Commentationes Mathematicae Universitatis Carolinae

Luděk Zajíček

# On the symmetry of Dini derivates of arbitrary functions

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 22 (1981), No. 1, 195--209

Persistent URL: http://dml.cz/dmlcz/106064

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

22,1 (1981)

# ON THE SYMMETRY OF DINI DERIVATES OF ARBITRARY FUNCTIONS

L. ZAJÍČEK

Abstract: In the article the strongest relation connecting the Dini derivates of an arbitrary real function which holds except on a first category set is found. We further prove that the relation holds except on a $\sigma$-porous set. The mentioned result is an easy consequence of a Dolženko's theorem on angle cluster sets. An additional result on the symmetry of Dini derivates is proved by the Jarník-Blumberg method, too.

Key words: Dini derivates, $\sigma$-porous sets, boundary behaviour of functions, Jarník-Blumberg method.

Classification: 26A27

---

1. Introduction. The well known Denjoy-Young-Saks theorem (see [3], chap. IV, th. 4.4 or [12], p. 271) is the most important theorem concerning the Dini derivates of arbitrary functions. This theorem establishes certain relations (so called Denjoy relations) valid almost everywhere, which connect the Dini derivates of arbitrary functions. On the other hand, there exists a function (e.g. the function f from Example 1) for which the set of all x at which the Denjoy relations hold is a first category set.

We can ask what is the strongest relation connecting the Dini derivates which holds except on a first category set. By the Neugebauer's theorem [11] for any continuous function f the set of all points x at which

(1) $D^+f(x) \neq D^-f(x)$ or $D_+f(x) \neq D_-f(x)$

is a first category set. But the Neugebauer's theorem cannot be extended to the case of an arbitrary function f. To see this simple fact it is sufficient to choose a dense set $A \subset R$ for which R-A is also dense and put $f = C_A$. For f (1) holds at each point $x \in R$.

For this function f at each point of a residual set (in fact at each point) either $D_-f(x) = -\infty$, $D^+f(x) = \infty$ or $D^-f(x) = \infty$, $D_+f(x) = -\infty$. We shall show that this circumstance is not accidental. In fact, for an arbitrary function f except on a first category set at least one from the following relations holds:

(i) $D^+f(x) = D^-f(x)$ and $D_+f(x) = D_-f(x)$

(ii) $D_-f(x) = -\infty$, $D^+f(x) = \infty$ and $D_+f(x) \leqq D^-f(x)$

(iii) $D^-f(x) = \infty$, $D_+f(x) = -\infty$ and $D_-f(x) \leqq D^+f(x)$.

The examples in part 5 of the present article show that this assertion gives the strongest relation connecting the Dini derivates of an arbitrary function which hold except on a first category set. Denote by $A_f$ the set of all points x at which none from the relations (i),(ii),(iii) holds. We stated that $A_f$ is a first category set and the Denjoy-Young-Saks theorem yields that $A_f$ is also a null set. Actually we shall prove a little more precise result (Theorem 1) which asserts that $A_f$ is $\sigma$-porous. Note that the system of all $\sigma$-porous sets is smaller than the system of all null sets of the first category. For the definition of the $\sigma$-porous set and further informations see part 2. Example 8 shows that this result concerning the magnitude of the sets of the form $A_f$ cannot be strengthened.

Of course, for some "subsets" of $A_f$ the sharper results hold. We shall prove (Theorem 2) that for an arbitrary f the set of all x for which one from the numbers

$$\max (|D^+f(x)|, |D_+f(x)|), \max (|D^-f(x)|, |D_-f(x)|)$$

is finite and the other is infinite is $\sigma$-strongly porous.

We prove our theorems by the Jarník-Blumberg method. In other words, we obtain our theorems on derivates of a function f as easy consequences of theorems concerning the boundary behaviour of functions of two variables (the boundary behaviour of the function $g(x,y) = (f(x) - f(y)) (x - y)^{-1}$ in the half-plane $x > y$ is investigated). This elegant method was first used by Jarník [7],[8] and Blumberg [2]. For further informations concerning this method see e.g. [4],[6],[9]. In the present article we use theorems on the angle cluster sets. Theorem 1 is a consequence of the Dolženko's theorem [5] and Theorem 2 is proved by a new result (Proposition 1) on the boundary behaviour of functions of two variables.

