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ON SIMULTANEOUS INTEGRABILITY OF TWO COMMUTING ALMOST TANGENT STRUCTURES

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Abstract: Let M be a differentiable manifold of dimension 4p provided with two almost tangent structures f and g, regular in the sense that Ker f = Im f, Ker g = Im g, such that fg = gf and dim Ker f $\cap$ Ker g = p. We shall associate with the couple f, g in a natural manner a G-structure on M and give necessary and sufficient conditions for its integrability, i.e. simultaneous integrability of f and g. Two examples of the studied structure will also be given.

Key words: G-structure, distribution, Nijenhuis tensor.

Classification: 53C10

Introduction. Among different types of G-structures, the class of G-structures induced by a system of tensor fields of type (1,1) is very important.

Naturally, G-structures induced by a single tensor field were intensively studied and most important problems of this kind have been successfully generally solved (see [2]).

However, when we consider G-structures induced by a couple of tensor fields, a great variety of complications at once arises. Probably it is not possible to give a general method for a solution of the problem of integrability of such a G-structure. Until now, only several very special cases were solved. Houh and Hsu [3],[5],[6] studied different cases of a couple (h,k), where $h^2 = \pm I$, $k^2 = \pm I$, $hk = \pm kh$. Hashimoto [7]studied a couple (F,G), where $F^3 + F = 0$, $G^3 + G = 0$, $FG = -GF$, $F^2 = G^2$. Hatakeyama [4] studied, more generally, a couple of semisimple O-deformable tensor fields.

Relatively recently, Czech authors Bureš and Vanžura [10] studied simultaneous integrability of a couple (J,T), where $J^2 = -I$ (almost complex structure), $T^2 = 0$ (almost tangent structure), Ker T = Im T, $TJ = \pm JT$ or $TJ + JT = kI$ (k const.). Simultaneously, the author [8] studied a couple of J-related almost tangent structures f and g such that Ker f = Im f, Ker g = Im g, $fg = gf = 0$.

It can be mentioned that whenever at least one of two tensor fields is nilpotent, and especially in a case when tensor fields do not commute, usual assumptions about vanishing of Nijenhuis tensors are generally not sufficient. One has to look for further assumptions corresponding to various geometrical structures induced by the couple of the studied tensor fields.

Let us note that some of the problems mentioned above can be studied from the point of view of manifolds modelled over A-modules (where A is commutative associative algebra over real numbers), which is an approach of Soviet authors Širokov and Kručkovič.

0. All differentiable structures considered in this paper are supposed to be of class C^∞.

Let M be a differentiable manifold of dimension 4p, endowed with a couple f, g of almost tangent structures, i.e. tensor fields of type (1,1) such that $f^2 = 0$ and $g^2 = 0$. Let us suppose that:

(i) $(K_f)_u = (I_f)_u$, $(K_g)_u = (I_g)_u$ (regularity property), for every $u \in M$,

(ii) $fg = gf$,

(iii) $\dim (K_f)_u \cap (K_g)_u = p$ for every $u \in M$

($K_f = \operatorname{Ker} f$, $I_f = \operatorname{Im} f$ etc.).

1. We have $\dim M = 4p$, $\dim K_f = \dim K_g = 2p$, $\dim K_f \cap \cap K_g = p$. Therefore it will be convenient to divide every linear frame on M into four p-tuples, e.g. $(X_1,\dots,X_p, Y_1,\dots \dots,Y_p, Z_1,\dots,Z_p, U_1,\dots,U_p)$, and then to use abbreviated notation (X,Y,Z,U). By the equality $fX = Z$ we shall mean $fX_i = = Z_i$, $i = 1,\dots,p$, etc.

We shall call (f,g)-adapted frame every linear frame (X,Y,Z,U) on M such that

$$(1) \qquad \begin{aligned} fX &= Z, \quad fZ = 0, \\ fY &= U, \quad fU = 0 \end{aligned}$$

and

$$(2) \qquad \begin{aligned} gX &= Y, \quad gY = 0, \\ gZ &= U, \quad gU = 0 \end{aligned}$$

(0 denotes here a p-tuple consisted of p zero tangent vectors).

