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SOME FIXED POINT THEOREMS IN LOCALLY CONVEX SPACES AND APPLICATIONS TO DIFFERENTIAL AND INTEGRAL EQUATIONS

Bogdan RZEPECKI

Abstract: Krasnoselskii [8] has given the following theorem: Let E be a Banach space, K a non-empty bounded closed convex subset of E, and A, B operators on K into E such that $Ax + By \in K$ for all x, y in K. If A is a contraction and B is completely continuous, then the equation $Ax + Bx = x$ has a solution in K. We present some modification and generalizations of this result for locally convex space, and give their applications to the theory of differential and integral equations. Our modification in question is connected with the well-known method of norm changing in the theory of differential equations.

Key words: Fixed point theorems in locally convex spaces, applications to differential-like equations, Bielecki method of norm changing, $\mathcal{L}^*$-spaces.

Classification: 4685, 3404, 3495, 4530

Introduction. Let E be a Banach space, K a non-empty bounded closed convex subset of E, and A, B operators on K into E such that $Ax + By \in K$ for all x, y in K. Krasnoselskii [8] proved that if A is a contraction and B is completely continuous, then the equation $Ax + Bx = x$ has a solution in K.

In this paper we establish some modifications and generalizations of this result for locally convex spaces, and give their applications to the theory of differential-like equations. The modification in question is connected with Bielecki's method ([1],[16, p. 34]) of norm changing in the theory of differential equations. Our fixed point theorems extend those from [8],[11],[12],[4] and [17]. For other generalizations of [8] see [14],[15],[18] and [19].

In the sequel we shall use the notations of $\mathcal{L}^*$-space, the $\mathcal{L}^*$-product of $\mathcal{L}^*$-spaces and a continuous mapping of one $\mathcal{L}^*$-space into another (see e.g. [10]). Finally, note that the following extension of Tychonoff's fixed point theorem is due to Singbal [2] (see also [6]) and is used in the proof of Theorem 2 and Theorem 3:

Let E be a Hausdorff locally convex topological vector space, let K be a closed and convex subset of E and let f be a continuous mapping of K into itself such that f[K] is contained in a compact set. Then f has a fixed point in K.

Part I: Results. Throughout this part, E will denote a Hausdorff locally convex topological vector space with a saturated family P of seminorms which generates the topology of E ([13],[3]).

First, assume that X is a sequentially complete set in E. Let Γ be an index set. Suppose that $(f_\gamma : \gamma \in \Gamma)$ is a net of mappings of X into itself such that there exists $\lim_{\gamma\in\Gamma} f_\gamma x$ for every $x \in X$ and $p(f_\gamma x - f_\gamma y) \leq k_p \cdot p(x - y)$ for all $p \in P$, $\gamma \in \Gamma$ and x, y in X, where k_p is a constant (depending of a seminorm p) with $0 \leq k_p < 1$. Moreover, let us put $f_0 x = \lim_{\gamma\in\Gamma} f_\gamma x$ for x in X.

Since X is a sequentially complete space, so by Cain and Nashed theorem [4, Th. 2.2] (cf. [12]) we obtain that f_γ $(\gamma \in \Gamma)$ and f_0 has a unique fixed point x_γ and x_0, respectively. Further, if $y_0 = x_0$, $y_n = f_0 y_{n-1}$ and $y_n^{(\gamma)} =$

$= f_\gamma y_{n-1}^{(\gamma)}$ for $n \geq 1$ and $\gamma \in \Gamma$, then

$$p(x_\gamma - y_n^{(\gamma)}) \leq (1 - k_p)^{-1} k_p^n \cdot p(y_1^{(\gamma)} - x_0)$$

for $p \in P$, $\gamma \in \Gamma$ and $n \geq 1$. It can easily be seen that $\lim_{\gamma \in \Gamma} x_\gamma = x_0$.

