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ON THE DIFFERENTIABILITY OF MULTIVALUED MAPPINGS, I

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Abstract: The concept of H.T. Banks and M.Q. Jacobs [2] of differentials of multivalued mappings is extended from reflexive Banach spaces to locally convex spaces. Moreover, some properties of differentiable multivalued mappings are derived.

Key words: Locally convex spaces, differentiable mappings, multivalued mappings.

Classification: Primary 47H99
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1. Preliminaries. In this paper, we shall consider only real locally convex spaces. Let X be a locally convex space (l.c.s.), whose topology τ is induced by a family of continuous seminorm P. We denote the family of all bounded (bounded closed, bounded convex closed, respectively) non-empty subsets of X by $\mathcal{B}(X)$ ($\mathcal{C}(X)$, $\mathcal{C}_0(X)$ resp.). For each $p \in P$ we define a pseudometric dp on $\mathcal{B}(X)$ by

$$dp(A,B) = \inf\{\lambda > 0 \; A \subseteq \overline{B + \lambda Sp} \text{ and } B \subseteq \overline{A + \lambda Sp}\}$$
$$= \max\{\sup_{x \in A} \inf_{y \in B} p(x-y), \sup_{y \in B} \inf_{x \in A} p(x-y)\},$$

where $Sp = \{x \in X \mid p(x) \leq 1\}$. We denote the closure of a set A+B by $A +^* B$. Put $\hat{X} = \mathcal{C}_0(X) \times \mathcal{C}_0(X)/\sim$, where the equivalence $\sim$ is defined by: $(A,B) \sim (C,D)$ iff $A +^* D = B +^* C$. Denote the class containing (A,B) by [A,B] and define

$$[A,B] + [C,D] = [A+^*C, B+^*D] \text{ for } [A,B],[C,D] \in \hat{X},$$

$\lambda[A,B] = [\lambda A, \lambda B]$ if $\lambda \geq 0$ and $\lambda[A,B] = [|\lambda| B, |\lambda| A]$ if $\lambda < 0$.

We use the following

<u>Embedding theorem</u> [4].

1) $\hat{X}$ is a linear space.

2) The family $\hat{P} = \{\hat{p} | p \in P\}$ of seminorms on $\hat{X}$ defined by $\hat{p}([A,B]) = dp\,(A,B)$ induces a locally convex topology $\hat{\tau}$ on $\hat{X}$.

3) The map $\varkappa: \mathcal{C}_0(X) \longrightarrow \hat{X}$ defined by $\varkappa(A) = [A,\{0\}]$ is isometric in the following sense: $\hat{p}(\varkappa(A) - \varkappa(B)) = dp\,(A,B)$ for all $A,B \in \mathcal{C}_0(X)$ and for continuous seminorms p on X.

Now we turn to the definition of differentiability of multivalued mappings.

Let M be a set and let F be a map of M into $\mathcal{C}_0(X)$; then we define a map $\hat{F}$ of M into $\hat{X}$ by:

$\hat{F}(m) = \varkappa(F(m)) = [F(m),\{0\}]$ for all $m \in M$.

If F is a map of M into $\hat{X}$, then it is clear that there exist maps A, B of M into $\mathcal{C}_0(X)$ such that $F(m) = [A(m),B(m)]$ for all $m \in M$ and we write $F = [A,B]$.

<u>Definition 1</u>. Let X, Y be locally convex spaces. A map F of X into $\mathcal{C}_0(Y)$ is said to be positively homogeneous if $F(tx) = tF(x)$ for all $x \in X$ and $t \geq 0$.

In the remainder of this section we always suppose that X, Y are locally convex spaces, Ω is an open subset of X, F is a map of Ω into $\mathcal{C}_o(Y)$.

