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PRIME EXTENSIONS AND NEARNESS STRUCTURES

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Abstract: An extension Y of X is called a prime extension if the open trace filter on X for each point in Y-X is a prime open filter. An ultrafilter generated nearness space is one for which the closure of each near collection is contained in a near ultrafilter. These spaces are shown to be induced by prime extensions. When the nearness structure is also concrete then the prime strict extension that induced it can be recovered, up to a homeomorphism, using Herrlich's completion. The category of ultrafilter generated nearness spaces is bicoreflective in NEAR.

Key words: Nearness space, extensions, ultrafilter complete, coreflective subcategory, H-closed.

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An extension Y of X is called a prime extension if each open trace filter on X corresponding to a point in Y-X is a prime open filter. An ultrafilter generated nearness structure is one for which the closure of every near collection is contained in some near ultrafilter. This concept is introduced in this paper and it is shown that they play the same role in the study of prime extensions as the concrete nearness structures play in the study of strict extensions. Specifically, one can recover, up to the usual equivalence, a T_1 prime strict extension of a space as the completion of the space equipped with an ultrafilter generated nearness structure.

1. Preliminaries. Herrlich's completion of a nearness space was presented in [11]. A brief description of it appears in [4] which we provide here for the convenience of the reader. Let (X, ξ) be a nearness space and let Y be the set of all X-clusters A with empty adherence. Set $X^* = X \cup Y$. For each $A \subset X$, define $c\ell\, A = \{y \in Y: A \in y\} \cup c\ell_{\xi}\, A$. A nearness structure ξ^* is defined on X^* as follows: $\mathcal{B} \in \xi^*$ provided $\mathcal{A} = \{A \subset X$: there exists $B \in \mathcal{B}$ with $B \subset c\ell\, A\} \in \xi$. (X^*, ξ^*) is a complete nearness space with $c\ell_{\xi^*} X = X^*$. Also, for $A \subset X$, $c\ell_{\xi^*} A = c\ell\, A$.

Let (X, ξ) be a nearness space. For any $A \subset X$, $\bar{A}$ will denote the closure of A in X, even when X is embedded in a larger space. For any ultrafilter $\mathcal{F}$ on X, $\mathcal{O}(\mathcal{F})$ will denote the collection of all the open subsets of X that belong to $\mathcal{F}$. Also, let $\mathcal{G}(\mathcal{F}) = \{A \subset X: \bar{A} \cap F \neq \emptyset$ for each $F \in \mathcal{F}\}$. Then $\mathcal{G}(\mathcal{F}) = \{A \subset X: \bar{A} \in \mathcal{F}\}$. Define:

$Y = \{\mathcal{G}(\mathcal{F}) : \mathcal{F}$ is a free near ultrafilter in (X, ξ)

$X' = X \cup Y$

$c\ell\, A = \bar{A} \cup \{\mathcal{G}(\mathcal{F}) \in Y: A \in \mathcal{G}(\mathcal{F})\}$ for $A \subset X$

$c\ell_{X'}(A) = A \cup c\ell(A \cap X)$ for $A \subset X'$.

Define ξ' as those $\mathcal{A} \subset \mathcal{P}(X')$ such that $\bigcap c\ell_{X'} \mathcal{A} \neq \emptyset$ or $\{A \cap X: A \in \mathcal{A}\} \in \xi$. Then (X', ξ') is an ultrafilter completion of (X, ξ), [9].

2. Basic construction

Theorem 2.1. Let (X,t) be a symmetric topological space. Let $\mathcal{G}$ be a collection of free ultrafilters on X. Define $\xi(\mathcal{G})$ by

$\xi(\mathcal{G}) = \{\mathcal{A} \subset \mathcal{P}(X) : \cap \bar{\mathcal{A}} \neq \emptyset$ or there exists $\mathcal{F} \in \mathcal{G}$ with $\mathcal{A} \subset \mathcal{G}(\mathcal{F})\}$.

Then $\xi(\mathcal{G})$ is a nearness structure on X compatible with the topology t.