Indeed, let $p_1, p_2, \ldots, p_l \in P$, $k = \max_{1 \leq i \leq l} k_{p_i}$ and $\varepsilon > 0$. Let $\mathcal{V}$ and $\mathcal{V}'$ be the sets of all x in X such that $p_i(x) < \varepsilon$ and $p_i(x) < \varepsilon(1 - k)$ for $1 \leq i \leq l$, respectively. Since $\lim_{\gamma \in \Gamma} f_\gamma x_0 = f_0 x_0$, there is an index $\gamma_0 \in \Gamma$ with $f_\gamma x_0 - f_0 x_0 \in \mathcal{V}'$ for all $\gamma \succcurlyeq \gamma_0$. Therefore, for $1 \leq i \leq l$ and $\gamma \succcurlyeq \gamma_0$ we have

$$\begin{aligned} p_i(x_\gamma - x_0) &\leq p_i(x_\gamma - f_\gamma x_0) + p_i(f_\gamma x_0 - x_0) \leq \\ &\leq (1 - k_{p_i})^{-1} \cdot k_{p_i} \cdot p_i(f_\gamma x_0 - f_0 x_0) + p_i(f_\gamma x_0 - f_0 x_0) = \\ &= (1 - k_{p_i})^{-1} \cdot p_i(f_\gamma x_0 - f_0 x_0) < (1 - k_{p_i})^{-1} \cdot \varepsilon(1 - k) \leq \varepsilon . \end{aligned}$$

This implies $x_\gamma - x_0 \in \mathcal{V}$ for all $\gamma \succcurlyeq \gamma_0$ and completes the proof.

Now, we give the following theorem (cf. [17]) of the type of Banach contraction principle:

<u>Theorem 1</u>. Let A be an arbitrary set, let T be a transformation from A into E such that $T[A]$ is a sequentially complete set, and let $(g_\gamma : \gamma \in \Gamma)$ be a net of transformations defined on A with the values in E and $g_\gamma[A] \subset T[A]$ for all $\gamma \in \Gamma$. Assume that there exists $\lim_{\gamma \in \Gamma} g_\gamma x$ for each $x \in A$ and $p(g_\gamma x - g_\gamma y) \leq k_p \cdot p(Tx - Ty)$ for all $p \in P$ and x, y in A, where k_p is a constant (depending of a seminorm p) such that $0 \leq k_p < 1$. Further, let $g_0 x = \lim_{\gamma \in \Gamma} g_\gamma x$ for all $x \in A$ and let μ denote the index such that $\mu = 0$ or $\mu \in \Gamma$.

Then, for every y in $T[A]$ the set $g_{\mu}[T_{-1}y]$ ($T_{-1}y$ denotes the inverse image of y under T) contains only one element and the mapping f_{μ} defined by $f_{\mu}y = g_{\mu}[T_{-1}y]$ has a unique fixed point y_{μ} in $T[A]$ (given as a limit of a sequence of successive approximations) with the following properties:

(i) $g_{\mu}x = Tx$ for every x in $T_{-1}y_{\mu}$;

(ii) if $g_{\mu}x^{(i)} = Tx^{(i)}$ for $i = 1,2$, then $Tx^{(1)} = Tx^{(2)}$;

(iii) $\lim_{\gamma \in \Gamma} Tx_{\gamma} = Tx_0$ for every x_{μ} in $T_{-1}y_{\mu}$.

Proof. Fix y in $T[A]$. Suppose that $v_i = g_{\mu}x_i$ for $i = 1,2$, where $Tx_i = y$. We have $p(v_1 - v_2) \leq k_p \cdot p(Tx_1 - Tx_2) = 0$ for every p in P. Since P is a saturated family of seminorms on E, for $v_1 \neq v_2$ there exists p' in P with $p'(v_1 - v_2) > 0$. Consequently, $v_1 = v_2$ and $g_{\mu}[T_{-1}y]$ contains only one element.