Definition 2. (H.T. Banks and Q.M. Jacobs [2].) The mapping F is said to be **directionally differentiable** at $x_0 \in \Omega$ iff the mapping $\hat{F}$ has directional derivatives in every direction h of X; i.e. for each $h \in X$ there exists

$$\lim_{t \to 0^+} \frac{\hat{F}(x_0+th) - \hat{F}(x_0)}{t} = D_+\hat{F}(x_0)(h).$$

This means that there exist positively homogeneous maps $A(x_0)(\cdot)$, $B(x_0)(\cdot)$ of X into $\mathcal{C}_0(Y)$ such that for each continuous seminorm p on Y, for each $h \in X$ and $t > 0$ such that $x_0+th \in \Omega$, the function $\omega_p(h,t)$ defined by

$$\omega_p(h,t) = dp(F(x_0+th) + B(x_0)(th), F(x_0) + A(x_0)(th))$$

satisfies the condition $\lim_{t \to 0^+} \frac{\omega_p(h,t)}{t} = 0$.

If $D_+\hat{F}(x_0) = [A(x_0), B(x_0)] \in L(X,\hat{Y})$ and $\lim_{t \to 0^+} \frac{\omega_p(h,t)}{t} = 0$ uniformly with respect to h on each bounded subset of X for each continuous seminorm p on Y, then F is said to be **Fréchet differentiable** at x_0; in this case we write $D\hat{F}(x_0)(h) = D_+\hat{F}(x_0)(h)$.

We say that F is strictly conically differentiable at x_0 if F is directionally differentiable at x_0 and $D_+\hat{F}(x_0)(h) \in \mathcal{K}(\mathcal{C}_0(Y))$ for each $h \in X$; i.e. there exists a positively homogeneous map $A(x_0)$ of X into $\mathcal{C}_0(Y)$ such that $D_+\hat{F}(x_0)(h) = [A(x_0)(h), \{0\}]$ for all $h \in X$. In this case we write $D_+F(x_0)(h) = A(x_0)(h)$.

2. Some properties of differentiable mappings. Throughout this section X, Y, Z denote locally convex spaces, Ω an open subset of X, F a map of Ω into $\mathcal{C}_0(Y)$. Let $T \in L(X,Y)$

and define maps $T_c: \mathscr{C}_0(X) \longrightarrow \mathscr{C}_0(Y)$ and $\hat{T} \in L(\hat{X},\hat{Y})$ by $T_c(A) = \overline{T(A)}$ and $\hat{T}([A,B]) = [T_c(A), T_c(B)]$ for $A \in \mathscr{C}_0(X)$ and $[A,B] \in \hat{X}$ (see [5]).

Lemma 1. Let $T \in L(Y,Z)$ be given and let F be directionally differentiable at x_0. Then the map $T_c \circ F$ of Ω into $\mathscr{C}_0(Z)$, defined by $(T_c \circ F)(x) = T_c(F(x))$ for all $x \in \Omega$, is directionally differentiable at x_0 and $D_+(\widehat{T_c \circ F})(x_0)(h) = \hat{T}(D_+\hat{F}(x_0)(h))$.

If F is strictly conically differentiable at x_0, then $T_c \circ F$ is also strictly conically differentiable at x_0 and

$$D_+(T_c \circ F)(x_0)(h) = T_c(D_+F(x_0)(h)).$$

Proof. The proof is obvious, since $\widehat{T_c \circ F} = \hat{T} \circ \hat{F}$.

Theorem 1. Suppose that F is directionally differentiable at x_0 and $D_+\hat{F}(x_0)(h) = [A(x_0)(h), B(x_0)(h)]$. Assume that F satisfies the following condition:

(1) There exists a map C of Ω into $\mathscr{C}_0(Y)$ such that for each continuous seminorm p on Y and for each $h \in X$, and each $t > 0$ such that $x_0 + th \in \Omega$ we have $\lim_{t \to 0^+} \frac{\omega_p(h,t)}{t} = 0$, where

$$\omega_p(h,t) = dp(F(x_0+th), F(x_0) + C(x_0+th)).$$

Then F is strictly conically differentiable at x_0 if one of the following two conditions is satisfied:

a) Y is a semireflexive space or a space of the type LF,

b) for each $h \in X$, one of the sets $A(x_0)(h), B(x_0)(h)$ is weakly compact.

Moreover, if Y is normable and each map F which is directionally differentiable at x_0 and satisfies the condition (1), is strictly conically differentiable at x_0, then Y is complete.

