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RELATIONS AMONG VARIOUS CRITERIA OF UNIQUENESS FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: The paper contains the discussions of relations among some criteria of uniqueness of solutions of ordinary differential equations. Some generalizations of earlier results of several authors are given.

Key words: Ordinary differential equation, uniqueness criterion, Kamke comparison function.

Classification: 34A10

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1. Introduction. We will consider an ordinary differential equation

(1) $$x' = f(t,x)$$

with the initial condition

(2) $$x(0) = x_0,$$

where we assume that the function $f(t,x)$ is given and defined on the Cartesian product of the interval $\langle 0,T\rangle$ and the set $\Omega \subset R^n$ and taking values in the space R^n.

The problem of uniqueness of solution of (1)-(2) is one of the basic problems in the theory of ordinary differential equations along with such problems as existence, continuation of solutions, convergence of successive approximations, for example (cf. [5],[7],[15]). This problem has been discussed in several works and, roughly speaking, it was initiated by Kamke [8], among others.

It is worth to mention that several criteria sufficient for uniqueness of (1)-(2) are known (cf. [4],[15]). It seems that the most convenient criteria of uniqueness are these of Kamke type. In order to their formulation let us consider a real function $\omega(t,u),(t,u)\in\langle 0,T\rangle\times\langle 0,+\infty)$ (or $(t,u)\in(0,T\rangle\times\times\langle 0,+\infty)$) and denote by $|x|$ one of the norms in the Euclidean space R^n. A function $\omega(t,u)$ will be called the Kamke comparison function if the inequality

(3) $$|f(t,x) - f(t,y)| \leq \omega(t, |x-y|)$$

along with some additional conditions guarantees uniqueness of solution of the Cauchy problem (1)-(2).

Obviously, the class of Kamke comparison functions comprises the well known criteria of Lipschitz, Nagumo, Osgood, Coddington and Levinson, for instance [15].

The aim of this paper is to discuss relations between some classes of Kamke comparison functions. Some considerations of such type may be found in a lot of works ([4],[10],[13],[14], [15]). In this paper we will give some generalizations of those results.

2. Some classes of Kamke comparison functions. For simplicity of considerations denote by J the interval $\langle 0,T\rangle$ and by J_0 the interval $(0,T\rangle$. Moreover, let $R_+ = \langle 0,+\infty)$. We will examine the following classes of Kamke comparison functions.

Class $\mathcal{A}$. This class contains all functions $\omega: J\times R^+ \to R^+$ such that $\omega(t,0) = 0$, $\omega(t,u)$ is continuous and

$u(t) \equiv 0$ is the only continuous and differentiable on J function which satisfies on J the equation

$$u' = \omega(t,u)$$

and the condition $u(0) = 0$.

Class $\mathcal{B}$. A function $\omega : J_0 \times R_+ \longrightarrow R_+$ belongs to the class $\mathcal{B}$ provided it is continous, $\omega(t) = 0$ and $u(t) \equiv 0$ is the only differentiable on J_0 and continuous on J function such that

$$u' = \omega(t,u), \ t \in J_0,$$

$$\lim_{t \to 0^+} \frac{u(t)}{t} = \lim_{t \to 0^+} u(t) = 0.$$

Class $\mathcal{C}$. $\omega(t,u) = \omega : J \times R_+ \longrightarrow R_+$ belongs to the class $\mathcal{C}$ if it satisfies the Carathéodory conditions i.e. it is Lebesgue measurable with respect to t for any fixed u and continuous with respect to u for any fixed t. Moreover, $\omega(t,u)$ is locally Lebesgue integrable, which means that for any $t_0 \in J_0$ and $u_0 > 0$ there exists a Lebesgue integrable on the interval $\langle t_0, T\rangle$ function h(t) such that $\omega(t,u) \leq h(t)$ for $(t,u) \in \langle t_0, T\rangle \times \times \langle 0, u_0\rangle$. Further we assume that $\omega(t,0) = 0$ and $u(t) \equiv 0$ is the only absolutely continuous function which satisfies the equation

$$u' = \omega(t,u) \text{ for almost all } t \in J,$$

and the condition

$$\lim_{t \to 0^+} \frac{u(t)}{t} = u(0) = 0.$$

Class $\mathcal{D}$. This class comprises all functions $\omega(t,u) = = \omega : J_0 \times R_+ \longrightarrow R_+$, $\omega(t,0) = 0$ which are continuous and for which the only continuous on J function satisfying the integral inequality

$$u(t) \leq \int_0^t \omega(s,u(s))ds, \quad t \in J_0,$$

and the conditions

$$\lim_{t \to 0^+} \frac{u(t)}{t} = \lim_{t \to 0^+} u(t) = 0$$

is the function $u(t) \equiv 0$.

