Commentationes Mathematicae Universitatis Carolinae

Moses A. Boudourides
On bounded solutions of nonlinear ordinary differential equations

Commentationes Mathematicae Universitatis Carolinae, Vol. 22 (1981), No. 1, 15--26

Persistent URL: http://dml.cz/dmlcz/106050

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

22,1 (1981)

ON BOUNDED SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

Moses A. BOUDOURIDES

Abstract: We prove the existence and an asymptotic property of bounded solutions of the nonlinear differential equation (in a Banach space E and with the independent variable $t\in[0,\infty)$)

$$x' = A(t)x + f(t,x)$$

under the assumption that the non-homogeneous linear equation

$$x' = A(t)x + b(t)$$

has at least one bounded solution for each b belonging to a function Banach space B.

Key words: Ordinary differential equations in Banach spaces, function spaces, admissibility, successive approximations.

Classification: 34A34, 34G20, 34C11

---

1. Introduction. The object of the present article is the study of the relations between the solutions of the following equations

$$x' = A(t)x \tag{1}$$

$$x' = A(t)x + b(t) \tag{2}$$

$$x' = A(t)x + f(t,x) \tag{3}$$

where $t\in J=[0,\infty)$; x, b, $f\in E$, a real Banach space; A(t), for every fixed t, is a continuous linear operator (endomorphism) of E into itself; A(t), b(t) are locally integrable (in the Bochner sense).

In the years 1930-1935, O. Perron, K.P. Persidskii and I.G. Malkin (cf. [4] for references) established (among other results) the equivalence of the following properties (in the case $\dim E < \infty$, $A(t)$ continuous)

(P1) for each bounded continuous b all the solutions of (2) are bounded;

(P2) for each f continuous, $\| f(t,x) \| \leq \beta$, $\| f(t,x) - f(t,y) \| \leq \gamma \| x-y \|$, with sufficiently small β, γ, all the solutions of (3) with sufficiently small $\| x(0) \|$ are bounded;

(P3) there exist positive constants N, ν such that for any solution x of (1) and for any $t \geqq t_0 \geqq 0$ we have

$$\| x(t) \| \leqq N e^{-\nu(t-t_0)} \| x(t_0) \| .$$

In the years 1958-1959, J.L. Massera and J.J. Schaffer (cf. [3],[4]) generalized these properties (in the case of $\dim E = \infty$ and of Carathéodory type conditions), considering a general category of function spaces.

The purpose of this article is to establish the equivalence of (P1) and (P2) in the frame of the general function spaces of [4] and in the case when f is such that $\| f(t,x) - f(t,y) \| \leqq \omega(t, \| x-y \|)$, where $\omega(t,\cdot)$ is an appropriate non-decreasing function. To this end, we first extend Coppel's equivalent criterion to (P1). Finally, we obtain sufficient conditions such that for every bounded solution x of (3) $\lim_{t\to\infty} \| x(t) \| = 0$.

2. Notation and preliminaries. Let X be a generic Banach space with norm $\| \cdot \|_X$. We denote by X^* its dual and by $(\cdot,\cdot)$

the duality pairing of X and X^*; the norm of X^* is denoted again by $\|\cdot\|_{X^*}$. We denote by $\tilde{X}$ the space of continuous endomorphisms of X and again by $\|\cdot\|_{\tilde{X}}$ the norm of $\tilde{X}$. If $A \in \tilde{X}$, we denote by $A^* \in \tilde{X}^*$ its adjoint operator.

For the Banach space E we write $\|\cdot\|_E = \|\cdot\|$. For any $a > 0$, we write $S_a = \{x \in E; \|x\| < a\}$.

By $C = C(E)$ we denote the Banach space of bounded continuous functions $u:J \longrightarrow E$ with the norm $\|u\|_C = \sup\{\|u(s)\| : s \in J\}$. For any $a > 0$, we write $\Sigma_a = \{u \in C: \|u\|_C < a\}$.