Proof. 1. The condition (1) can be written as follows:

$$\hat{p}(\hat{F}(x_0+th) - \hat{F}(x_0) - \hat{C}(x_0+th)) = \omega_p(h,t) = o(t) \text{ if } t \to 0^+.$$

Then
$$D_+\hat{F}(x_0)(h) = \lim_{t\to 0^+} \frac{\hat{F}(x_0+th) - \hat{F}(x_0)}{t} = \lim_{t\to 0^+} \frac{\hat{C}(x_0+th)}{t}$$

$$= \lim_{n\to\infty} \frac{\hat{C}(x_0+n^{-1}h)}{n^{-1}} \in \overline{\varkappa(\mathscr{C}_0(Y))}^s$$

where A^s denotes the sequential closure of the set A. Now the assertion of the first part of our Theorem follows from Corollaries 1,4 [5].

2. Let Y be a normed space and let the assumption of the part 2 of Theorem be satisfied. We shall prove that the space Y coincides with the completion $\widetilde{Y}$ of Y. Let $y \in \widetilde{Y}$ be given; then there exist $y_n \in Y$ $(n=1,2,\dots)$ such that $y = \sum_1^\infty y_n$ and $\sum_1^\infty \| y_n \| < \infty$. Put

$$\alpha(t) = \begin{cases} 1 - 3n|t| & \text{for } t: |t| \le \frac{1}{3n} \\ \frac{1}{2} - \frac{3}{2} n|t| & \text{for } t: \frac{1}{3n} \le |t| \le \frac{2}{3n} \\ -\frac{3}{2} + \frac{3}{2} n|t| & \text{for } t: \frac{2}{3n} \le |t| \le \frac{1}{n} \\ 0 & \text{for } t: |t| \ge \frac{1}{n} \end{cases}$$

$\beta_n(t) = \int_0^t \alpha_n(\tau)d\tau$ for $n = 1,2,\dots$, $f(t) = \sum_1^\infty \beta_n(t)y_n \in \widetilde{Y}$.

Then it is easy to verify that $f'(t) = Df(t)(1) = \sum \alpha_n(t)y_n$.

Let $h_0 \in X$, $h_0 \neq 0$, $X_1 = \{th_0 \mid t \in R\}$. We define a map u of X_1 into $\mathscr{C}_0(Y)$ by

$u(th_0) = \{\sum \beta_n(t)y_n\} \in \mathscr{C}_0(Y)$ (as $\beta_n(0) = 0$ for all n and for $t \neq 0$, $\beta_n(t) \neq 0$ only for a finite number of n). Let i be the inclusion of Y into $\widetilde{Y}$. Then the map $\widehat{i_c \circ u}$ is Fréchet differentiable on X_1, since $\widehat{i_c \circ u}(th_o) = [\{f(t)\},\{0\}]$ and $D(\widehat{i_c \circ u})(th_o)(h_o) = [\{f'(t)\},\{0\}]$ for all $t \in R$. Furthermore, we know that the map $\hat{i}$ is an isomorphism of $\hat{Y}$ onto $\hat{\widetilde{Y}}$ (see Remark 3 after Theorem 3 [5]). Hence the map $\hat{u} = (\hat{i})^{-1}(\widehat{i_c \circ u})$ is Fréchet differentiable on X_1. By the Definition 2, it follows that u is Fréchet differentiable on X_1. Let π be the projection of X onto X_1. We define a map F of Ω into $\mathscr{C}_o(Y)$ by $F(x) = u(\pi(x-x_o))$ for all $x \in \Omega$. Then, of course, F satisfies the condition (1) with $C(x) = F(x)$, and F is Fréchet differentiable on Ω (so at x_o) and $D\hat{F}(x_o)(h) = D\hat{u}(o)(\pi h)$. By the assumption, F is strictly conically differentiable at x_o, so there exists an $A \in \mathscr{C}_o(Y)$ such that $D\hat{F}(x_o)(h_o) = [A,\{0\}]$. Then

$$[\bar{A},\{o\}] = \hat{i}(D\hat{F}(x_o)(h_o)) = \hat{i}(D\hat{u}(o)(h_o)) = D(\widehat{i_c \circ u})(o)(h_o) = [\{y\},\{o\}],$$

where $\bar{A}$ denotes the closure of A in $\widetilde{Y}$. Hence: $y \in \{y\} = \bar{A} = A \subseteq Y$. This means that $\widetilde{Y} \subseteq Y$ and this completes the proof.