<u>Proposition</u>. Let G be a set of all regular matrices A of type $4p \times 4p$, such that $A\tilde{I} = \tilde{I}A$, $AH = HA$, where

$$(3) \qquad \tilde{I} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$(4) \qquad H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix}$$

(0 and I means the zero and unit matrix of type $p \times p$, respectively). Then G is a Lie group and the set B of all (f,g)-

adapted frames on M is a G-structure.

Proof. Apparently G is a closed subgroup of $GL(4p,R)$ and therefore it is a Lie group. One can easily verify that G operates on B in the needed manner.

Let us show in several steps that in a neighbourhood of any point $u \in M$ there exists a local section of B, i.e. local differentiable "moving" frame (X,Y,Z,U), satisfying (1) and (2). We shall leave to the reader to verify that the following construction is correct:

a) Let us choose local vector fields $Z_1,\dots,Z_p$ in such a way that $Z_1,\dots,Z_p$ is a local basis of K_f over $K_f \cap K_g$.

b) Let us put $U_i = gZ_i$, $i = 1,\dots,p$. Clearly, vector fields $U_1,\dots,U_p$ are linearly independent and $U_i \in K_f \cap K_g$, $i = 1,\dots,p$.

c) Let us choose vector fields $X_1,\dots,X_p$ in such a way that $fX_i = Z_i$, $i = 1,\dots,p$. It is not difficult to show that $X_1,\dots,X_p$ are linearly independent over $K_f + K_g$.

d) Let us put $Y_i = gX_i$, $i = 1,\dots,p$. It is easy to show that $Y_1,\dots,Y_p$ are linearly independent over K_f and $Y_i \in K_g$, $i = 1,\dots,p$.

Apparently (X,Y,Z,U) is a local differentiable (f,g)-adapted frame.

QED.

We shall say that f and g are simultaneously integrable if the above defined G-structure is integrable.

2. In a theorem about simultaneous integrability of f and g, which will be formulated in the following section, we shall need the theorem about simultaneous integrability of an almost

tangent structure and a distribution. This theorem was proved by J. Vanžura in [1], and we shall recall here a simplier version of it.

Let N be a differentiable manifold of dimension 2n provided with a regular almost tangent structure F. Let D be a distribution of dimension d on N, invariant with respect to F (i.e. $F(D) \subset D$). Let us suppose that D is regularly situated with respect to F, i.e. that $K_F \cap D$ has constant dimension, say b, at every point of N.

We shall say that F and D are simultaneously integrable, if for every $u \in M$ there exists a local coordinate neighbourhood U of u with local coordinates $(u_1, \dots, u_{2n})$ on it such that

$$(5) \quad F \frac{\partial}{\partial u_i} = \frac{\partial}{\partial u_{i+n}}, \; F \frac{\partial}{\partial u_{i+n}} = 0, \; i = 1, \dots, n, \text{ and}$$

$$(6) \quad \frac{\partial}{\partial u_{n-d+b+1}}, \dots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial u_{2n-b+1}}, \dots, \frac{\partial}{\partial u_{2n}} \text{ is}$$

a local basis of D.

Theorem A. (Vanžura) F and D are simultaneously integrable if and only if the following conditions are satisfied:

1. F is integrable,
2. D is involutive,
3. distributions $K_F + D$ and $F^{-1}D$ are involutive,
4. for every vector field $X \in D$ the Lie derivative $L_X F$ maps the tangent bundle TN into D.

Let us recall that a regular almost tangent structure F is said to be integrable if for every $u \in N$ there exists a local coordinate neighbourhood with local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ such that

$F\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$, $F\frac{\partial}{\partial y_i} = 0$, $i = 1,\dots,n$.

Let us present here the following already known result.

Theorem B. F is integrable if and only if $\{F,F\} = 0$.

Remark. $\{\ ,\ \}$ is a Nijenhuis bracket defined for commutating tensor fields h, k of type $(1,1)$ by the formula

$$\{h,k\}(X,Y) = [hX,kY] + hk[X,Y] - h[X,kY] - k[hX,Y],$$

where X, Y are vector fields.