Class $\mathcal{E}$. $\omega \in \mathcal{E}$ if and only if $\omega : J \times R_+ \longrightarrow R_+$, $\omega(t,0) = 0$ and similarly as in the class $\mathcal{C}$ it satisfies the Carathéodory conditions and is locally Lebesgue integrable. Besides $u(t) \equiv 0$ is the only absolutely continuous on J function for which

$$u(t) \leq \int_0^t \omega(s,u(s))ds, \quad t \in J,$$

and

$$\lim_{t \to 0^+} \frac{u(t)}{t} = u(0) = 0.$$

Class $\mathcal{F}$. This class contains functions $\omega : J \times R_+ \longrightarrow R_+$, $\omega(t,0) = 0$ which similarly to those from the class $\mathcal{C}$ satisfy the Carathéodory conditions and are locally integrable. Furthermore, we assume that the only continuous on J function which satisfies the inequality

$$u(\bar{t}) - u(t) \leq \int_t^{\bar{t}} \omega(s,u(s))ds, \quad 0 \leq t \leq \bar{t} \leq T,$$

and such that $\lim_{t \to 0^+} \frac{u(t)}{t} = u(0) = 0$, is the function $u(t) \equiv 0$.

In what follows, let us denote by $\mathcal{A}^+, \mathcal{B}^+, \ldots, \mathcal{F}^+$ the subclasses of the classes $\mathcal{A}, \mathcal{B}, \ldots, \mathcal{F}$, respectively, consisting of all functions $\omega(t,u)$ which are increasing with respect to u.

Notice that we may also consider more general classes, for instance

Class $\mathcal{A}_0$. $\omega \in \mathcal{A}_0$ provided $\omega : J \times R_+ \to R_+$ is continuous, $\omega(t,0) = 0$ and for every $a \in (0,T)$, the function $u(t) \equiv 0$ is the only differentiable on $\langle 0,a\rangle$ function which satisfies the equation $u' = \omega(t,u)$ for $t \in \langle 0,a\rangle$ and the initial condition $u(0) = 0$.

In the same way we may define the classes $\mathcal{A}_0$, $\mathcal{B}_0, \ldots$ $\ldots, \mathcal{F}_0$.

Notice first that each of the classes $\mathcal{A}, \mathcal{B}, \ldots, \mathcal{F}$ is the Kamke class i.e. if the inequality (3) is satisfied with some function $\omega \in \mathcal{A}, \ldots, \omega \in \mathcal{F}$, respectively, then the Cauchy problem (1)-(2) admits at least one solution. Such theorems with respect to the class $\mathcal{A}$ have been proved by Perron [12] and with respect to $\mathcal{B}$ by Kamke [8]. The theorem that the class $\mathcal{C}$ is sufficient for uniqueness is due to Coddington and Levinson [4]. By this regard that $\mathcal{D} \subset \mathcal{B}$, $\mathcal{E} \subset \mathcal{C}$ (see Theorem 1 below) the classes $\mathcal{D}$ and $\mathcal{E}$ are also sufficient for uniqueness. Moreover, the class $\mathcal{F}$ is equal to the class $\mathcal{C}$ (see [1]) so that $\mathcal{F}$ is also sufficient for uniqueness.

Notice also that owing to Walter results [14] we have the following equalities

$$\mathcal{A} = \mathcal{A}_0, \ \mathcal{B} = \mathcal{B}_0, \ldots, \ \mathcal{F} = \mathcal{F}_0.$$

Moreover, let us remark that if the right side of the equation (1) is continuous then the classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are equivalent in some sense. This result was first proved by Olech [10] (cf. also [14]).

Notice now that we have the following inclusions

$$\mathcal{A}^+ \subset \mathcal{A}, \ \mathcal{B}^+ \subset \mathcal{B}, \ldots, \mathcal{F}^+ \subset \mathcal{F}$$

and generally these inclusions are strong i.e. it may happen that the function f satisfies the inequality (3) with some function $\omega(t,u)$ which belongs to $\mathcal{B}$, for example, but there do not exist any functions from class $\mathcal{B}^+$ such that the inequality (3) is satisfied. The examples of such type may be found in [11].

On the other hand if we want to obtain a theorem about convergence of successive approximation to a solution then we generally have to assume the classes $\mathcal{A}^+,\dots,\mathcal{F}^+$ ([10],[11]). Further let us mention that at first the uniqueness criterion of Kamke type was investigated by Bompiani [2]. Namely, he has considered the class $\mathcal{A}^+$. The classes similar to $\mathcal{D}$ and $\mathcal{E}$ were investigated by Coddington and Levinson [4], Walter [15], Kisyński [9], Goebel and Rzymowski [6].