<u>Theorem 2</u>. **(The mean value theorem.)** Suppose that F is directionally differentiable on Ω, $D_*\hat{F}(x)(h) = [A(x)(h), B(x)(h)]$ for $x \in \Omega$, $h \in X$ and let $x_o, x_1 \in \Omega$ be given such that $\{tx_o+(1-t)x_1 \mid 0 \le t \le 1\} \subseteq \Omega$. Put $k = x_1 - x_o$. Then:

1) If $D_+\hat{F}(x_o+tk)(k) = [A(x_o+tk)(k)),\{0\}] \in \varkappa(\mathscr{C}_o(Y))$ for all $t \in [0,1]$ and if Y is a space of the type LF, then there exists a set $Q(x_o,x_1) \in \mathscr{C}_o(Y)$ such that $F(x_1) = F(x_o) +^* +^* Q(x_o,x_1)$.

2) If Y is a regular inductive limit of a sequence of metrizable locally convex spaces and $M = \overline{\text{conv}}\,\{A(x_o+tk)(k) \mid 0 \le$

$\leq t \leq 1\}$ and $N = \overline{\mathrm{conv}}\, \{B(x_0+tk)(k) \mid 0 \leq t \leq 1\}$ are separable and weakly compact, then there exist sets $A(x_0,x_1), B(x_0,x_1) \in$ $\in \mathscr{C}_0(Y)$, $A(x_0,x_1) \subseteq M$, $B(x_0,x_1) \subseteq N$ such that $F(x_1) +^* B(x_0,x_1) =$ $= F(x_0) +^* A(x_0,x_1)$.

Proof. By the mean value theorem for singlevalued mappings (see [1]) it follows that:

$$[F(x_1),F(x_0)] = \hat{F}(x_1) - \hat{F}(x_0) = \hat{F}(x_0+k) - \hat{F}(x_0) \in$$
$$\in \overline{\mathrm{conv}}\, \{D_+\hat{F}(x_0+tk)(k) \mid 0 \leq t \leq 1\}.$$

1) Let $Y = \varinjlim_{n} Y_n$ be a space of the type LF and let $D_+\hat{F}(x_0+tk)(k) = [A(x_0+tk)(k),\{0\}] \in \varkappa(\mathscr{C}_0(Y))$ for all $t \in [0,1]$. If we put $G(t) = F(x_0+tk)$ for $t \in [0,1+2\delta]$, where δ is a positive number such that $x_0+tk \in \Omega$ for all $t \in [0,1+2\delta]$, then we obtain a map G of $[0,1+2\delta]$ into $\mathscr{C}_0(Y)$ which is directionally differentiable on $[0,1+2\delta)$. It implies that $\hat{G}$, so as G, is continuous on $[0,1+\delta]$. It is easy to verify that the set $\bigcup\{G(t) \mid 0 \leq t \leq 1+\delta\} = \bigcup\{F(x_0+tk) \mid 0 \leq t \leq 1+\delta\}$ is bounded in Y. By Theorem 6.5 ([7], chapt. II) there exists n_0 such that $\bigcup\{F(x_0+tk) \mid 0 \leq t \leq 1+\delta\} \subseteq Y_{n_0}$, i.e. $\hat{F}(x_0+tk) \in$ $\in \hat{i}_{n_0}(\hat{Y}_{n_0})$ where $\hat{i}_{n_0}$ is the inclusion of $\hat{Y}_{n_0}$ into $\hat{Y}$, for all $t \in [0,1+\delta]$. Then, of course, we have $[A(x_0+tk)(k),\{0\}] =$ $= D_+F(x_0+tk)(k) \in \overline{\hat{i}_{n_0}(\hat{Y}_{n_0})}$ for all $t \in [0,1]$. We claim that $A(x_0+tk)(k) \in \mathscr{C}_0(Y_{n_0})$. Suppose that it is not true, then there exist $t_0 \in [0,1]$ and a point $y \in A(x_0+t_0k)(k)$ such that $y \notin$ $\notin Y_{n_0}$. Since Y_{n_0} is a closed subspace of Y, there exists a convex circled closed 0-neighborhood N in Y such that $(y + 2N) \cap Y_{n_0} = \emptyset$ i.e. $A(x_0+t_0k)(k) \nsubseteq Y_{n_0} + 2N$. Then it follows that $([A(x_0+t_0k)(k),\{0\}] + \hat{U}_N) \cap \hat{i}_{n_0}(\hat{Y}_{n_0}) = \emptyset$, where