Now, applying the above remarks to the mappings f_{μ} of $T[A]$ into itself, we can conclude the proof of the first part of our theorem. Let $f_{\mu}y_{\mu} = y_{\mu}$ and let $\lim_{\gamma \in \Gamma} y_{\gamma} = y_0$. If x_{μ} is such that $Tx_{\mu} = y_{\mu}$, then $Tx_{\mu} = g_{\mu}x_{\mu}$ and $\lim_{\gamma \in \Gamma} Tx_{\gamma} = \lim_{\gamma \in \Gamma} y_{\gamma} = y_0 = Tx_0$. Finally, if $g_{\mu}x = Tx$ for some $x \in A$ then $f_{\mu}(Tx) = Tx$. It means the points $Tx^{(1)}$, $Tx^{(2)}$ from (ii) are fixed points of the transformation f_{μ} and the unicity of fixed points implies (ii). This completes the proof.

Theorem 2. Suppose we are given: X - a subset of E; K - a convex closed subset of E; T - a mapping from X to E such that $T[X]$ is sequentially complete and $T[X] \subset K$; Q - a continuous mapping from K into a compact subset of E. Assume, moreover, that F is a mapping from $X \times K$ into E satisfying the following conditions:

(i) $F[X \times K] \subset T[X]$;

(ii) for each p in P, there is a constant k_p, $0 \leq k_p < 1$, such that $p(F(x_1,y) - F(x_2,y)) \leq k_p\, p(Tx_1 - Tx_2)$ for all x_1, x_2 in X and $y \in K$;

(iii) for each p in P, there is a constant $C_p > 0$ such that $p(F(x,y_1) - F(x,y_2)) \leq C_p \cdot p(Qy_1 - Qy_2)$ for all $x \in X$ and y_1, y_2 in K.

Then there exists a point x in X such that $F(x,Tx) = Tx$.

Proof. Let us fix y in K. Theorem 1 implies there exists a point u_y in X such that $F(u_y,y) = Tu_y$. Now, we define an operator f as $y \longmapsto Tu_y$. Then f maps K into itself.

First, we prove that f is continuous: Let $(x_\alpha : \alpha \in \Delta)$ be a convergent net in K and $\lim_{\alpha \in \Delta} x_\alpha = x_0$. Further, let us put $g_\alpha x = F(x,x_\alpha)$ and $g_0 x = F(x,x_0)$ for x in X. Then, $g_\alpha[X] \subset$ $\subset T[X]$ and $p(g_\alpha x_1 - g_\alpha x_2) \leq k_p \cdot p(Tx_1 - Tx_2)$ for $p \in P$ and x_1, x_2 in X. Now, let $\varepsilon > 0$ and let us fix $p_1, p_2, \dots, p_k$ in P. Since $V = \{x \in E : p_i(x) < \varepsilon \ \text{ for } 1 \leq i \leq k\}$ is a neighbourhood of the origin in E, there is an $\alpha_0 \in \Delta$ such that $p_i(Qx_\alpha - Qx_0) < \varepsilon$ $(1 \leq i \leq k)$ for all $\alpha \geqslant \alpha_0$. From this and (iii) it follows that $\lim_{\alpha \in \Delta} g_\alpha x = \lim_{\alpha \in \Delta} F(x,x_\alpha) = F(x,x_0) = g_0 x$ for every x in X. Therefore, by Theorem 1, $\lim_{\alpha \in \Delta} fx_\alpha = \lim_{\alpha \in \Delta} Tu_{x_\alpha} = Tu_{x_0} = fx_0$ and the mapping f is continuous on K.

