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THERE IS NO UNIVERSAL SEPARABLE FRÉCHET OR SEQUENTIAL COMPACT SPACE

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Abstract: A space $X \in \mathcal{K}$ is called a universal in a class $\mathcal{K}$ if every space from $\mathcal{K}$ is a continuous image of X. We prove that there is no universal space in the following classes of separable spaces: Fréchet compact, sequential compact, spaces of Mrówka, of Isbell and of Franklin. Generalizations for some uncountable cardinals are given.

Key words: Fréchet space, Isbell space, the density, almost disjoint family.

Classification: 54A25, 54C05, 54D55, 54D99

All spaces are assumed Hausdorff and mappings continuous. Our terminology follows [3].

A family $\mathcal{F}$ of spaces is called a universal family for a class $\mathcal{K}$ if for each space $Y \in \mathcal{K}$ there are a space $X \in \mathcal{F}$ and a mapping of X onto Y.

For any set X we define a family of countable infinite subsets of X to be an almost disjoint family (denote by ADF) over X iff its elements are pairwise almost disjoint, i.e. the intersection of any two of its elements is finite. We shall sometimes use the notation ADF(k) when $|X| = k$. Every ADF R over X determines the so called Mrówka space M(R) in the following way [5]: M(R) is the disjoint union $X \cup R$ topologized as below. Each $x \in X$ is declared to be isolated, and a neighbourhood base at a point $\{V\}$, $V \in R$, is formed by sets $\{V\} \cup V \setminus F$, where F is a finite subset of X. A Mrówka space M(R) is first-countable, locally countable, locally compact; it is non-compact if R is infinite. If R is a maximal ADF then M(R) is called an Isbell space and denoted by I(R), Each I(R) is pseudocompact, moreover, M(R) is pseudocompact iff R is maximal [5]. The one-point Alexandroff compactification of I(R) is called a Franklin space and denoted by F(R). It is sequential non-Fréchet compact space [4].

In [2] the following proposition was proved.

Proposition. Let X be a dense subspace of a dense in itself metrizable space Z. Let V(z) be a fixed sequence of distinct points of X which converges to z, for every $z \in Z$. Then $R = \{V(z) : z \in Z\}$ is ADF over X and the one-point compactification $\omega M(R)$ of the Mrówka space M(R) is Fréchet.

Theorem 1. Let $k^{\omega_0} = \exp k$. Then there is no universal family of the cardinality k^{ω_0} of Mrówka spaces of the density k for the class of compact Fréchet spaces of the density k.

Proof. Let Z be a dense in itself metrizable space of the cardinality k^{ω_0} with a dense subspace X of the cardinality k. Let $\mathcal{R}$ be a collection of the cardinality k^{ω_0} of ADF(k)'s. We can realize every $R \in \mathcal{R}$ as ADF over X. A compact Fréchet space of the density k which is not an image of M(R), $R \in \mathcal{R}$, will be constructed as $\omega M(P)$ for some ADF P of convergent in Z sequences of points of X.

Consider the set of all pairs (R,f), where $R \in \mathcal{R}$ and f is an arbitrary map of X into $X \cup \exp k \cup \{\omega\}$ such that $fX \supset X$. This set is of the cardinality exp k, hence, it can be indexed by ordinals $< \exp k: \{(R_\beta, f_\beta): \beta \in \exp k\}$. Applying transfinite induction we define transfinite sequences $P_1 \subset P_2 \subset \subset \ldots \subset P_\beta \subset \ldots$ and $i_\beta: P_\beta \longrightarrow \exp k$, $\beta \in \exp k$, of ADF's P_β over X and of imbeddings i_β such that for each $\beta \in \exp k$ the following conditions are fulfilled:

1) $|P_\beta| \leq \max\{k, |\beta|\}$,

2) every $W \in P_\beta$ is a convergent in Z sequence,

3) for each $z \in Z$ there is at most one $W \in P_\beta$ convergent to z,

4) $i_\gamma = i_\beta | P_\gamma$ for all $\gamma < \beta$,

5) $i_\beta(P_\beta) \supset f_\beta(X) \cap \exp k$,

6) a mapping $g_\beta: X \longrightarrow \omega M(P_\beta)$ defined by the following formula

$$g_\beta x = \begin{cases} f_\beta x & \text{if } f_\beta x \in X \cup \{\omega\}, \\ i_\beta^{-1} f_\beta x & \text{if } f_\beta x \in \exp k \end{cases}$$

has no extension $\tilde{g}_\beta$ of $M(R_\beta)$ onto $\omega M(P_\beta)$ (note that there is at most one extension $\tilde{g}_\beta$).