By $L^p = L^p(E)$, $1 \leqq p < \infty$, we denote the Banach space of strongly measurable functions $u:J \longrightarrow E$ such that $\int_J \|u(s)\|^p ds < \infty$ with the norm $\|u\|_{L^p} = \{\int_J \|u(s)\|^p ds\}^{1/p}$. By $L^\infty = L^\infty(E)$ we denote the Banach space of strongly measurable functions $u:J \longrightarrow E$ such that ess sup $\{\|u(s)\| : s \in J\} < \infty$ with the norm $\|u\|_{L^\infty} = \text{ess sup}\{\|u(s)\| : s \in J\}$.

By $L = L(E)$ we denote the space of strongly measurable functions $u:J \longrightarrow E$, Bochner integrable in every finite subinterval I of J, with the topology of the convergence in the mean on every such I.

Let $B(R)$ be a Banach space of measurable functions $u:J \longrightarrow R$ such that

(i) $B(R)$ is stronger than $L(R)$ (cf. [4], p.35);

(ii) if $u \in L^\infty(R)$ with compact support, then $u \in B(R)$;

(iii) if $u \in B(R)$ and $v:J \longrightarrow R$ measurable and such that $|v| \leqq |u|$, then $v \in B(R)$ and $\|v\|_{B(R)} \leqq \|u\|_{B(R)}$.

By the associate space $B^*(R)$ we denote the Banach space of all measurable functions $v:J \longrightarrow R$ such that

$$\sup\{\int_J |u(s)v(s)|ds : u \in B(R), \|u\|_{B(R)} \leqq 1\} < \infty$$

with norm $\|v\|_{B^*(R)} = \sup\{\int_J |u(s)v(s)|ds : u \in B(R), \|u\|_{B(R)} \leqq \leqq 1\}$. According to Theorem 22.M of [4], the following "Hölder's Inequality" holds: if $u \in B(R)$ and $v \in B^*(R)$, then $|uv| \in L^1(R)$ and

$$\int_J |u(s)v(s)|ds \leqq \|u\|_{B(R)} \|v\|_{B^*(R)}.$$

We denote by $B = B(E)$ ($B^* = B^*(E)$) the Banach space of all strongly measurable functions $u : J \longrightarrow E$ such that $\|u\| \in \in B(R)$ ($\|u\| \in B^*(R)$) provided with the norm $\|u\|_B = = \| \|u\| \|_{B(R)}$ ($\|u\|_{B^*} = \| \|u\| \|_{B^*(R)}$).

Let $A \in L(\widetilde{E})$ and let E_0 be the set of all points of E which are values for $t = 0$ of bounded solutions of (1).

We assume that E_0 is closed. Then according to Theorem 4.1 of [3], there exists $S > 0$ such that every bounded solution x of (1) satisfies the estimate

$$\|x\|_C \leqq S \|x(0)\| .$$

Moreover, we assume that E_0 has a closed complement E_1. Let P be the projection of E onto E_0. Furthermore, let $U(t)$ be the fundamental solution of (1) such that $U(0) = I$. For any $t \in J$ we define a function $G(t,\cdot) \in L(\widetilde{E})$ by

$$G(t,s) = \begin{cases} U(t)PU^{-1}(s) & \text{for } 0 \leqq s \leqq t \\ -U(t)(I-P)U^{-1}(s) & \text{for } s \geqq t. \end{cases}$$

3. <u>The Non-homogeneous Linear Equation</u>. The pair of Banach spaces (B,C) is called admissible (cf. [4], p. 127), if for every $b \in B$ there exists at least one bounded solution of (2). Then by Theorem 51.E of [4] there exists a constant $K > 0$

such that for every $b \in B$ the equation (2) has a unique bounded solution x with $x(0) \in E_1$ and $\| x\|_C \leqq K \| b\|_B$. Moreover, by Theorem 52.J of [4] for every $b \in B$ with compact support the unique bounded solution x of (2) with $x(0) \in E_1$ and $\| x\|_C \leqq K \| b\|_B$ is represented by $x(t) = \int_J G(t,s)b(s)ds$.