We prove that $f[K]$ is conditionally compact in E: Let $(x_\alpha : \alpha \in \Delta)$ be a net in K. We have

$$p(F(u_{x_\alpha}, x_\alpha) - F(u_{x_\beta}, x_\beta)) \leq k_p \cdot p(Tu_{x_\alpha} - Tu_{x_\beta}) + \\ + C_p \cdot p(Qx_\alpha - Qx_\beta),$$

hence

$$p(F(u_{x_\alpha}, x_\alpha) - F(u_{x_\beta}, x_\beta)) \leq (1 - k_p)^{-1} \cdot C_p \cdot p(Qx_\alpha - Qx_\beta)$$

for all $p \in P$ and α, β in Δ. Further, let $\varepsilon > 0$, let us fix $p_1, p_2, \dots, p_k$ in P, and let us put

$$V = \{x \in E : p_i(x) < C^{-1} \cdot \varepsilon(1 - L) \text{ for } 1 \leq i \leq k\}$$

with $C = \max_{1 \leq i \leq k} C_{p_i}$ and $L = \max_{1 \leq i \leq k} k_{p_i}$. Since $(Qx_\alpha : \alpha \in \Delta)$ has a convergent subnet $(Qx_\delta : \delta \in \Delta_1)$, so there exists $\delta_0 \in \Delta_1$ such that $Qx_\alpha - Qx_\beta \in V$ for all α, β in Δ_1 with $\alpha, \beta \succcurlyeq \delta_0$. From the above

$$p_i(F(u_{x_\alpha}, x_\alpha) - F(u_{x_\beta}, x_\beta)) \leq (1 - k_{p_i})^{-1} \cdot C_{p_i} \cdot p_i(Qx_\alpha - Qx_\beta) \leq (1 - L)^{-1} \cdot C \cdot p_i(Qx_\alpha - Qx_\beta) < \varepsilon$$

for $1 \leq i \leq k$ and α, β in Δ_1 with $\alpha, \beta \succcurlyeq \delta_0$. Consequently, $(F(u_{x_\delta}, x_\delta) : \delta \in \Delta_1)$ is a Cauchy net and therefore $(F(u_{x_\delta}, x_\delta) : \delta \in \Delta_1)$ is a convergent subnet of the net $(F(u_{x_\alpha}, x_\alpha) : \alpha \in \Delta)$.

Finally, by Singbal result (given in Introduction), there exists x in K such that $fx = x$. Hence $Tu_x = F(u_x, x) = F(u_x, fx) = F(u_x, Tu_x)$ and we are done.

Theorem 3. Suppose that we are given: X - a subset of E; K - a convex closed subset of E; T - a mapping from X to E such that T[X] is sequentially complete and $T[X] \subset K$; Q - a continuous mapping from K into a compact subset of E. Assume, moreover, that F is a mapping from $X \times \overline{Q[K]}$ into E satisfying the following conditions:

(i) $F(x,y) \in T[X]$ for every $(x,y) \in X \times \overline{Q[K]}$;

(ii) for each p in P, there is a constant k_p, $0 \leq k_p < 1$, such that $p(F(x_1,y) - F(x_2,y)) \leq k_p \cdot p(Tx_1 - Tx_2)$ for all x_1, x_2 in X and y in $\overline{Q[K]}$;

(iii) for every x in X the mapping $y \longmapsto F(x,y)$ is continuous on $\overline{Q[K]}$.

Then there exists a point x in X such that $F(x,Q(Tx)) = Tx$.

Proof. Essentially the same proof as that of Theorem 2 yields that there exists a mapping $y \longmapsto Tu_y$ from $\overline{Q[K]}$ to K which is continuous and $F(u_y,y) = Tu_y$ for every y in $\overline{Q[K]}$. Now, the operator f defined on K by $fx = Tu_{Qx}$ is continuous, maps K into K and $f[K]$ is a conditionally compact set. Therefore, by the Singbal fixed point theorem, f has a fixed point z in K, and $Tu_{Qz} = F(u_{Qz},Qz) = F(u_{Qz},Q(fz)) = F(u_{Qz},Q(Tu_{Qz}))$. This completes the proof.