2. Preliminaries. We denote by R the set of all real numbers and put $\overline{R} = R \cup \{-\infty, \infty\}$. The symbol $\mu$ stands for the outer Lebesgue measure in R. If $M \subset R^2$ then $M'$ is the set of all points of accumulation of M. The open circle of the centre $x \in R^2$ and the radius r is denoted by $B(x,r)$. For $M \subset R$ we put $-M = \{x; -x \in M\}$. For $x \in R$ and $M \subset R$ we mean by $\rho(x,M)$ the distance from the point x to the set M.

Let $M \subset R$, $x \in R$, $r > 0$. Denote by $p(M,x,r)$ the maximum of the lengths of connected sets I such that $I \subset (x-r,x+r) - M$. Obviously $0 \leqq p(M,x,r) \leqq 2r$ and if $x \in M$, then $0 \leqq p(M,x,r) \leqq r$.

A set $M \subset R$ is said to be porous at a point $x \in R$ if $\limsup_{r \to 0_+} p(M,x,r)\, r^{-1} > 0$.

A set $M \subset R$ is said to be strongly porous at a point $x \in R$ if $\limsup_{r \to 0_+} p(M,x,r)\, r^{-1} = 1$.

A set $M \subset R$ is termed a porous set (resp. a strongly porous set) if M is porous (resp. strongly porous) at any point $x \in M$.

A subset of R is termed a $\sigma$-porous set (resp. a $\sigma$-strongly porous set) if it is the union of a sequence of porous sets (resp. strongly porous sets). The notion of a $\sigma$-porous set was defined by E.P. Dolženko [5]. Each $\sigma$-porous set is obviously a first category set and by the density theorem it is also a null set. On the contrary, there exist (perfect) null sets of the first category which are not $\sigma$-porous. This assertion was stated in [5] and proved in [14]. The notion of a $\sigma$-strongly porous set is identical with the notion of a $\sigma$-(x,1/2)-porous set from [14]. There exist (perfect) porous sets which are not $\sigma$-strongly porous [14]. Note that exceptional sets which are first category measure zero sets are frequently also $\sigma$-porous sets (see e.g. [13],[1]).

3. <u>The boundary behaviour of functions of two variables.</u> In the following we denote the open half plane $\{(x,y);\ y > 0\}$ by H and the open half plane $\{(x,y);\ x > y\}$ by $H^*$.

For $0 < \theta < \pi$ and $t \in R$ we denote by $L(t,\theta)$ the ray in H which terminates at (t,0) and makes an angle $\theta$ with the x-axis. By $L^*(t,\theta)$ we denote the ray in $H^*$ which terminates at (t,t) and makes an angle $\theta$ with the half line $\{(x,y);\ x=y,\ x < t\}$.

For $0 < \alpha < \beta < \pi$ and $t \in R$ we denote by $A(t,\alpha,\beta)$ the open angle in H determined by $L(t,\alpha)$ and $L(t,\beta)$. By

$A^*(t,\alpha,\beta)$ we mean the open angle in $H^*$ determined by $L^*(t,\alpha)$ and $L^*(t,\beta)$. An angle of the form $A(t,\alpha,\beta)$ (resp. $A^*(t,\alpha,\beta)$) is termed an angle at $(t,0)$ (resp. $(t,t)$).

If f is a real function defined in H (resp. $H^*$) and A is an angle at $(t,0)$ (resp. $(t,t)$), then we denote by $C(f,t,A)$ the cluster set of f at $(t,0)$ (resp. $(t,t)$) with respect to A. Thus $C(f,t,A)$ is the set of all $y \in \bar{R}$ such that $(t,0)$ (resp. $(t,t)$) is a point of accumulation of the set $f^{-1}(V) \cap A$ for any neighbourhood V of y.

The following theorem is a special case of a Dolženko's theorem [5].

Theorem D. Let f be an arbitrary function defined on H. Let M be the set of all $t \in R$ for which there exist angles $A_1$, $A_2$ at $(t,0)$ such that $C(f,t,A_1) \neq C(f,t,A_2)$. Then M is $\sigma$-porous.

It is easy to see that Theorem D has the following consequence.