Suppose that for all $\gamma < \beta$ P_γ have been constructed. Put $P'_\beta = \bigcup\{P_\gamma: \gamma < \beta\}$ and define an imbedding i'_β in the natural way. It is clear that the conditions 1) - 4) for P'_β and i'_β are fulfilled. Since $|P'_\beta| < \exp k = |Z|$ we can choose $Z' \subset Z$ of the cardinality k every point of which is not the limit of any sequence $W \in P'_\beta$. Let W(z) be a sequence of points of X convergent to z and $Q = \{W(z): z \in Z'\}$. Fix $z_0 \in Z$ and define an imbedding $q: Q \longrightarrow \exp k$ such that $qQ \cap i'_\beta P'_\beta = \emptyset$, $qQ \cup i'_\beta P'_\beta \supset f_\beta X \not\ni \not\ni qW(z_0)$. Now put $P''_\beta = P'_\beta \cup Q$ and $i''_\beta | P'_\beta = i'_\beta$, $i''_\beta | Q = q$. Then the conditions 1) - 5) for P''_β and i''_β are fulfilled. Let $g_\beta: X \longrightarrow \omega M(P''_\beta)$ be defined as in 6). If g_β has no extension over $M(R_\beta)$ onto $\omega M(P''_\beta)$ we put $P_\beta = P''_\beta$, $i_\beta = i''_\beta$. Otherwise, there exists $V \in R_\beta$ such that $(f_\beta V)^* \subset W(z_0)^*$ [1]. Then we take $W'(z_0)$ such that $(f_\beta V)^* \supset (W'(z_0))^* \neq (f_\beta V)^*$. Determing $P_\beta = P''_\beta \setminus \{W(z_0)\} \cup \{W'(z_0)\}$ and i_β as i''_β changed at one point we see that the conditions 1) - 6) are fulfilled.

Put $P = \cup\{P_\beta : \beta \in \exp k\}$ and $i: P \to \exp k$ determined by i_β $(i|P_\beta = i_\beta)$. Evidently, P is ADF and i is a one-to-one map onto exp k. We shall prove that for any $R \in \mathcal{R}$ there is no mapping of M(R) onto $\omega M(P)$. Suppose the opposite: let f be a mapping of M(R) onto $\omega M(P)$ for some $R \in \mathcal{R}$. Then there exists $\beta \in \exp k$ such that $R = R_\beta$ and $f|A = f_\beta | A$, $i \circ f|(X \setminus A) = f_\beta | (X \setminus A)$, where $A = f_\beta^{-1}(X \cup \{\omega\})$. Then the composition of f and the natural projection of $\omega M(P)$ onto $\omega M(P_\beta)$ is an extension of g_β. Contradiction.

<u>Theorem 2</u>. Let k be a cardinal. If there is a cardinal m such that $m \leq k \leq m^{\omega_0} = \exp m$ then there is no universal family of the cardinality k^{ω_0} of Isbell spaces of the density $\leq k$ for the class of Fréchet compact spaces of the density k.

Proof. We have $m^{\omega_0} \leq k^{\omega_0} \leq k^m \leq (\exp m)^m = \exp m = m^{\omega_0}$.

Let $\mathcal{R}$ be a collection of the cardinality k^{ω_0} of maximal ADF(k)'s over a set X. For every subset $Y \subset X$ of the cardinality m the closure $cl_{I(R)} Y$ is an Isbell space, the set of all

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1) The set X is of a dual character: being a subspace of Z it is dense in itself metrizable space and it is a discrete space considered as a subspace of Mrówka spaces. Here we consider the Stone-Čech compactification βX of X and use the standard notation
$A^* = cl_{\beta X} A \setminus A$ whenever $A \subset X$. Then M(R) can be expressed as the quotient space of $X \cup \cup R^*$ by collapsing each element of $R^* = \{V^* : V \in R\}$ to a point.

isolated points of which coincides with Y. Let us denote by $\mathcal{F}$ the collection of all such subspaces of all I(R), $R \in \mathcal{R}$. Then $|\mathcal{F}| = k^{\omega_0} k^m = \exp m$, because each I(R) has k^m such subspaces. From the proof of Theorem 1 it follows that there is an ADF(m) P_0 over a set Z disjoint with X such that $\omega M(P_0)$ is Fréchet and it is not an image of any space of $\mathcal{F}$. Let P_1 be an ADF(k) over X for which $\omega M(P_1)$ is Fréchet. Consider an ADF(k) $P = P_0 \cup P_1$ over $Z \cup X$. Then $\omega M(P)$ is Fréchet, too, and $cl_{\omega M(P)} Z = \omega M(P_0)$. We shall prove that it is impossible to map I(R), $R \in \mathcal{R}$, onto $\omega M(P)$. Indeed, let f be a mapping of I(R) onto $\omega M(P)$. The preimage of each isolated point is a clopen set, hence, it has an isolated point. Therefore, we can choose a set $A \subset I(R)$ of the cardinality m of isolated points the image of which is equal to Z. But $f(cl_{I(R)} A)$ is a subspace pseudocompact and dense in $\omega M(P_0)$, hence, $f(cl_{I(R)} A) = \omega M(P_0)$. Contradiction.