Part II: Applications. Throughout this part $J = [0,\infty)$, $I = [0,a]$ and $I_h = [0,h]$ with $0 < h \leq a$. Moreover, we shall denote by $\mathbb{R}^k$ the k-dimensional Euclidean space, and by $C_0(\Omega,E)$ the Banach space of all bounded continuous functions from a subset Ω of J to a Banach space E. In particular, let us put $\mathbb{R} = \mathbb{R}^1$ and $C(\Omega) = C_0(\Omega,\mathbb{R})$.

1. Suppose that $(E,\|\cdot\|)$ is a Banach space and $\mathcal{L}(E)$ is a Banach algebra of all linear continuous operators from E into itself with the standard norm $|\cdot|$. Moreover, let us denote:

by $C(J,E)$ - the set of all continuous functions defined on J with the values in E;

by $\mathfrak{X}$ - the set of all mappings A from J into $\mathcal{L}(E)$ such that $t \mapsto A(t)$ is a continuous operator-valued function (i.e., $t \mapsto A(t)x$ is a strongly continuous E-valued function for each x in E).

The set $C(J,E)$ be considered as a vector space endowed with the topology of uniform convergence on compact subsets of J. This topology is determined by the sequence (p_n) of seminorms given by $p_n(y) = \sup_{0 \leq t \leq n} \| y(t) \|$ for $y \in C(J,E)$, and therefore (see [20]) $C(J,E)$ is a Fréchet space.

In the sequel, we shall deal with the set $\mathcal{X}$ as an $\mathcal{L}^*$-space endowed with the following convergence: (A_n) is a convergent sequence, if $\sup_{n \geq 1} \sup_{t \in \Omega} \| A_n(t) \| < \infty$ on compact subsets Ω of J and $(A_n(t)y(t))$ converges uniformly on compact subsets of J for each $y \in C(J,E)$. Moreover, $E \times \mathcal{X}$ will be considered as an $\mathcal{L}^*$-product [10, p.86] of the spaces $E, \mathcal{X}$.

For example, $\mathcal{X}$ endowed with almost uniform convergence (i.e., uniform convergence on every compact subset of J) is an $\mathcal{L}^*$-space satisfying the above conditions. Indeed, let Ω be a compact set of J and $\lim_{n \to \infty} \sup_{t \in \Omega} \| A_n(t) - A_0(t) \| = 0$. Then $\lim_{n \to \infty} \sup_{t \in \Omega} \| A_n(t)x - A_0(t)x \| = 0$ for each $x \in E$, and therefore $(A_n(t))$ is uniformly bounded for $n \geq 1$ and $t \in \Omega$. Further, by Lemma 3.4 in [9, p. 22], $\lim_{n \to \infty} \sup_{t \in \Omega} \| A_n(t)y(t) - A_0(t)y(t) \| = 0$ for every $y \in C(J,E)$, and we are done.

<u>Proposition 1</u>. For an arbitrary $x \in E$ and $A \in \mathcal{X}$ there exists a unique function $y_{(x,A)}$ in $C(J,E)$ such that $y_{(x,A)}(0) = x$ and

$$y'_{(x,A)}(t) = A(t)y_{(x,A)}(t) \text{ for } t \geq 0.$$

Moreover, the transformation $(x,A) \longmapsto y_{(x,A)}$ from $\mathcal{L}^*$-product $E \times \mathcal{X}$ into $C(J,E)$ is continuous.

The above type result is contained in [9] (without the proof of continuous dependence). Using Theorem 1 we obtain a very simple proof.