Theorem D*. Let f be an arbitrary function defined on $H^*$. Let M be the set of all $t \in R$ for which there exist angles $A_1^*$, $A_2^*$ at $(t,t)$ such that $C(f,t,A_1^*) \neq C(f,t,A_2^*)$. Then M is $\sigma$-porous.

Lemma 1. Let $S \subset H$ be a set and $0 < \alpha < \beta < \pi$. Denote by M the set of all $x \in R$ such that $(x,0) \notin (S \cap A(x,\alpha,\beta))'$ and $(x,0) \in (S \cap A(x,\alpha,\gamma))'$ for each $\beta < \gamma < \pi$. Then M is $\sigma$-strongly porous.

Proof. We identify $R^2$ with the set of all complex numbers by the standard way. Thus we have $(x,0) = x$ and the symbol arg z is defined for $z \in H$. For a positive integer n denote by $M_n$ the set of all $x \in M$ for which $A(x,\alpha,\beta) \cap B(x,1/n) \cap S = \emptyset$. Let $x \in M_n$. Then there exists a sequence $x_k \to x$, $x_k \in S$ such that

$\arg(x_k - x) \searrow \beta$ . Let $x_k^{\alpha}$ and $x_k^{\beta}$ be the real numbers for which $\arg(x_k - x_k^{\alpha}) = \alpha$ and $\arg(x_k - x_k^{\beta}) = \beta$ . It is easy to see that for sufficiently large k the interval $(x_k^{\alpha}, x_k^{\beta})$ does not contain points from $M_n$, $x_k^{\alpha} \to x$ and

$$\lim_{k\to\infty} (x_k^{\beta} - x_k^{\alpha})(x - x_k^{\alpha})^{-1} = 1.$$

Consequently $M_n$ is strongly porous at x. Therefore each $M_n$ is strongly porous and $M = \bigcup_{n=1}^{\infty} M_n$ is $\sigma$-strongly porous.

Proposition 1. Let f be an arbitrary function in H, $z \in \overline{R}$ and $0 < \alpha < \beta < \pi$ . Denote by M the set of all $x \in R$ such that $z \notin C(f,x,A(x,\alpha,\beta))$ and $z \in C(f,x,A(x,\alpha,\gamma))$ for each $\beta < \gamma < \pi$. Then M is $\sigma$-strongly porous.

Proof. It is sufficient to choose a neighbourhood V of z for which $(x,0) \notin (f^{-1}(V) \cap A(x,\alpha,\beta))'$ and to apply Lemma 1 to $S = f^{-1}(V)$.

It is easy to see that Proposition 1 has the following consequence.

Proposition 1* . Let f be an arbitrary function in H* , $z \in \overline{R}$ and $0 < \alpha < \beta < \pi$ . Denote by M the set of all $x \in R$ such that $z \notin C(f,x,A^*(x,\alpha,\beta))$ and $z \in C(f,x,A^*(x,\alpha,\gamma))$ for each $\beta < \gamma < \pi$ . Then M is $\sigma$-strongly porous.

4. The main results. In the following f is an arbitrary real function defined on R and $g(x,y) = (f(x) - f(y))(x-y)^{-1}$.

Lemma 2. Let $D^- f(t) < D^+ f(t) < \infty$ . Then there exists an angle A at (t,t) such that $D^+ f(t) \notin C(g,t,A)$.

Proof. We shall show that it is possible to choose $A = A^*(t, \pi/4, \pi/2) = \{(x,y);\ x > t, y < t, t-y > x-t\}$.

Choose an $\varepsilon > 0$ such that $D^-f(t) + 4\varepsilon < D^+f(t)$. Let $(x,y) \in A$. If $(x,y)$ is sufficiently near to $(t,t)$, we have

$$(f(x) - f(t)) / (x-t) < D^+f(t) + \varepsilon \quad \text{and} \quad (f(t)-f(y)) / (t-y) < D^+f(t) - 4\varepsilon .$$

For such $(x,y)$ we have

$$g(x,y) = \frac{f(x)-f(y)}{x-y} = \frac{(f(x)-f(t)) + (f(t)-f(y))}{(x-t) + (t-y)} \leqq$$

$$\leqq \frac{(D^+f(t) + \varepsilon)(x-t) + (D^+f(t) - 4\varepsilon)(t-y)}{(x-t) + (t-y)} = D^+f(t) +$$

$$+ \frac{\varepsilon(x-t) - 4\varepsilon(t-y)}{(x-t) + (t-y)} .$$

Using the inequality $t-y > x-t$ we obtain

$$g(x,y) \leqq D^+f(t) + (\varepsilon - 4\varepsilon)/2 < D^+f(t) - \varepsilon .$$

Therefore $D^+f(t) \notin C(g,t,A)$.