$\hat{U}_N = \{[A,B] \in \hat{Y} | A \subseteq \overline{B + N}$ and $B \subseteq \overline{A + N}\}$ is an $\mathcal{O}$-neighborhood in $\hat{Y}$ (see [5]). This contradicts the fact that $[A(x_o+t_o k)(k), \{0\}] \in \overline{\hat{i}(\hat{Y}_{n_o})}$. This means that $D_* \hat{F}(x_o+tk)(k) = [A(x_o+tk)(k), \{0\}] \in \overline{\hat{i}_{n_o}(\varkappa(\mathcal{C}_o(Y_{n_o})))}$ for all $t \in [0,1]$. By the mean value theorem for singlevalued mappings we have that $[F(x_1),F(x_o)] \in \overline{\hat{i}_{n_o}(\varkappa(\mathcal{C}_o(Y_{n_o})))}$. But we know that $\varkappa(\mathcal{C}_o(Y_{n_o}))$ is a complete subset of $\hat{Y}_{n_o}$ (see [4]), as Y_{n_o} is an F-space, and $\hat{i}_{n_o}$ is an isomorphism of $\hat{Y}_{n_o}$ into $\hat{Y}$ (see [5]). Hence $[F(x_1),F(x_o)] \in \hat{i}_{n_o}(\varkappa(\mathcal{C}_o(Y_{n_o})))$. Thus, there exists a set $Q(x_o,x_1) \in \mathcal{C}_o(Y_{n_o}) \subseteq \mathcal{C}_o(Y)$ such that $[F(x_1),F(x_o)] = [Q(x_o,x_1),\{0\}]$. Then

$$F(x_1) = F(x_o) +^* Q(x_o,x_1).$$

2. Put $\mathfrak{M} = \{[A,B] \in \hat{Y} | A \subseteq M, B \subseteq N\}$, then $\mathfrak{M}$ is a convex subset of $\hat{Y}$ and by Proposition 2 [5] is $\sigma(Y,Y')$-compact. Therefore $\mathfrak{M}$ is $\sigma(Y,Y')$-closed and it implies that $\mathfrak{M}$ is closed in $\hat{Y}$ in topology $\hat{\tau}$, where τ is the topology of Y. It is clear now that $[F(x_1),F(x_o)] \in \mathfrak{M}$, since $D_* \hat{F}(x_o+tk)(k) \in \mathfrak{M}$ for all $t \in [0,1]$. Therefore there exist sets $A(x_o,x_1) \subseteq M$, $B(x_o,x_1) \subseteq N$ such that $[F(x_1),F(x_o)] = [A(x_o,x_1),B(x_o,x_1)]$, which means that $F(x_1) +^* B(x_o,x_1) = F(x_o +^* A(x_o,x_1))$.
This completes the proof.

<u>Theorem 3</u>. Suppose that F is strictly conically differentiable on Ω (i.e. F is strictly conically differentiable at each point $x \in \Omega$). Then

1. $D_+F(x)(h)$ is a singleton for all $x \in \Omega$ and $h \in X$;

2. if Ω is connected and Y is quasi-complete, then for each $x_o \in \Omega$ there exists a unique singlevalued mapping f of

Ω into Y such that:

$$F(x) = F(x_0) + f(x)$$

and

$$D_+F(x)(h) = \{D_+f(x)(h)\}$$

for all $x \in \Omega$; $h \in X$.

Proof. We divide the proof in two steps.