Let $x \in E$ and $A \in \mathcal{K}$. To prove the first part of our proposition, we define

$$(Ty)(t) = \exp(-r \int_0^t \| A(s) \| ds) \cdot y(t),$$

$$(Fy)(t) = \exp(-r \int_0^t \| A(s) \| ds) \cdot (x + \int_0^t A(s)y(s)ds)$$

for $y \in C(J,E)$, where $r > 1$ is constant. Further, let Ω be a compact subset of J. Assume that $\lim_{n\to\infty} A_n = A_o$ and $\lim_{n\to\infty} \| x_n - x_o \| = 0$, where $(x_m, A_m) \in E \times \mathcal{K}$ for $m = 0,1,\ldots$. Now, let us put:

$$(Ty)(t) = \exp(-rt) \cdot y(t);$$

$$(F_m y)(t) = \exp(-rt) \cdot (x_m + \int_0^t A_m(s)y(s)ds) \quad (m = 0,1,\ldots)$$

for $y \in C_o(\Omega, E)$, where $r > \sup_{n\geq 1} \sup_{t\in\Omega} \| A_n(t) \|$. Then, by Theorem 1, there exists a unique y_m in $C_o(\Omega,E)$ $(m = 0,1,\ldots)$ such that $y_{(x_m,A_m)}|_\Omega = y_m$ and $\sup_{t\in\Omega} \| y_{(x_n,A_n)}(t) - y_{(x_o,A_o)}(t) \| \longrightarrow 0$ as $n \longrightarrow \infty$. This completes the proof of the theorem.

2. In the vector space $E_\infty = C_o(I) \times C_o(I) \times \ldots$ define a sequence (p_n) of seminorms $p_n(x) = \sup_{t\in I} | x_n(t) |$, where $x = (x_1, x_2, \ldots)$. It is known that the space E_∞ equipped with a topology generated by a saturated family $P = \{ p_n : n \geq 1 \}$ is a locally convex space, and as stated in [20], is a Fréchet space.

Let n_i $(i = 1,2,\ldots)$ be positive integers with $\sup_{i\geq 1} n_i = +\infty$. We consider in E_∞ the infinite system of integral equations

$$(++) \qquad x_i(t) = \int_0^t g_i(s, x_i(s))ds + \int_0^t f_i(s, x_1(s), x_2(s), \ldots, x_{n_i}(s))ds \qquad (i = 1,2,\ldots)$$

where f_i, g_i are defined on $I \times \mathbb{R}^{n_i}$ and $I \times \mathbb{R}$, respectively.

<u>Proposition 2</u>. Suppose that the functions f_i, g_i

$(i = 1,2,\ldots)$ are continuous and there exist integrable on I functions A_i, B_i and L_i such that

$$|f_i(t,u_1,u_2,\ldots,u_{n_i})| \le A_i(t), \quad |g_i(t,u)| \le B_i(t),$$
$$|g_i(t,u) - g_i(t,v)| \le L_i(t)\ |u - v|$$

for every $t \in I$, $(u_1,u_2,\ldots,u_{n_i}) \in \mathbb{R}^{n_i}$ and for u, v in $\mathbb{R}$. Then the system (++) has at least one solution defined on the interval I with continuous coordinates x_i.

The assumptions of Proposition 2 do not allow to use it for linear or weakly linear cases (i.e. $g_i(t,x_i) = \alpha_i(t)x_i$, $\alpha_i(t) \neq 0$). Therefore we write it without proof.

3. In this section we assume that E is a Hausdorff locally convex complete topological vector space and P is a saturated family of seminorms which generates the topology of E. Further, let $p_1,p_2,\ldots,p_m$ from P be fixed and let B be a set of all x in E such that $p_i(x) \le b$ for $1 \le i \le m$. In the sequel we shall deal with the integral in the sense of [12] or [7].

Denote by $C(I_h,E)$ the vector space of all continuous functions from I_h to E with the topology of uniform convergence. Then, $\{x \longmapsto \sup_{t \in I_h} p(x(t)) : p \in P\}$ is a saturated family of seminorms defining the topology of $C(I_h,E)$ and this space is complete.

Applying the Theorem 3 we shall prove the following result:

<u>Proposition 3</u>. Let f be a continuous function from $I \times B \times E$ to E such that $M = \sup_{1 \le i \le m} \sup_{I \times B \times E} p_i(f(t,x,y)) < \infty$ and $p(f(t,x_1,y) - f(t,x_2,y)) \le L(t) \cdot p(x_1 - x_2)$ for all $p \in P$, $t \in I$, $y \in E$ and x_1, x_2 in B, where L is an integrable function on I.