Lemma 3. Let $-\infty < D_-f(t) \leqq D^-f(t) < \infty$ . Then there exists an angle $A$ at $(t,t)$ such that $\infty \notin C(g,t,A)$.

Proof. We shall show that it is possible to choose

$$A = A^*(t, \pi/4 - \text{arctg } 1/2, \pi/4) = \{(x,y);\ x < t, y < t, 2(t-x) < t-y\}.$$

Choose $K > 0$ such that $\max(|D_-f(t)|, |D^-f(t)|) < K$. If $(x,y) \in A$ is sufficiently near to $(t,t)$, then we have

$$(f(x) - f(t)) / (x-t) \geqq -K \text{ and } (f(t) - f(y)) / (t-y) \leqq K.$$

For such $(x,y)$ we obtain

$$g(x,y) = \frac{f(x)-f(y)}{x-y} = \frac{(f(x) - f(t)) + (f(t) - f(y))}{(x - t) + (t - y)} \leqq$$

$$\leqq \frac{-K(x-t) + K(t-y)}{(x-t) + (t-y)} \leqq K \ \frac{3(t-y)/2}{(t-y)/2} = 3K.$$

Therefore $\infty \notin C(g,t,A)$.

Theorem 1. Let f be an arbitrary function on R. Then there exists a $\sigma$-porous set P such that for any $x \in R-P$ at least one from the following relations holds:

(i) $D^+f(x) = D^-f(x)$ and $D_+f(x) = D_-f(x)$

(ii) $D_-f(x) = -\infty$, $D^+f(x) = \infty$ and $D_+f(x) \leqq D^-f(x)$

(iii) $D^-f(x) = \infty$, $D_+f(x) = -\infty$ and $D_-f(x) \leqq D^+f(x)$.

Proof. If h is a function on R then we denote by A(h) the set of all $x \in R$ at which $D^-h(x) < D^+h(x)$ and the relation (ii) does not hold. Further put $B(h) = \{x;\ D^-h(x) < D^+h(x) < \infty\}$, $C(h) = \{x;\ D^+h(x) = \infty$ and $-\infty < D_-h(x) \leqq D^-h(x) < \infty\}$ and $D(h) = \{x;\ D_+h(x) > D^-h(x)\}$. Obviously $A(h) \subset B(h) \cup C(h) \cup D(h)$. Let h be a function and $x \in B(h)$. Put $g(x,y) = (h(x) - h(y)) / (x - y)$. Then by Lemma 2 there exists an angle A at (x,x) such that $D^+h(x) \notin C(g,x,A)$. Since $D^+h(x) = \limsup_{z \to (x,y),\, z \in L^+(x, 3\pi/4)} g(z)$ we have $D^+h(x) \in C(g,x,A^*(x, \pi/2, 4\pi/5))$. Therefore by Theorem D* we obtain that B(h) is a $\sigma$-porous set. Quite similarly we obtain by Lemma 3 that C(h) is a $\sigma$-porous set. Since D(h) is countable (see [12], p. 261) we have that A(h) is $\sigma$-porous. Now let P be the set of all points at which none from the relations (i),(ii),(iii) holds. Then

$$P = A(f(x)) \cup A(-f(x)) \cup -A(f(-x)) \cup -A(-f(-x))$$

and therefore P is a $\sigma$-porous set.

Corollary. For an arbitrary function the set on which one unilateral derivative exists and is finite but the derivative does not exist is a $\sigma$-porous set.

Lemma 4. Let $-\infty < D_-f(t) \leqq D^-f(t) < \infty$ and $D^+f(t) = \infty$. Then $\infty \notin C(g,t,A^*(t, \pi/4 - \operatorname{arctg} 1/2, \pi/4))$ and

$\infty \in C(g,t,A^*(t,\pi/4 - \text{arctg } 1/2, \gamma))$ for each $\pi/4 < \gamma < \pi$.