Proof. For each $A \subset X$ it is apparent that $c\ell_{\xi(\mathcal{G})}A = c\ell_X A$. Axioms (N1) and (N4) are clearly satisfied. Axiom (N2). Suppose $\bar{\mathcal{A}} \in \xi$; then either $\cap \bar{\bar{\mathcal{A}}} \neq \emptyset$, which implies that $\cap \bar{\mathcal{A}} \neq \emptyset$ and $\mathcal{A} \in \xi$ or there exists $\mathcal{F} \in \mathcal{G}$ with $\bar{\mathcal{A}} \subset \mathcal{G}(\mathcal{F})$ which implies that $\mathcal{A} \subset \mathcal{G}(\mathcal{F})$ and hence $\mathcal{A} \in \xi$.

Axiom (N3). Suppose $\mathcal{A} \notin \xi$ and $\mathcal{B} \notin \xi$. Then $\cap \overline{\mathcal{A} \vee \mathcal{B}} = \emptyset$. Let $\mathcal{F} \in \mathcal{G}$, then there exists $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\bar{A} \notin \mathcal{F}$ and $\bar{B} \notin \mathcal{F}$. Thus $\overline{A \cup B} = \bar{A} \cup \bar{B} \notin \mathcal{F}$ since $\mathcal{F}$ is an ultrafilter. Therefore $\mathcal{A} \vee \mathcal{B} \notin \xi$.

Definition 2.2. A nearness space (X, ξ) is called ultrafilter generated if there exists a collection $\mathcal{G}$ of free ultrafilters on X such that $\xi = \xi(\mathcal{G})$ as defined in theorem 2.1.

Let (X, ξ) be a nearness space. The following statements are equivalent.

(1) ξ is ultrafilter generated.

(2) (X', ξ') is topological.

(3) For each $\mathcal{A} \in \xi$ there exists a near ultrafilter $\mathcal{F}$ such that $\bar{\mathcal{A}} \subset \mathcal{F}$.

The above result is found in [9] and the following comment follows easily from the definitions.

Let (X, ξ) be an ultrafilter generated nearness space. Then $\mathcal{A} \in \xi$ implies $\bar{\mathcal{A}}$ has the finite intersection property.

Theorem 2.3. Let (X, ξ) be a nearness space. Then the following conditions are equivalent:

(1) (X, ξ) is concrete and ultrafilter generated.

(2) $\xi = \xi(\mathcal{S})$ for some collection $\mathcal{S}$ of free ultrafilters satisfying the following equivalent conditions:

(A) If $\mathcal{F}$ and $\mathcal{H}$ belong to $\mathcal{S}$ then $\mathcal{F} \not\subset \mathcal{G}(\mathcal{H})$ and $\mathcal{H} \not\subset \mathcal{G}(\mathcal{F})$.

(B) If $\mathcal{F}$ and $\mathcal{H}$ belong to $\mathcal{S}$ then $\mathcal{O}(\mathcal{F}) \not\subset \mathcal{H}$ and $\mathcal{O}(\mathcal{H}) \not\subset \mathcal{F}$.

Two ultrafilters $\mathcal{F}$ and $\mathcal{H}$ are said to have the open intersection property if each open set in $\mathcal{F}$ meets every open set in $\mathcal{H}$.

Corollary 2.4. Let (X, ξ) be an ultrafilter generated nearness space with $\xi = \xi(\mathcal{S})$.

(1) If $\mathcal{S}$ is finite then (X, ξ) is concrete.

(2) If for each pair $\mathcal{F}$ and $\mathcal{H}$, members of $\mathcal{S}$, we have that $\mathcal{F}$ and $\mathcal{H}$ do not have the open intersection property then (X, ξ) is concrete.

(3) If $\mathcal{O}(\mathcal{F})$ is an open ultrafilter for each $\mathcal{F} \in \mathcal{S}$ then (X, ξ) is concrete.

For each free ultrafilter $\mathcal{F}$, $\mathcal{G}(\mathcal{F})$ is a grill. Moreover, $\bar{A} \in \mathcal{G}(\mathcal{F})$ implies $A \in \mathcal{G}(\mathcal{F})$. Hence we have the following theorem.

Theorem 2.5. Every ultrafilter generated nearness space is bunch determined and hence subtopological.