Step I. First of all we suppose that Y is the space of the type LF. For each $x \in \Omega$, we take a convex neighborhood U(x) of x contained in Ω . By the mean value Theorem 2 for each $z \in U(x)$ there exist sets A(x,z) and A(z,x) of $\mathcal{C}_0(Y)$ such that $F(z) = F(x) +^* A(x,z)$ and $F(x) = F(z) +^* A(z,x)$. Then $F(z) = F(z) +^* A(x,z) +^* A(z,x)$ or $A(x,z) +^* A(z,x) = \{0\}$. The latter identity holds if and only if A(x,z), A(z,x) are singletons and $A(x,z) = - A(z,x)$. Let g(x,z) be a unique element of A(x,z). Then $F(z) = F(x) + g(x,z)$ for all $z \in U(x)$. It is easy to verify that the map $g(x_1, \cdot)$ of U(x) into Y is directionally differentiable on U(x) and $D_+F(x)(h) = \{D_+g(x,x)(h)\}$ for all $h \in X$. This shows that $D_+F(x)(h)$ is a singleton for all $x \in \Omega$ and $h \in X$.

If Ω is connected, put $G = \{x; x \in \Omega$ and there exists a point $f(x) \in Y$ such that $F(x) = F(x_0) + f(x)\}$. One can verify that G is open and closed in Ω . From connectedness of Ω it follows that $\Omega = G$ and it is clear that f is unique.

Step II. We denote the bidual space of Y by Y" and let Y" be endowed with the σ-topology τ'' , where σ is the family of all equicontinuous subsets of Y'. Then the canonical embedding (evaluation map) J of Y into Y" is an isomorphism of Y into Y". Let $y' \in Y'$, then by Lemma 1 the mapping $y'_c \circ F$ is strictly conically differentiable on Ω and by step I,

$D_+(y_c' \circ F)(x)(h) = y_c'(D_+F(x)(h))$ is a singleton for all $x \in \in \Omega$, $h \in X$, $y' \in Y'$, since R is an F-space. It follows that $D_+F(x)(h)$ is a singleton, since Y' distinguishes points of Y. Denote the unique element of $D_+F(x)(h)$ by $u(x)(h)$, then for each $x \in \Omega$, $u(x)(\cdot)$ is a positively homogeneous mapping of X into Y. If Ω is connected, then for each $y' \in Y'$ there exists a map $f_{y'}$ of Ω into R such that $y_c'(F(x)) = y_c'(F(x_0)) + f_{y'}(x)$ and $D_+f_{y'}(x)(h) = y'(u(x)(h))$. Now we define a mapping $v(x)$ of Y' into R by $v(x)(y') = f_{y'}(x)$ for each $x \in \Omega$. Then we claim that $v(x) \in Y''$ and $D_+v(x)(h) = J(u(x)(h))$ for each $x \in \Omega$, $h \in X$. We prove that

i) $v(x)$ is a linear functional of Y' into R. Let $y', z' \in \in Y'$ and $\alpha, \beta \in R$ be given, then

$$D_+(f_{\alpha y'+\beta z'} - \alpha f_{y'} - \beta f_{z'})(x)(h) = D_+f_{\alpha y'+\beta z'}(x)(h) - \alpha D_+f_{y'}(x)(h) - \beta D_+f_{z'}(x)(h) = (\alpha y' + \beta z')(u(x)(h)) - \alpha y'(u(x)(h)) - \beta z'(u(x)(h)) = 0$$

and $(f_{\alpha y'+\beta z'} - \alpha f_{y'} - \beta f_{z'})(x_0) = 0$.

It follows that $f_{\alpha y'+\beta z'}(x) = \alpha f_{y'}(x) + \beta f_{z'}(x)$ for all $x \in \in \Omega$. Then $v(x)(\alpha y' + \beta z') = \alpha v(x)(y') + \beta v(x)(z')$, i.e. $v(x)(\cdot)$ is linear.

ii) $v(x) \in Y''$. For this purpose set

$$V = (F(x) \cup F(x_0))^0 = \{y' \in Y' \mid |\langle y', y\rangle| \leq 1 \text{ for all } y \in F(x) \cup F(x_0)\}.$$