Proof. In the proof of Lemma 3 we have proved that $\infty \notin C(g,t,A^*(t,\pi/4 - \text{arctg } 1/2, \pi/4))$. Let $\pi/4 < \gamma < \pi$. Choose $\pi/4 < \delta < \min(\gamma, 3\pi/4)$. It is sufficient to prove that

$$(2) \qquad \limsup_{z \to (t,t), z \in L^*(t,\delta)} g(z) = \infty .$$

If we put $p = \text{tg}(\delta - \pi/4)$, then $p > 0$ and we easily obtain that $(x,y) \in L^*(t,\delta)$ if and only if $y < t < x$ and $x-t = p(t-y)$. Since $D^+f(t) = \infty$ there exists a sequence $x_n \searrow t$ such that $\lim_{n\to\infty} g(x_n,t) = \infty$. Define the sequence $(y_n)$ by the equation

$$(3) \qquad x_n - t = p(t - y_n).$$

Obviously $(x_n,y_n) \in L^*(t,\delta)$ and $(x_n,y_n) \longrightarrow (t,t)$. Choose an arbitrary $K > 0$. Then for sufficiently large n we have $g(t,y_n) > D_-f(t) - 1$ and $g(x_n,t) > K$. Using (3) we obtain for these n:

$$g(x_n,y_n) = \frac{f(x_n) - f(y_n)}{x_n - y_n} = \frac{(f(x_n) - f(t)) + (f(t) - f(y_n))}{x_n - y_n} \geqq$$

$$\geqq \frac{(x_n - t)K + (t-y_n)(D_-f(t) - 1)}{(x_n - t) + (t - y_n)} = \frac{pK + D_-f(t) - 1}{p + 1} .$$

Thus we have proved (2) and the proof is complete.

Theorem 2. Let f be an arbitrary function on R. Denote by M the set of all $t \in R$ for which one from the numbers

$$\max(|D^+f(t)|, |D_+f(t)|), \max(|D^-f(t)|, |D_-f(t)|)$$

is finite and the other is infinite. Then M is $\sigma$-strongly porous.

Proof. Using the function $f(-x)$, $-f(x)$, $-f(-x)$ as in the proof of Theorem 1 we easily see that it is sufficient to prove that the set $B = \{t;\ -\infty < D_-f(t) \leqq D^-f(t) < \infty,\ D^+f(t) = \infty\}$ is $\sigma$-strongly porous. But the last assertion is an immediate consequence of Lemma 4 and Proposition 1* .

5. Examples. The function f from the following example is well known (see e.g. [10]). I do not know the origin of this simple construction (it is possible that it is due to Z. Zahorski).

Example 1. Let $M \subset R$ be a measurable set such that $\mu(M \cap I) > 0$, $\mu(I - M) > 0$ for each compact interval I and $\mu(M \cap (0,\infty)) = \mu(M \cap (-\infty,0)) = \infty$. Put

$f(x) = \mu(M \cap \langle 0,x \rangle)$ for $x \geq 0$ and

$f(x) = -\mu(M \cap \langle x,0 \rangle)$ for $x < 0$.

Then f is an increasing homeomorphism of R onto itself. Put $g(x,y) = (f(x) - f(y)) / (x - y)$. Using the Lebesgue density theorem we easily obtain that for any $t \in R$

$$\limsup_{z \to (t,t),\, z \in H^*} g(z) = 1 \quad \text{and} \quad \liminf_{z \to (t,t),\, z \in H^*} g(z) = 0.$$

By Theorem 1 from [4] we obtain that

$$\limsup_{z \to (t,t),\, z \in L^*(t,\pi/4)} g(z) = D^-f(t) = 1,$$

$$\liminf_{z \to (t,t),\, z \in L^*(t,\pi/4)} g(z) = D_-f(t) = 0$$

$$\limsup_{z \to (t,t),\, z \in L^*(t,3\pi/4)} g(z) = D^+f(t) = 1,$$

$$\liminf_{z \to (t,t),\, z \in L^*(t,3\pi/4)} g(z) = D_+f(t) = 0$$

hold for each t from a residual set $G \subset R$. Let the real numbers $a \leqq b$ be given. Define

$$f_1(x) = (b-a)\, f(x) + ax.$$

Then obviously $D^+f_1(t) = D^-f_1(x) = b$ and $D_+f_1(t) = D_-f_1(t) =$ $= a$ for $t \in G$.

