(t), \{f(t) \mid f \in \mathcal{F}\}) = 0$ and this completes the proof.

Theorem 3. Let (a_i,b_i) $i = 1,2,\dots$ be the intervals constructed in Lemma 3 $\{a_i\},\{b_i\}$ sequences with at most two limit points a and b. If $\operatorname{int} F(t) \neq \emptyset$ for all $t \in [a,b]$ then there exists a convex family $\mathcal{F}$ of differentiable selections of F such that:

1) f' is a continuous selection of $\hat{F}'$ on $[a,b]$ for all $f \in \mathcal{F}$

2) $F(t) = \overline{\{f(t) \mid f \in \mathcal{F}\}}$ for all $t \in [a,b]$
$\operatorname{int} F(t) = \{f(t) \mid f \in \mathcal{F}\}$ for all $t \in (a,b)$

3) For each $t \in [a,b]$ either $A(t) = \{f'(t) \mid f \in \mathcal{F}\}$ or $B(t) = \{-f'(t) \mid f \in \mathcal{F}\}$.

Proof. It is sufficient to prove the case b is a unique limit point of $\{b_i\}$ (so as of $\{a_i\}$), i.e. we can suppose:

$$a = a_1 < b_1 \leqq a_2 < \dots \leqq a_n < b_n \leqq \dots \leqq \lim a_n = \lim b_n = b$$

Then F is positively or negatively conically differentiable on $[a_i,a_{i+1}]$ for $i = 1,2,\dots$. By the Remark 1 there exist convex families $\mathcal{F}_n$ of differentiable selections of F on $[a_n,a_{n+1}]$, which satisfy the conditions 1),2),3) on $[a_n,a_{n+1}]$. Let $\mathfrak{M} = \{(f_m) \in \prod^{\infty} \mathcal{F}_n \mid f_n(a_{n+1}) = f_{n+1}(a_{n+1})$ for all $n = 1,2,\dots\}$. For each $(f_m) \in \mathfrak{M}$ put $\tilde{h}_{(f_m)}(t) = f_n(t)$ for all $t \in [a_n,a_{n+1}]$ and all $n = 1,2,\dots$. Then $\tilde{h}_{(f_m)}$ is continuously differentiable on $\bigcup_n [a_n,a_{n+1}] = [a,b)$. It is clear that $\hat{F}'(b) \in \varkappa(X)$. Let $\hat{F}'(b) = \varkappa(\{x_b\})$, then $\lim_{t\to b^-} \tilde{h}'_{(f_m)}(t) = x_b$. It implies that there exists $\lim_{t\to b^-} \tilde{h}_{(f_m)}(t)$, too.
We define:

$$h_{(f_m)}(t) = \begin{cases} \tilde{h}_{(f_m)}(t) \text{ for } t \in [a,b] \\ \lim_{t \to b^-} \tilde{h}_{(f_m)}(t) \text{ for } t = b \end{cases}$$

$$\mathcal{F} = \{ h_{(f_m)} \mid (f_m) \in \mathcal{M} \}$$

Then it is easy to see that $\mathcal{F}$ satisfies the conditions 1) 2) and 3). This completes the proof.

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