Then V is an σ-neighborhood in the strong topology $\beta(Y', Y)$ on Y'. For each $y' \in V$ we have:

$$|v(x)(y')| = |f_{y'}(x)| = d(\{f_{y'}(x)\}, \{0\}) = d(y_c'(F(x_0)) + f_{y'}(x), y_c'(F(x_0))) = d(y_c'(F(x)), y_c'(F(x_0))) \leq 2.$$

This shows that $v(x)$ is a linear continuous functional on

$(Y', \beta(Y',Y))$ and therefore $v(x) \in Y''$.

iii) $D_+v(x)(h) = J(u(x)(h))$ for all $h \in X$. Let p'' be a continuous seminorm on (Y'', τ''), $S_{p''} = \{y'' \in Y'': p''(y'') \leqq 1\}$. Then there exists an equicontinuous subset E of Y' such that $S_{p''} = E^0 = \{y'' \in Y'': |\langle y'', y'\rangle| \leqq 1$ for all $y' \in E\}$. Let $S_p = {}^0E = \{y \in Y: |\langle y', y\rangle| \leqq 1$ for all $y' \in E$ and let p be the gauge functional of the set S_p. Then for each $x \in \Omega$, $h \in X$, $t > 0$, $x + th \in \Omega$ and for each $y' \in E$ we have:

$$|\langle v(x) + th) - v(x) - J(u(x)(th)), y'\rangle| = |f_y(x + th) - f_y(x) - y'(u(x)(th))| = d(y_c'(F(x_0)) + f_y(x + th), y_c'(F(x_0)) + f_y(x) + y_c'(D_+F(x)(th))) = d(y_c'(F(x + th)), y_c'(F(x) + D_+F(x)(th))) \leqq dp(F(x + th), F(x) + D_+F(x)(th)) = \omega_p(h,t)$$

and $\lim_{t \to 0^+} \frac{\omega_p(h,t)}{t} = 0$, since p is a continuous seminorm on Y and F is directionally differentiable at x and $D_+\hat{F}(x)(h) = \varkappa(D_+F(x)(h))$. Then $p''(v(x + th) - v(x) - J(u(x)(th))) \leqq \sup\{|\langle v(x + th) - v(x) - J(u(x)(th)), y'\rangle|, y' \in E\} \leqq \omega_p(h,t)$.

This means that $D_+v(x)(h) = J(u(x)(h))$. Put $G(x) = J_c(F(x)) - v(x)$. Then

$$\hat{G}(x) = \widehat{J_c \circ F}(x) - \hat{v}(x), \text{ where } \hat{v}(x) = [\{v(x)\}, \{0\}]$$

$$D_+\hat{G}(x)(h) = D_+(\widehat{J_c \circ F})(x)(h) - [\{J(u(x)(h))\}, \{0\}] = 0.$$

This means that $\hat{G}$ (and simultaneously G), is constant on Ω. This implies that $J_c(F(x)) - v(x) = J_c(F(x_0))$,

$$v(x) \in J_c(F(x)) - J_c(F(x_0)).$$

On the other hand, $J_c(F(x)) = \overline{J(F(x))} = J(F(x))$, since $F(x)$ is a complete subset of Y and J is an isomorphism of Y into Y''. Then $v(x) \in J(F(x) - F(x_0)) \subseteq J(Y)$. Put $f(x) =$

$= J^{-1}(v(x))$, then $F(x) = F(x_0) + f(x)$. Of course, $D_+F(x)(h) = \{u(x)(h)\} = \{D_+f(x)(h)\}$. This completes the proof.

<u>Remark 1</u>. We can define the differentiation of the mapping $F: \Omega \longrightarrow \mathcal{B}(Y)$ in the same way as De Blasi [3].

<u>Definition 3</u>. The map F is said to be directionally differentiable at $x_0 \in \Omega$ iff there exists a positively homogeneous map $D_+F(x_0)$ of X into $\mathcal{C}_0(X)$ such that for each continuous seminorm p on Y and for each $h \in X$ and $t > 0$ such that $x_0 + th \in \Omega$ we have $\lim_{t \to 0^+} \frac{\omega_p(h,t)}{t} = 0$, where

$$\omega_p(h,t) = dp(F(x_0 + th), F(x_0) + D_+F(x_0)(th)).$$

It is easy to see that if F is directionally differentiable at x_0, then the map co F of Ω into $\mathcal{C}_0(Y)$ defined by $(\text{co } F(x) = \overline{\text{conv}}\, F(x)$ is strictly conically differentiable at x_0, and $D_+(\text{co } F)(x)(h) = D_+F(x)(h)$.