Suppose that g is a continuous function from $I\times I\times B$ to E, $g[I\times I\times B]$ is conditionally compact and the mapping $t \longmapsto g(t,s,x)$ is continuous on I uniformly with respect to $(s,x)\in I\times B$. Then there exists at least one solution of the equation

$$x(t) = \int_0^t f(s,x(s), \int_0^s g(s,\sigma,x(\sigma))d\sigma)ds$$

defined on the interval I_h with $h = \min(a, M^{-1}b)$.

<u>Proof</u>. Let us put:

$X = \{x\in C(I_h,E): x(t)\in B \text{ for } t \text{ in } I_h\}$;

$K = \{x\in C(I_h,E): p_i(x(t)) \le b\cdot\exp(-r\int_0^t L(s)ds) \text{ for } 1\le i\le m$ and t in $I_h\}$;

$(Tx)(t) = \exp(-r\int_0^t L(s)ds)\cdot x(t)$ for $x\in X$;

$(Qx)(t) = \int_0^t g(t,s,\exp(r\int_0^s L(\sigma)d\sigma)\cdot x(s))ds$ for $x\in K$;

$F(x,y)(t) = \exp(-r\int_0^s L(s)ds)\cdot\int_0^t f(s,x(s),y(s))ds$ for $x\in X$ and $y\in C(I_h,E)$,

where $r>1$ is a constant. It can be easily seen that $T[X]\subset K$, $F[X\times C(I_h,E)]\subset T[X]$, K is a closed convex set and $T[X]$ is complete. For every $p\in P$, x_1, $x_2\in X$ and $y\in C(I_h,E)$ we have

$$p\left(\int_0^t f(s,x_1(s),y(s))ds - \int_0^t f(s,x_2(s),y(s))ds\right) \le$$
$$\le \sup_{t\in I_h}\left[\exp(-r\int_0^t L(s)ds)\cdot p(x_1(t)-x_2(t))\right]\cdot\int_0^t L(s)\cdot \exp(r\int_0^s L(\sigma)d\sigma)ds \le r^{-1}\cdot\exp(r\int_0^t L(s)ds)\cdot$$
$$\cdot\sup_{t\in I_h} p((Tx_1)(t) - (Tx_2)(t))$$

and this yields $\sup_{t\in I_h} p(F(x_1,y)(t) - F(x_2,y)(t)) \le$ $\le r^{-1}\cdot\sup_{t\in I_h} p((Tx_1)(t) - (Tx_2)(t))$.

Let us fix x in X. Suppose that $\{y_\alpha : \alpha\in\Gamma\}$ is a convergent net in $C(I_h,E)$ with $\lim_{\alpha\in\Gamma} y_\alpha = y_0$. Assume, moreover, that

$q_1, q_2, \dots, q_k$ are fixed in P, $\varepsilon > 0$, Z is a compact set and $y_\alpha[I_h] \subset Z$ for all $\alpha \in \Gamma$. The function $f|_{I_h \times x[I_h] \times Z}$ is uniformly continuous so there exist $q_1', q_2', \dots, q_l'$ in P, $\varepsilon' > 0$ such that $q_i(f(t,x(t),u) - f(t,x(t),v)) < h^{-1}\varepsilon$ for every $1 \leq i \leq k$, $t \in I_h$ and $u,v \in Z$ with $q_j'(u - v) < \varepsilon'$ ($j = 1,2,\dots,l$). Since $\mathcal{W} = \{x \in C(I_h,E) : \sup_{t \in I_h} q_j'(x(t)) < \varepsilon'$ for $1 \leq j \leq l\}$ is a neighbourhood of the origin in $C(I_h,E)$, there is an index α_0 in Γ such that $y_\alpha - y_0 \in \mathcal{W}$ for all $\alpha \succcurlyeq \alpha_0$. From the previous facts we have