The following theorem is a generalization of an analogous result concerning continuous images of separable Isbell spaces [1]. The proof is the same.

Theorem 3. A space X is an image of an Isbell space of the density k if and only if X has a sequentially dense and sequentially compact in X subset of the cardinality $\leq k$, i.e. X has a subset Z with the following properties:

0) $|Z| \leq k$,

1) every sequence of points of Z has a convergent subsequence,

2) every point of X is the limit of a sequence of points of Z.

Evidently, every Fréchet compact space of the density $\leq k$ has such properties. Hence, we obtain the following theorem.

Theorem 4. Let $m \leq k \leq m^{\omega_0} = \exp m$ for some m. There is no universal family of the cardinality k^{ω_0} in the following subclasses of the class of all spaces of the density $\leq k$:

(i) of Isbell spaces,

(ii) of Fréchet compact spaces.

Theorem 5. Let $k^{\omega_0} = \exp k$. Then there is no universal family of the cardinality exp k of Isbell spaces of the density k for the class of Franklin spaces of the density k.

Proof. Let $\mathcal{R}$ be a family of the cardinality $\leq \exp k$ of maximal ADF(k)'s such that $\{I(R): R \in \mathcal{R}\}$ is a universal family for the class of all Franklin spaces of the density k. Let f be a mapping of I(R) onto F(P). We can choose a subset A of isolated points of I(R) such that f(A) is the set of all isolated points of F(P) and $f|A$ is an injection. Then $cl_{I(R)}A$ is an Isbell space and $f(cl_{I(R)}A) = I(P)$. Hence, the family of all such subspaces $\{cl_{I(R)}A: R \in \mathcal{R}\}$ is of the cardinality $\exp k = k^{\omega_0}$ and it is universal in the class of Isbell spaces. Contradiction with Theorem 4.

Theorem 6. Let $\exp k = k^{\omega_0}$. Then there is no universal family of the cardinality exp k in the following subclasses of the class of all spaces of the density $\leq k$:

(i) of Mrówka spaces,

(ii) of Franklin spaces,

(iii) of sequential compact spaces.

Proof. (i) it follows from Theorems 1 and 3.

(ii) Note that every Isbell space is homeomorphic to the discrete union of itself and a one-point space. Hence, I(R) can be mapped onto F(R) (if the adjoint point is mapped onto the point at infinity). Therefore, (ii) follows from Theorem 5.

(iii) Let Z be a sequential compact space with an everywhere dense subspace X of the cardinality k. Then we can construct (as in Theorem 3) a mapping of some Isbell space of the density k onto the set of limit points of sequences in X. Suppose that there is an universal family of the cardinality exp k in the class of all sequential compact spaces of the density k. For each space of this family choose I(R) as above. Let us recall that Franklin spaces are sequential and compact. Hence, we have a family of the cardinality exp k of Isbell spaces of the density $\leq$k which can be mapped onto everywhere dense subspaces of each Franklin space of the density k. But any such image is pseudocompact, therefore it is either the whole Franklin space, or the corresponding Isbell space. In (ii) we have noted that I(R) can be mapped onto F(R). Hence, this family of Isbell spaces is universal for the class of all Franklin spaces of the density $\leq$ k. This contradicts Theorem 5.

Remark. Since $\omega_0^{\omega_0} = \exp \omega_0$, Theorems 1, 5 and 6 for a countable case and Theorems 2 and 4 for cardinals not greater than c are valid without additional set-theoretical assumptions.

Question. Are these theorems valid for all cardinals?

Since the cardinality of every sequential space of the density k is not greater than k^{ω_0}, we see that for cardinals $k = k^{\omega_0}$ there exists a universal Mrówka space of the density k for the class of all sequential spaces of the density k. It is clear that any Mrówka space which has clopen discrete subspace of the cardinality k is a universal space for the clas of all spaces of the cardinality $\leq k$. Hence, under GCH Theorem 1 and Theorem 6 (i) are true only for cardinals k such that $\exp k = k^{\omega_0}$.

References

[1] A.I. BASHKIROV: On continuous maps of Isbell spaces and strong 0-dimensionality, Bull. Pol. Acad. Sci. 27,7(1979), 605-611.

[2] A.I. BASHKIROV: On Fréchet compactifications of discrete spaces, ibid. (to appear).

[3] R. ENGELKING: General Topology, Warszawa, 1977.

[4] S.P. FRANKLIN: Spaces in which sequences suffice, II, Fund. Math. 61(1967), 51-56.

[5] S. MRÓWKA: On completely regular spaces, ibid. 41(1954), 105-106.

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