Example 2. Put $h = f_1^{-1}$. Then for each $x$ from the residual set $H = f_1(G)$ we have

$$D^+h(x) = D^-h(x) = \infty \quad \text{and } D_+h(x) = D_-h(x) = 1.$$

Let $a$ be an arbitrary real number. Put

$$f_2(x) = h(x) + (a-1)\, x.$$

Then obviously $D^+f_2(x) = D^-f_2(x) = \infty$ and $D_+f_2(x) = D_-f_2(x) =$ $= a$ for $x \in H$.

Example 3. Let $a, b, G, f_1$ be as in Example 1. Choose a countable dense set $C \subset R$ and put

$$f_3(x) = f_1(x) \text{ for } x \in R-C,$$
$$f_3(x) = f_1(x) + 1 \text{ for } x \in C.$$

Then obviously $D^+f_3(x) = \infty$, $D_-f_3(x) = -\infty$, $D_+f_3(x) = a$, $D^-f_3(x) = b$ for each $x$ from the residual set $G-C$.

Example 4. Let $a, H, f_2$ be as in Example 2. Choose a countable dense set $C \subset R$ and put

$$f_4(x) = f_2(x) \text{ for } x \in R-C,$$
$$f_4(x) = f_2(x) + 1 \text{ for } x \in C.$$

Then obviously $D^+f_4(x) = \infty$, $D_+f_4(x) = a$, $D^-f_4(x) = \infty$, $D_-f_4(x) = -\infty$ for each $x$ from the residual set $H-C$.

Example 5. Choose in $R$ three pairwise disjoint sets $M$, $L$, $K$ such that $M$ is residual and $L$, $K$ are dense. Put

$$f_5(x) = 1 \text{ for } x \in L,$$
$$f_5(x) = 0 \text{ for } x \in M,$$
$$f_5(x) = -1 \text{ for } x \in K.$$

Then obviously $D^+f_5(x) = D^-f_5(x) = \infty$ and $D_+f_5(x) = D_-f_5(x) = -\infty$ for each x from the residual set M.

Example 6. Let $G \subset R$ be a residual measure zero set. Then there exists (see e.g. [3], chap. 14, Th. 3.2) a continuous function $f_6$ such that $f_6'(x) = \infty$ on G.

Example 7. Let $G, f_6$ be as in Example 6. Choose a countable dense set $C \subset R$ and put

$f_7(x) = f_6(x)$ for $x \in R-C$,

$f_7(x) = f_6(x) + 1$ for $x \in C$.

Then obviously $f_+'(x) = \infty$ , $D^-f(x) = \infty$ , $D_-f(x) = -\infty$ .

Considering the functions $f_i(x)$, $-f_i(x)$, $f_i(-x)$, $-f_i(-x)$, $i = 1,\dots,7$, we obtain the following proposition.

Proposition 2. Let $D^+$, $D_+$, $D^-$, $D_-$ be elements of $\bar{R}$ such that at least one from the following relations holds:

(i) $D^+ = D^-$, $D_+ = D_-$

(ii) $D_- = -\infty$ , $D^+ = \infty$ and $D_+ \leq D^-$

(iii) $D^- = \infty$ , $D_+ = -\infty$ and $D_- \leq D^+$.

Then there exist a function f and a residual set G such that $D^+ = D^+f(x)$, $D^- = D^-f(x)$, $D_+ = D_+f(x)$, $D_- = D_-f(x)$ for each $x \in G$.

The following example shows that Theorem 1 is the sharpest result on the magnitude of the sets $A_f$.

Example 8. Let $A \subset R$ be a $\sigma$-porous set. We shall construct a Lipschitz function f such that $D^-f(x) < D^+f(x)$ for each $x \in A$ and consequently $A \subset A_f$. By the definition of $\sigma$-porous sets there exist porous sets $A_n$ such that $A = \bigcup_{n=1}^{\infty} A_n$. It is easy to construct closed sets $F_n \supset A_n$ with the following properties:

(i) $F_n$ is porous at each point of $A_n$

(ii) If $(a,b)$ is an interval contiguous to $F_n$ then $b-a<1$ and $(b+a)/2 \notin A$.