3. Relations among Kamke classes. In this section we are going to discuss some relations among the introduced classes of Kamke comparison functions. At first we prove the following simple theorem.

Theorem 1. $\mathcal{D} \subset \mathcal{B}$, $\mathcal{E} \subset \mathcal{C}$.

Proof. We prove, for example, the second inclusion (the proof of the first one is similar). To do it let us suppose that the function $u(t)$, $t \in J$, is absolutely continuous and such that

(4) $$u' = \omega(t,u) \text{ for almost all } t \in J,$$

and

(5) $$\lim_{t \to 0+} \frac{u(t)}{t} = u(0) = 0.$$

Because of absolute continuity of the function u(t) from (4) we obtain

$$\int_0^t u'(s)ds = u(t) - u(0) = \int_0^t \omega(s,u(s))ds.$$

Thus in view of the assumption that $\omega \in \mathscr{E}$ and (5) we have that $u(t) \equiv 0$ and the proof is complete.

Now we prove the main theorem of this paper.

<u>Theorem 2</u>. Let $\omega(t,u) \in \mathscr{C}$ and let there exist a function $\tilde{\omega}(t,u) \in \mathscr{C}^+$ such that $\omega(t,u) \leq \tilde{\omega}(t,u)$ for all $t \in J$ and $u \in R^+$. Then $\omega(t,u) \in \mathscr{E}$.

<u>Proof</u>. Suppose that u(t) satisfies the integral inequality

$$u(t) \leq \int_0^t \omega(s,u(s))ds \tag{6}$$

and is absolutely continuous and satisfies (5). Then with respect to the assumptions, we obtain

$$u(t) \leq \int_0^t \tilde{\omega}(s,u(s))ds. \tag{7}$$

Let us suppose that u(t) does not vanish identically on the interval J so that there exists $t_1 \in J_0$ such that $u(t_1) = \alpha > 0$. Now consider the following differential equation on J:

$$g' = \tilde{\omega}(t,g) \tag{8}$$

with the initial condition $g(T) = \alpha$.

Let g(t) be the minimal solution (from T to left) of this equation. Obviously g(t) is increasing, because of the function $\tilde{\omega}(t,u)$ being nonnegative. Hence

$$g(t_1) \leq u(t_1) = \alpha \,. \tag{9}$$

From (7) we obtain

$$u(t) \leq \alpha + \int_0^t \tilde{\omega}(s,u(s))ds.$$

Now taking the interval $\langle 0,t_1\rangle$ into consideration and using the well known theorems on integral and differential inequalities (see for example [15], p. 43 or [3]) we obtain from (9) that

$$g(t) \leq u(t) \text{ for } t \in \langle 0,t_1\rangle.$$

Finally, we obtain that g(t) is the solution of (8) which does not vanish identically on J and such that $\lim_{t \to 0^+} \frac{g(t)}{t} = g(0) = 0$, which contradicts the fact that $\tilde{\omega}(t,u) \in \mathcal{C}^+$. Thus the proof is complete.

Corollary 1. $\mathcal{E}^+ = \mathcal{C}^+$.

Indeed, Theorem 2 implies that $\mathcal{C}^+ \subset \mathcal{E}^+$. The converse inclusion follows easily from Theorem 1.

Next, for an arbitrary Kamke comparison function $\omega(t,u)$, denote by $\omega^*(t,u)$ its smallest majorant which is increasing with respect to u, i.e.

$$\omega^*(t,u) = \sup\,[\omega(t,v) : 0 \leq v \leq u].$$

Thus we have

Corollary 2. If for a function $\omega(t,u) \in \mathcal{C}$ the function $\omega^*(t,u) \in \mathcal{C}^+$ then $\omega(t,u) \in \mathcal{E}$.

In a similar way as in the proof of Theorem 2, we may prove the next theorem.

Theorem 3. $\mathcal{B}^+ = \mathcal{D}^+$.

Now we return to the Cauchy problem (1)-(2). Let us assume that the function f(t,x) is defined and continuous on the set $J \times P$, where $P = [x \in R^n : |x - x_0| \leq b]$, $b > 0$, b = const. Then we have

Theorem 4. If the function f satisfies the Kamke comparison condition (3) for all (t,x), $(t,y) \in J \times P$ and if $\omega(t,u) \in \mathcal{D}$ (or $\omega(t,u) \in \mathcal{E}$) then there exists a function $h(t,u) \in \mathcal{A}$ such that

$$|f(t,x) - f(t,y)| \leq h(t,|x-y|), \text{ for } (t,x),(t,y) \in J \times P.$$

Proof. Let, for example, the function $\omega(t,u)$ belong to the class $\mathcal{D}$. Then, by virtue of Theorem 1, this function belongs to the class $\mathcal{B}$. Thus the existence of the desired function $h(t,u)$ follows immediately from Olech theorem [10]. The proof of the second part of our theorem is the same.