<u>Theorem 1</u>. Let (B,C) be admissible. Then there exists a constant $K > 0$ such that, for any $t \in J$, $G(t,\cdot) \in B^*(E)$ and $\| G(t,\cdot)\|_{B^*(\tilde{E})} \leqq K$.

Proof. Let $b \in B$ with compact support. Suppose that b vanishes for $t > T$, where T is arbitrarily fixed. By the remarks preceding the theorem, there exists a constant $K > 0$ such that for any $t \in J$

$$\left\| \int_0^T G(t,s)b(s)ds \right\| \leqq K \| b\|_B .$$

However, for any $x^* \in E^*$, $\| x^*\| \leqq 1$, and any $t \in J$

$$\begin{aligned} \left| \int_0^T (b(s), G^*(t,s)x^*)ds \right| &\leqq \left| \left(\int_0^T G(t,s)b(s)ds, x^*\right)\right| \\ &\leqq \| x^* \| \left\| \int_0^T G(t,s)b(s)ds \right\| \\ &\leqq K \| b\|_B \end{aligned}$$

and as $G^*(t,\cdot)x^* \in L(E^*)$ for $t \in J$, Theorem 22.U of [4] implies that $G^*(t,\cdot)x^* \in B^*(E^*)$ for $t \in J$ and

$$\| G^*(t,\cdot)x^* \|_{B^*(E^*)} \leqq K \text{ for } t \in J.$$

Therefore, we obtain, for any $t \in J$,

$$\begin{aligned} \| G(t,\cdot)\|_{B^*(\tilde{E})} &= \| G^*(t,\cdot)\|_{B^*(\tilde{E}^*)} \\ &= \sup \{\| G^*(t,\cdot)x^* \|_{B^*(E^*)} : x^* \in E^*, \| x^*\| \leqq 1\} \\ &\leqq K. \end{aligned}$$

<u>Remark 1</u>. Theorem 1 generalizes the results of Coppel

[2] for $B = C(R^n)$ and $B = L^1(R^n)$, Conti [1] for $B = L^p(R^n)$, $1 \leqq p \leqq \infty$, and Szufla [5] for $B = L_\Phi(E)$ (Orlicz spaces).

In particular, the above theorem implies that if (B,C) is admissible, then the (Bochner) integral $\int_J G(t,s)b(s)ds$ exists for any $b \in B$.

<u>Theorem 2</u>. Let (B,C) be admissible. If $b \in B$, then a function $x:J \longrightarrow E$ is a bounded solution of (2) if and only if

$$x(t) = U(t)Px(0) + \int_J G(t,s)b(s)ds.$$

<u>Proof</u>. Since the sufficiency is easily seen to hold, we will only prove the necessity. So, let x a bounded solution of (2) and let $b \in B$. Writing

$$y(t) = x(t) - U(t)Px(0) - \int_J G(t,s)b(s)ds,$$

it is clear that y is a bounded solution of (1) with

$$y(0) = x(0) - Px(0) + (I-P) \int_J U^{-1}(s)b(s)ds,$$

i.e. $y(0) \in E_1$. Therefore, $y = 0$.

4. <u>The Nonlinear Equation</u>. Consider the nonlinear equation (3), where we assume that $f:J \times S_a \longrightarrow E$, $0 < a \leqq \infty$, is such that

(f1) $f(t,x)$ is strongly measurable in t for all $x \in S_a$ and continuous in x for $t \in J$;

(f2) $f(\cdot,0) \in B$.

Let $\omega : J \times [0,2a) \longrightarrow R$ be such that

(ω1) $\omega(\cdot,r) \in B(R)$ for all $r \in [0,2a)$;

(ω2) $\omega(t,r)$ is continuous nondecreasing in r for $t \in J$;

and defining $\Omega: [0,2a) \longrightarrow R$ by $\Omega(r) = K \|\omega(\cdot,r)\|_{B(R)}$ (where K as in Theorem 1) we assume

(ω3) $r = 0$ is the only fixed point of Ω in $[0,2a)$;

(ω4) for each $r \in [0,2a)$, $\Omega(r) \leqq r$.