$$q_i(F(x,y_\alpha)(t) - F(x,y_0)(t)) \leq$$

$$\leq \int_0^t q_i(f(s,x(s),y_\alpha(s)) - f(s,x(s),y_0(s)))ds < h^{-1} \cdot \varepsilon \cdot t$$

for every $t \in I_h$, $1 \leq i \leq k$ and $\alpha \succcurlyeq \alpha_0$. So we proved that for every neighbourhood $\mathcal{V}$ of the origin in $C(I_h,E)$ there exists $\alpha_0 \in \Gamma$ such that $F(x,y_\alpha) - F(x,y_0) \in \mathcal{V}$ for all $\alpha \succcurlyeq \alpha_0$. Therefore, $\lim_{\alpha \in \Gamma} F(x,y_\alpha) = F(x,y_0)$.

Arguments similar to the above imply that the mapping Q is continuous. Now, we prove that the set Q[K] is compact.

For $p \in P$, $x \in K$ and t_1, t_2 in I_h, we have

$$p((Qx)(t_2) - (Qx)(t_1)) \leq$$

$$\leq \int_0^t p(g(t_2,s,\exp(r\int_0^s L(\sigma)d\sigma)\cdot x(s)) -$$

$$- g(t_1,s,\exp(r\int_0^s L(\sigma)d\sigma)\cdot x(s)))ds +$$

$$+ \int_{t_1}^{t_2} p(g(t_2,s,\exp(r\int_0^s L(\sigma)d\sigma)\cdot x(s)))ds.$$

Let $\varepsilon > 0$, let $q_1, q_2, \dots, q_k$ be fixed in P, and let us put $M' = \sup_{1 \leq i \leq k} \sup_{I \times I \times B} q_i(g(t,s,u))$. Since the function $t \longmapsto g(t,s,u)$ is continuous on a compact I uniformly with respect to $(s,u) \in I \times B$, so there exists $\delta' > 0$ such that

$$\sup_{1 \le i \le k} \sup_{(s,u)\in I\times B} q_i(g(t_1,s,u) - g(t_2,s,u)) < \frac{\varepsilon}{2h}$$

for every $|t_1 - t_2| < \delta'$. From this it follows

$$q_i((Qx)(t_2) - (Qx)(t_1)) \le$$

$$\le \int_0^{t_1} \sup_{1\le i\le k} \sup_{(s,u)\in I\times B} q_i(g(t_2,s,u) - f(t_1,s,u))ds +$$

$$+ \int_{t_1}^{t_2} \sup_{1\le i\le k} \sup_{(t,s,u)\in I\times I\times B} q_i(g(t,s,u))ds < \frac{\varepsilon}{2} + M' \,|t_2 - t_1|$$

for $1 \le i \le k$, $x \in K$ and $|t_1 - t_2| < \min(\delta', \varepsilon/2M')$. Consequently, for all neighbourhoods V of the origin in E there exists a number $\delta > 0$ such that $(Qx)(t_1) - (Qx)(t_2) \in V$ for every $|t_1 - t_2| < \delta$ and $x \in K$.

Further, by the Mazur theorem [5] the set $\overline{\text{conv}}\ (\overline{g[I\times I\times B]})$ is compact. Using the integral mean-value theorem (cf. [12]) we obtain

$$\int_0^t g(t,s,x(s))ds \in t\cdot\overline{\text{conv}}\ (g[I\times I\times B]) \text{ for } x\in K$$

and therefore $\{(Qx)(t): x \in K\}$ is a conditionally compact set for all t in I_h. Finally, by Ascoli-Arzela theorem the set Q[K] is conditionally compact in $C(I_h,E)$, and all the assumptions of Theorem 3 are satisfied. Whence, there exists at least one $x \in X$ such that $F(x,Q(Tx))(t) = (Tx)(t)$ for each $t \in I_h$, and we are done.

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