In view of the above theorem, in the case when the right side of the equation (1) is continuous, the classes $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ are in some sense equivalent to that of Perron (i.e. $\mathcal{A}$).

4. Examples and final remarks. In this section we shall show that the classes $\mathcal{B}$ and $\mathcal{C}$ are generally essentially wider than the classes $\mathcal{D}$ and $\mathcal{E}$.

Example 1. Consider a decreasing sequence $t_0, t_1, t_2, \ldots$ such that $t_n \in \langle 0,T \rangle$ and $t_0 = T$, $\lim\limits_{n\to\infty} t_n = 0$. Let us take a sequence $r_1, r_2, \ldots$ such that

$$0 < r_n \leq \frac{1}{3} \min [t_{n-1} - t_n,\ t_n - t_{n+1},\ t_{n-1}^2 - t_n^2,\ t_n^2 - t_{n+1}^2]$$

for $n = 1,2,\ldots$. Furthermore, denote by

$$\beta_n = \{ \int_{Z_n} (1 - \frac{1}{r_n^2} [(t - t_n)^2 + (t^2 - t_n^2)^2]) dt \}^{-1},$$

where $Z_n = [t : (t-t_n)^2 + (t^2 - t_n^2)^2 \leq r_n^2]$ and $n = 1,2,\ldots$.

Finally, let us define the function $\omega(t,u)$ as follows:

$$\omega(t,u) = \begin{cases} \beta_n\left(1 - \frac{1}{r_n^2}\left[(t - t_n)^2 + (u - t_n^2)^2\right]\right) \text{ if } \\ \qquad (t,u) \in K((t_n,t_n^2),r_n) \\ \qquad \text{for some } n = 1,2,\ldots \\ 0 \text{ if } (t,u) \in J \times R_+ - \bigcup_{n=1}^{\infty} K((t_n,t_n^2),r_n) \end{cases}$$

where $K((t_n,t_n^2),r_n)$ denotes the circle on the plane R^2 centered at the point (t_n,t_n^2) and with radius r_n.

It is easy to check that the function $\omega(t,u)$, defined above, is continuous on the set $J \times R_+ - \{(0,0)\}$ and $\omega(t,0) = 0$. Moreover, it belongs to the class $\mathcal{B}$ and consequently to $\mathcal{C}$ because the function $u(t) \equiv 0$ is the only function which satisfies the equation $u' = \omega(t,u)$ and the initial condition $u(0) = 0$. Indeed, if the function $u = u(t)$, $t \in J$, satisfies the equation $u' = \omega(t,u)$, then in the case if $[(t,u(t)):t \in J] \cap \bigcup_{n=1}^{\infty} K((t_n,t_n^2),r_n) = \emptyset$, it must be constant so that the initial condition $u(0) = 0$ implies that $u(t) \equiv 0$. In the converse case the curve $u = u(t)$ meets some circle $K((t_n,t_n^2),r_n)$ and it must be constant in some interval $\langle a,T_1\rangle$, $a > 0$ which lies on the left of the point t_n. Because of continuity of the function $u(t)$ we have that $a = 0$, so that the initial condition $u(0) = 0$ again implies that $u(t) \equiv 0$. On the other hand, for the function $\varphi(t) = t^2$ we obtain

$$\varphi(t) \not\leq \int_0^t \omega(s, \varphi(s))ds = +\infty$$

for any $t \in J_0$.

The above example shows that the following theorem is true.

<u>Theorem 5</u>. There exists a function $\omega(t,u)$ such that $\omega(t,u) \in \mathcal{B}$ and $\omega(t,u) \in \mathcal{C}$ but $\omega(t,u) \notin \mathcal{D}$ and $\omega(t,u) \notin \mathcal{E}$.

In the example below we show that the assumption of monotonicity in the Theorem 2 is not necessary.

Example 2. Let us consider the function $\omega : J_0 \times R_+ \rightarrow R_+$, defined by the formula

$$\omega(t,u) = \frac{u}{t + u^2} .$$

It is easy to check that $\omega(t,u)$ belongs to the classes $\mathcal{B}$ and $\mathcal{D}$ simultaneously and it is not increasing with respect to u.

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