Theorem 3. Let (B,C) be admissible. Suppose that f satisfies (f1) and (f2) and that there exists a function ω satisfying (ω1)-(ω4) such that for any $t \in J$ and $x,y \in S_a$

$$(4) \qquad \|f(t,x) - f(t,y)\| \leqq \omega(t, \|x - y\|).$$

Then, if

$$(5) \qquad \|f(\cdot,0)\|_B < K^{-1}(a - \Omega(a)),$$

there exists, for any $\xi \in E_0$ such that

$$\|\xi\| < b = S^{-1}(a - \Omega(a) - K\|f(\cdot,0)\|_B),$$

a unique bounded solution $x(\cdot;\xi)$ of (3) such that $x(\cdot;\xi) \in \Sigma_a$ and $Px(0;\xi) = \xi$. Moreover, the mapping $\xi \mapsto x(0;\xi)$ is continuous in $F_0 = \{\xi \in E_0 : \|\xi\| < b\}$ and it can be extended to a homeomorphism H of $F_0 + E_1$ onto itself which leaves the affine subspaces $\xi + E_1$, $\xi \in F_0$, invariant.

Proof. First we remark that if $z \in \Sigma_a$, then $f(\cdot,z) \in B$ and

$$\|f(\cdot,z)\|_B < \|\omega(\cdot,a)\|_{B(R)} + \|f(\cdot,0)\|_B.$$

Let $\rho = a^{-1}(\Omega(a) + K\|f(\cdot,0)\|_B) < 1$. Let $\xi \in F_0$ be given arbitrarily. Clearly we have

$$\|U(\cdot)\xi\|_C \leqq S\|\xi\| < Sb = (1 - \rho)a.$$

Consider the following sequence of successive approximations in C

$$z_1(t) = \int_J G(t,s)f(s,U(s)\xi)ds$$

$$z_{n+1}(t) = \int_J G(t,s)f(s,z_n(s) + U(s)\xi)ds, \quad n = 1,2,\ldots$$

Note that the above integrals exist (since (B,C) is admissible, according to Theorem 1), provided that they are all well defined. Indeed, it can be shown (inductively) that

$$\|z_n\|_C < \rho a, \quad n = 1,2,\ldots .$$

Now we define a sequence $\{r_n\}$ in $[0,2a)$ as it follows

$$r_1 = 2\rho a$$

$$r_{n+1} = \Omega(r_n), \quad n = 1,2,\ldots .$$

It is easily seen, using $(\omega 2)$, $(\omega 3)$ and $(\omega 4)$ that $\lim_{n\to\infty} r_n = 0$.

Moreover, once again by induction it can be shown that

$$\|z_{n+1} - z_n\|_C \leq r_n, \quad n = 1,2,\ldots .$$

Therefore, $\{z_n\}$ is a Cauchy sequence in C and there exists $z \in C$, $z = \lim_{n\to\infty} z_n$. Clearly $\|z\|_C \leq \rho a$. Consequently, the function $x(t;\xi) = U(t)\xi + z(t)$ would be bounded, since

$$\|x(\cdot;\xi)\|_C < (1-\rho)a + \rho a = a$$

and would solve the integral equation

$$x(t;\xi) = U(t)\xi + \int_J G(t,s)f(s,x(s;\xi))ds.$$

By a simple differentiation it results that $x(\cdot;\xi)$ is a bounded solution of (3) and

$$\begin{aligned} Px(0;\xi) &= P(\xi - (I-P)\int_J U^{-1}(s)f(s,x(s;\xi))ds) \\ &= P\xi = \xi . \end{aligned}$$