3. Let us formulate now the main result of this paper.

Theorem. Almost tangent structures f and g on a differentiable manifold M of dimension $4p$, satisfying (i) - (iii), are simultaneously integrable if and only if

$$\{f,f\} = 0,\ \{f,g\} = 0,\ \{g,g\} = 0.$$

Proof. We shall prove that the above condition is sufficient. At first we shall show that the almost tangent structure f and the distribution K_g are simultaneously integrable. Let us verify the conditions of the theorem of Vanžura.

Clearly $fK_g \subset K_g$. Integrability of f follows from $\{f,f\} = 0$. One can easily deduce from $\{f,f\} = 0$ and $\{g,g\} = 0$ that K_f and K_g are involutive distributions.

Let (X,Y,Z,U) be a local (f,g)-adapted linear frame. Apparently $K_f = \mathrm{Span}\{Z,U\}$, $K_g = \mathrm{Span}\{Y,U\}$, $K_f + K_g = \mathrm{Span}\{Y,Z,U\}$, $f^{-1}K_g = K_f + K_g$. We shall show that for $i,j = 1,\dots,p$, vector fields $[Z_i,Y_j]$ are from $K_f + K_g$. It is possible to write (at least locally) $Z_i = fX_i$, $Y_j = gX_j$. Then

$$0 = \{f,g\}(X_i,X_j) = [Z_i,Y_j] + fg[X_i,X_j] - f[X_i,Y_j] -$$

$- g[Z_i, X_j]$ and the assertion is obvious. Now it is easy to see that $K_f + K_g$ is involutive.

The last condition of Vanžura's theorem follows from $\{f,g\} = 0$. Namely, if $X \in K_g$ and Y is any vector field, we have $0 = \{f,g\}(Y,X) = fg[Y,X] - g[fY,X]$ and so

$g((L_X f)Y) = g(L_X(fY) - fL_X Y) = g[X,fY] - gf[X,Y] = 0.$

Now we are ready to use the theorem of Vanžura. In a neighbourhood of any point $u \in M$ there exist local coordinates (x,y,z,u) such that

1. $fX = Z$, $fY = U$, $fZ = 0$, $fU = 0$ and
2. Y, U is a local basis of K_g,

where now

$$X_i = \frac{\partial}{\partial x_i}\ ,\ Y_i = \frac{\partial}{\partial y_i}\ \text{ and so on, } i = 1,\dots,p.$$

Apparently $gX_i \in K_g$ for $i = 1,\dots,p$. Therefore we can write $gX_i = a_i^j Y_j + c_i^j U_j$, $i,j = 1,\dots,p$, where $a = a(x,y,z,u)$ and $c = c(x,y,z,u)$ are some matrix-functions of type $p \times p$.

We have $gZ_i = g(fX_i) = f(gX_i) = f(a_i^j Y_j + c_i^j U_j) = a_i^j U_j$. Therefore the matrix of g with respect to the basis X,Y,Z,U is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & a & 0 \end{pmatrix}.$$

From the regularity of g it follows that the rank of this matrix has to be $2p$. One can easily conclude that the matrix a has to be regular. Let us prove the following lemma.

<u>Lemma 1</u>. Matrix functions $a = a(x,y,z,u)$ and $c = c(x,y,z,u)$ satisfy the following conditions :

1. a does not depend on y, z and u,

2. c does not depend on u,

3. $\frac{\partial a_k^l}{\partial x_i} = \frac{\partial c_k^l}{\partial z_i}$,

4. $a_j^i \frac{\partial c_k^l}{\partial y_i} = a_k^i \frac{\partial c_j^l}{\partial y_i}$,

$i,j,k,l = 1,\dots,p$.

In order to prove the lemma, we apply the condition $\{f,g\} = 0$ on couples (X_i,X_j), (Y_u,X_j), (U_i,U_j) and $\{g,g\} = 0$ on (X_i,X_j) and (X_i,Z_j). After the straightforward computation we get conditions 1. - 4.