<u>Theorem 3'</u>. Let $F: \Omega \longrightarrow \mathcal{B}(Y)$ be directionally differentiable on Ω. Then: 1) $D_+F(x)(h)$ is a singleton for all $x \in \Omega$ and $h \in X$; 2) if Ω is connected and Y is quasicomplete, $x_0 \in \Omega$, then there exists a unique map f of Ω into Y such that

$$\overline{F(x)} = \overline{F(x_0)} + f(x)$$

$$D_+F(x)(h) = \{D_+f(x)(h)\}$$

for all $x \in \Omega$ and $h \in X$.

<u>Proof</u>. 1. By Theorem 3, $D_+F(x)(h) = D_+(\text{co } F)(x)(h)$ is a singleton for all $x \in \Omega$, $h \in X$.

2. By Theorem 3 there exists a map f of Ω into Y such that $(\text{co } F)(x) = (\text{co } F)(x_0) + f(x)$; $D_+F(x)(h) = \{D_+f(x)(h)\}$. Set $G(x) = \overline{F(x)} - f(x)$. Let p be a continuous seminorm on Y.

Put $g(x) = dp(G(x),G(x_0))$. Using the same arguments as in the proof of Theorem 3.2 [3], one can prove that $D_+g(x)(h) = 0$ for all $x \in \Omega$, $h \in X$. It follows then $dp(G(x),G(x_0)) = 0$ for all $x \in \Omega$ and for all continuous seminorms p on Y. This means that $G(x) = G(x_0) = \overline{F(x_0)}$ and hence $\overline{F(x)} = \overline{F(x_0)} + f(x)$.

Remark 2. If Y is not quasicomplete, then the second part of Theorem 3 is not true. For instance, we take a normed space which is not complete. Let $\tilde{Y}$ be the completion of Y, $y \in \tilde{Y}$, $y \notin Y$. For each n, choose $z_n \in Y$ such that $\|y - z_n\| \leq (4n^2)^{-1} \cdot 2^{-n-2}$. Put $y_1 = z_1$, $y_n = z_n - z_{n-1}$ for $n = 1,2,\ldots$. Then

$$\sum y_n = y, \quad \sum 4n^2 \|y_n\| < +\infty .$$

Set

$$\alpha_n(t) = \begin{cases} -1 + \frac{1}{n} + t \text{ for } 1 - \frac{1}{n} \leq t \leq 1 - \frac{1}{2n} \\ 1 - t \text{ for } t\text{: } 1 - \frac{1}{2n} \leq t \leq 1 \\ 0 \text{ for } t\text{: } t \leq 1 - \frac{1}{n} \text{ or } t \geq 1 \end{cases}$$

$$\beta_n(t) = \int_0^t \alpha_n(\tau)d\tau \quad \text{for } n = 1,2,\ldots$$

Then $\beta_n(t) = 0$ for $t \leq 1 - \frac{1}{n}$; $\beta_n(t) = \frac{1}{4n^2}$ for $t \geq 1$.

Define

$$f(t) = \sum 4n^2 \beta_n(t) y_n$$

$$F(t) = (f(t) + S_1) \cap Y \in \mathscr{C}_0(Y),$$

where $S_1 = \{y \in \tilde{Y}: \|y\| \leq 1\}$. Then it is easy to verify that F is strictly conically differentiable on R and

$$D_+F(t)(1) = \{\sum \alpha_n(t)\, 4n^2 y_n\}; \; D_+F(t)(-1) = -D_+F(t)(-1) = \{-\sum \alpha_n \, 4n^2 y_n\} .$$

We suppose that there exists a map g of R into Y such that

$F(t) = F(0) + g(t)$. Then $\overline{F(t)} = \overline{F(0)} + g(t) = S_1 + f(t)$. Hence $y = f(1) = g(1) \in Y$ and this contradicts the assumption $y \notin Y$.

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