Furthermore, $x(\cdot;\xi)$ is the unique bounded solution of (3) with these properties. Indeed, let $\bar{x}(\cdot;\xi)$ be another bounded solution of (3) such that $\|\bar{x}(\cdot;\xi)\|_C < a$ and $P\bar{x}(0;\xi) = \xi$

and let $u(t) = \vartheta(x(t;\xi) - \bar{x}(t;\xi))$ for some fixed $\vartheta \in (0,1)$. Clearly u would solve the following integral equation

$$u(t) = \vartheta\int_J G(t,s)\{f(s,x(s;\xi)) - f(s,\vartheta x(s;\xi) - u(s))\}ds.$$

We define a sequence $\{\bar{r}_n\}$ in $[0,2a)$ by $\bar{r}_1 = 2\vartheta a$, $\bar{r}_{n+1} = \vartheta\Omega(\bar{r}_n)$, $n = 1,2,\ldots$. Clearly $\lim_{n\to\infty} \bar{r}_n = 0$. It is easily seen by induction that $\|u\|_C \leqq \bar{r}_n$, $n = 1,2,\ldots$, which implies $u = 0$.

Let $\varepsilon > 0$ be arbitrarily fixed ($\varepsilon < a$). If $f(\cdot,0) = 0$, we remark that what we have already shown implies that, for any $\xi \in E_0$, $\|\xi\| < S^{-1}(\varepsilon - \Omega(\varepsilon))$, there exists a unique bounded solution $x(\cdot;\xi)$ of (3) such that $\|x(\cdot;\xi)\|_C < \varepsilon$ and $Px(0;\xi) = \xi$.

Thus for any $\varepsilon > 0$ we put $\delta = S^{-1}(\varepsilon - \Omega(\varepsilon))$. Then for any $\xi, \eta \in E_0$ such that $\|\xi - \eta\| \leqq \delta$ the function $u(t) = x(t;\xi) - x(t;\eta)$ is a bounded solution of

$$u' = A(t)u + g(t,u),$$

where $g(t,u) = f(t,x(t;\xi)) - f(t,x(t;\xi) - u)$ satisfies (4) and $g(\cdot,0) = 0$. Since $\|Pu(0)\| < \delta$ the above remark implies that $\|u\|_C < \varepsilon$, i.e.

$$\|x(0;\xi) - x(0;\eta)\| \leqq \|x(\cdot,\xi) - x(\cdot;\eta)\|_C < \varepsilon ,$$

which shows the continuity of the mapping $\xi \mapsto x(0;\xi)$ of F_0 into itself.

Finally, the mapping H defined by

$$H(\xi) = x(0;P\xi) + (I-P)\xi ,$$

with the inverse

$$H^{-1}(\xi) = \xi - (I-P)x(0;P\xi),$$

extends the mapping $\xi \mapsto x(0;\xi)$ to a 1-1 mapping of $F_0 + E_1$ onto itself which leaves the affine subspace $\xi + E_1$ invariant and both it and its inverse are continuous. Therefore, it is a homeomorphism.

Remark 2. Theorem 3 is a generalization of the results of Massera-Schäffer [3] for $\omega(t,r) = \gamma(t)r$, $\gamma \in B(R)$, $K\|\gamma\|_{B(R)} < 1$, and of Szufla [6] for $\omega(t,r) = \gamma(t)\phi(r)$, $\phi(r)$ nondecreasing, $\phi(r) < r$, $\gamma \in B(R)$, $K\|\gamma\|_{B(R)} < 1$.

Given any subinterval I of J, we denote by χ_I the characteristic function of I, i.e. $\chi_I(t) = 1$ for $t \in I$ and $\chi_I(t) = 0$ for $t \in J \setminus I$. A function Banach space B is called lean (cf. [4], p. 48) if for any $b \in B$

$$\lim_{t\to\infty} \|\chi_{[t,\infty)} b\|_B = 0.$$

Theorem 4. Let (B,C) be admissible and f, ω satisfy (f1),(f2),(ω1)-(ω4),(4) and (5). If B is lean and B is not stronger than L^1, then for every bounded solution x of (3)

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

Proof. Theorem 3 guarantees the existence of bounded solutions of (3). We claim that if x is any bounded solution of (3), then x should solve the integral equation

$$x(t) = U(t)Px(0) + \int_J G(t,s)f(s,x(s))ds. \tag{6}$$

Indeed, writing $y(t) = x(t) - U(t)Px(0) - \int_J G(t,s)f(s,x(s))ds$, it is easy to see that y is a bounded solution of (1) such that

$$y(0) = x(0) - Px(0) + (I-P)\int_J U^{-1}(s)f(s,x(s))ds,$$

i.e. $y(0) \in E_1$. Therefore, $y = 0$, which proves our claim.