Let us introduce new local coordinates by the formulas

$$x_i' = x_i$$
$$y_i' = \mathring{a}_j^i(x)\, y_j$$
$$z_i' = z_i$$
$$u_i' = \mathring{a}_j^i(x)\, u_j + \frac{\partial \mathring{a}_j^i}{\partial x_k} y_j\, z_k, \text{ where } \mathring{a} = a^{-1}.$$

It can be easily verified that $gX_i' = Y_i' + \gamma_i^j U_i'$, $i,j = 1,\dots,p$, where $\gamma = \gamma(x',y',z',u')$ is a certain matrix function of type $p\times p$, $gZ' = U'$, $gY' = 0$, $gU' = 0$ (again $X_i' = \frac{\partial}{\partial x_i'}$, etc.). Let us denote the new coordinates again (x,y,z,u) and the corresponding linear frame (X,Y,Z,U).

Lemma 2. The matrix-function $\gamma = \gamma(x,y,z,u)$ satisfies the following conditions:

1. γ does not depend on z and u

2. $\frac{\partial \gamma_j^i}{\partial y_k} = \frac{\partial \gamma_k^i}{\partial y_j}$, $i,j,k = 1,\dots,p$.

The proof of the lemma 2 is the same as the proof of the lemma 1.

The condition 2. of the lemma 2 implies that there exist functions $l_i\ (x,y)$, $i = 1,\dots,p$, such that

$$\frac{\partial l_i}{\partial y_j} = - \gamma^i_j,\ j = 1,\dots,p.$$

Let us put

$$x'_i = x_i$$
$$y'_i = y_i$$
$$z'_i = z_i$$
$$u'_i = l_i(x,y) + u_i.$$

An easy computation shows that new coordinates are (f,g)-adapted, i.e. that the frame ($\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}, \frac{\partial}{\partial u'}$) is (f,g)-adapted.

As usual, we are not going to prove the obvious fact that vanishing of Nijenhuis brackets $'$ is necessary for simultaneous integrability of f and g.

QED.

4. Let us present here two examples of the above studied structure for $p = 1$. In both cases we shall take $M = \{(x,y,z,u) \in \in R^4 \mid x > 0,\ y > 0,\ z > 0,\ u > 0\}$ and in both cases we are going to choose linearly independent vector fields X, Y, Z, U and define f and g by the equations

$$fX = Z,\ fY = U,\ fZ = 0,\ fU = 0,$$
$$gX = Y,\ gY = 0,\ gZ = U,\ gU = 0.$$

<u>Example 1</u>. We shall put

$$X = z\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + u\frac{\partial}{\partial u},$$
$$Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + z\frac{\partial}{\partial u},$$

$$Z = x\frac{\partial}{\partial z} + y\frac{\partial}{\partial u},$$

$$U = x\frac{\partial}{\partial u}.$$

Vector fields X, Y, Z, U are chosen in such a way that

$$[Y,Z] = 0,\ [Y,U] = 0,\ [Z,U] = 0.$$

It can be easily computed that in this case

$$(9)\quad \{f,f\} = 0 \iff f\,[X,U] = 0,$$

$$(10)\quad \{g,g\} = 0 \iff g\,[X,U] = 0 \text{ and}$$

$$(11)\quad \{f,g\} = 0 \iff \begin{cases} f\,[X,U] = 0 \\ g\,[X,U] = 0 \\ f\,[X,Y] = g\,[X,Z]. \end{cases}$$

We have $[X,U] = (z - x)\frac{\partial}{\partial u}$, therefore (9) and (10) hold and both f and g are integrable almost tangent structures.

Let us notice that the condition $f\,[X,Y] = g\,[X,Z]$ implies $fg\,[X,Z] = 0$. It is $[X,Z] = -x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}$. From the formulas (8) it is possible to compute that

$$[X,Z] = -\frac{x}{z}X + W(x,y,z,u),$$

where $W(x,y,z,u)$ is a certain vector-valued function on M with values in Span$\{Y,Z,U\}$. Therefore $fg\,[X,Z] = -\frac{x}{z}U \neq 0$.
We can conclude that f and g are not simultaneously integrable.

<u>Example 2</u>. Let us put

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + (x + z)\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}$$

and let Y, Z, U be as in Example 1. In this case $[X,Y] = x\frac{\partial}{\partial u}$ and the remaining Lie brackets are equal to 0. The reader can easily verify that f and g are simultaneously integrable, because all the three above mentioned Nijenhuis brackets vanish.

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