Put $f_n(x) = n^{-2}\rho(x,F_n)$. The function $f_n$ has the following properties:

(a) $0 \leqq f_n(x) \leqq n^{-2}/2$

(b) $f_n$ is a Lipschitz function with the constant $n^{-2}$

(c) $D^-f_n(x) \leqq D_+f_n(x)$ for each $x \in A$

(d) $D^-f_n(x) < D^+f_n(x)$ for each $x \in A_n$.

In fact, (a) and (b) are obvious, (c) follows from (ii) and (d) is an easy consequence of (i). Put $f = \sum_{n=1}^{\infty} f_n$. Then $f$ is obviously a Lipschitz function. Choose an arbitrary $x \in A$. Then there exists a positive integer $k$ such that $x \in A_k$. Find an index $m$ for which

$$(4) \qquad 2\sum_{n=m+1}^{\infty} n^{-2} < D^+f_k(x) - D^-f_k(x).$$

Put $s = \sum_{n=1}^{m} f_n$ and $r = \sum_{n=m+1}^{\infty} f_n$. By (c) we easily obtain

$$(5) \qquad D^+f_k(x) - D^-f_k(x) \leqq D^+s(x) - D^-s(x).$$

From (b) follows

$$(6) \qquad D^-r(x) \leqq \sum_{n=m+1}^{\infty} n^{-2} \text{ and } D_+r(x) \geqq -\sum_{n=m+1}^{\infty} n^{-2}.$$

Since $f = s + r$, we obtain by (4),(5),(6) $D^-f(x) < D^+f(x)$.

Note. Using Theorems 12, 13 from [13] and the Jarník-Blumberg method we can obtain also some new results concerning the symmetry of approximate Dini derivates. These questions will be investigated in a subsequent article.

## R e f e r e n c e s

[1] C.L. BELNA and M.J. EVANS and P.D. HUMKE: Most directional cluster sets have common values, Fund. Math. 101(1978), 1-10.

[2] H. BLUMBERG: A theorem on arbitrary functions of two variables with applications, Fund. Math. 16(1930), 17-24.

[3] A.M. BRUCKNER: Differentiation of real functions, Lecture notes in Mathematics, No. 659, Springer Verlag, 1978.

[4] A.M. BRUCKNER and C. GOFFMAN: The boundary behaviour of real functions in the upper half plane, Rev. Roumaine Math. Pures Appl. 11(1966), 507-518.

[5] E.P. DOLŽENKO: The boundary properties of arbitrary functions, Russian, Izv. Akad. Nauk SSSR, Ser. Mat. 31(1967), 3-14.

[6] M.J. EVANS and P.D. HUMKE: Directional cluster sets and essential directional cluster sets of real functions defined in the upper half plane, Rev. Roumaine Math. Pures Appl. 23(1978), 533-542.

[7] V. JARNÍK: Sur les fonctions de la première classe de Baire, Bull. Internat. Acad. Sci. Boheme 1926.

[8] V. JARNÍK: Sur les fonctions de deux variables réelles, Fund. Math. 27(1936), 147-150.

[9] J. LUKEŠ and L. ZAJÍČEK: When finely continuous functions are of the first class of Baire, Comment. Math. Univ. Carolinae 18(1977), 647-657.

[10] F. MIGNOT: Controle dans les inéquations variationelles elliptiques, J. Functional Analysis 22(1976), 130-185.

[11] C. NEUGEBAUER: A theorem on derivatives, Acta Sci. Math. Szeged, 23(1962), 79-81.

[12] S. SAKS: Theory of the Integral, New York, 1937.

[13] L. ZAJÍČEK: On cluster sets of arbitrary functions, Fund.

Math. 83(1974), 197-217.

[14] L. ZAJÍČEK: Sets of $\sigma$-porosity and sets of $\sigma$-porosity (q), Časopis pěst. mat. 101(1976), 350-359.

Matematicko-fyzikální fakulta
Universita Karlova
Sokolovská 83, 18600 Praha 8
Československo

(Oblatum 3.10. 1980)