Since B is lean and $f(\cdot,x) \in B$ (as it has been shown in the proof of Theorem 3), there exists a $\tau_0 \in J$, so that for any $\varepsilon > 0$ arbitrarily fixed

$$\| \chi_{[t,\infty)} f(\cdot,x) \|_B < \varepsilon/2K, \text{ for } t \geqq \tau_0.$$

On the other hand, the assumption that B is not stronger than L^1 implies according to Theorem 62.D of [4] that there exist a positive valued function N defined on J and a positive constant ν such that every solution y of (1) with $y(0) \in E_0$ satisfies, for all $t \geqq t_0 \geqq 0$,

$$\| y(t) \| \leqq N(t_0) e^{-\nu(t-t_0)} \| y(t_0) \|$$

and the fundamental solution U of (1) satisfies

$$\| U(t)P \| \leqq N(0) e^{-\nu t}, \text{ for all } t \in J,$$

i.e.

$$\lim_{t\to\infty} \| U(t)P \| = 0.$$

Therefore, there exists a $\tau_1 \in J$ so that

$$\| U(t)P \| < \frac{\varepsilon}{2} \{ \| x(0) \| + \int_0^{\tau_0} \| U^{-1}(s) f(s,x(s)) \| ds \}^{-1} \text{ for } t \geqq \tau_1.$$

Consequently, (6) implies, for all $t \geqq \max\{\tau_0, \tau_1\}$,

$$\begin{aligned} \| x(t) \| &\leqq \| U(t)P \| \, \| x(0) \| + \| \int_J G(t,s) \chi_{[0,\tau_0]} f(s,x(s)) ds \\ &\quad + \int_J G(t,s) \chi_{[\tau_0,\infty)} f(s,x(s)) ds \| \\ &\leqq \| U(t)P \| \, \| x(0) \| + \| U(t)P \| \int_0^{\tau_0} \| U^{-1}(s) f(s, x(s)) \| ds \\ &\quad + K \| \chi_{[\tau_0,\infty)} f(\cdot,x) \|_B \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon , \end{aligned}$$

i.e. $\lim_{t\to\infty}\|x(t)\| = 0$.

<u>Remark 3</u>. If B is stronger than L^1, the above theorem holds if it is in addition assumed that $\lim_{t\to\infty}\|U(t)P\| = 0$.

<u>Remark 4</u>. Theorem 4 is a generalization of an analogous result of Coppel [2] for $B = L^{\infty}(R^n)$, $\omega(t,r) = \gamma r$, $K\gamma < 1$.

References

[1] R. CONTI: On the boundedness of solutions of ordinary differential equations, Funkcial. Ekvac. 9(1966), 23-26.

[2] W.A. COPPEL: Stability and Asymptotic Behavior of Differential Equations, Heath Math. Monographs, Boston (1965).

[3] J.L. MASSERA and J.J. SCHÄFFER: Linear differential equations and functional analysis, Ann. Math. 67(1958), 517-573.

[4] J.L. MASSERA and J.J. SCHÄFFER: Linear differential equations and function spaces, Academic, New York and London (1966).

[5] S. SZUFLA: On the boundedness of solutions of ordinary differential equations in Banach spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 17 (1969), 745-750.

[6] S. SZUFLA: On the boundedness of solutions of non-linear differential equations in Banach spaces, Ann. Soc. Math. Polon. Sér. I Comment. Math. XXI(1979), 381-387.

Department of Mathematics

Democritus University of Thrace

Xanthi, GREECE

(Oblatum 1.9. 1980)