Thus each ultrafilter generated nearness structure is induced by an extension. Indeed, each ultrafilter generated nearness structure is grill determined, [6] and [13].

3. Extensions. An extension Y of a space X is a space in which X is densely embedded. For notational convenience we

will assume that $X \subset Y$.

If Y is an extension of X then $\xi = \{\mathcal{A} \subset \mathcal{P}(X) : \bigcap cl_Y \mathcal{A} \neq \emptyset\}$ is called the nearness structure on X induced by Y.

Let (Y,t) be a topological space and $\overline{X} = Y$, $t(X)$ will denote the subspace topology on X. For each $y \in Y$, set $\mathcal{O}_y = \{O \cap X : y \in O \in t\}$. Then $\{\mathcal{O}_y : y \in Y\}$ is called the filter trace of Y on X.

Y will be called a prime (maximal) extension of X if for each $y \in Y-X$ it follows that $\mathcal{O}_y$ is a prime open filter (open ultrafilter).

Let t (strict) be the topology on Y generated by the base $\{O^* : O \in t(X)\}$ where $O^* = \{y \in Y : O \in \mathcal{O}_y\}$. Let t (simple) be the topology on Y generated by the base $\{O \cup \{y\} : O \in \mathcal{O}_y,\ y \in Y\}$. Then t (strict) and t (simple) are such that Y with either of these topologies is an extension of $(X,t(X))$, called a strict extension, or simple extension of X, respectively. Note that

$$t(\text{strict}) \leq t \leq t(\text{simple}).$$

Moreover, a topology s on Y with the same filter trace as t, forms an extension of $(X,t(X))$ if and only if it satisfies the above inequality (see Banaschewski [1]).

The following lemma, providing the crucial link between the trace filters of an extension and an ultrafilter generated nearness structure, is due to Frolík [10].

Lemma 3.1. Let (X,t) be a topological space and $\mathcal{O}$ a prime open filter on X. Then there exists an ultrafilter $\mathcal{F}$ such that $\mathcal{O}(\mathcal{F}) = \mathcal{O}$.

The following important theorem is due to Bentley and Herrlich [4].

Theorem A. For any T_1 nearness space the following conditions are equivalent:

(1) ξ is a nearness structure induced on X by a strict extension.

(2) The completion (X^*, ξ^*) of (X, ξ) is topological.

(3) Every nonempty X-near collection is contained in some X-cluster.

Recall a nearness space satisfying the above equivalent conditions is called concrete. Essentially the same relationship that exists for concrete nearness spaces and their completions exists between ultrafilter generated nearness spaces and their ultrafilter completions. This is demonstrated in the following theorem.

Theorem 3.2. For any nearness space (X, ξ) the following conditions are equivalent:

(1) ξ is induced on X by a prime extension.

(2) The ultrafilter completion (X', ξ') of (X, ξ) is topological.

(3) ξ is ultrafilter generated.

Proof. (2) and (3) are equivalent by section 2. (2) implies (1). By [9], we have for each $\mathcal{G}(\mathcal{F}) \in X'-X$ that the collection $\{\{\mathcal{G}(\mathcal{F})\} \cup O : O \in \mathcal{F}\}$ is a base for the open sets in X' containing $\mathcal{G}(\mathcal{F})$. Thus $\mathcal{O}_{\mathcal{G}(\mathcal{F})} = \mathcal{O}_{\mathcal{F}}$, and since $\mathcal{F}$ is an ultrafilter, $\mathcal{O}_{\mathcal{F}}$ is a prime open filter.

(1) implies (3). Let Y be a prime extension and $\xi = \{\mathcal{A} \subset \mathcal{P}(X) : \bigcap cl_Y \mathcal{A} \neq \emptyset\}$. For $y \in Y-X$ we have that $\mathcal{O}_y$, the trace filter on X, is a prime open filter. By lemma 3.1, there exists an ultrafilter $\mathcal{F}_y$ on X with $\mathcal{O}(\mathcal{F}_y) = \mathcal{O}_y$. Let $\mathcal{S} =$

$= \{\mathcal{F}_y : y \in Y-X\}$ and let $\xi(\mathcal{S})$ be the nearness structure on X generated by $\mathcal{S}$. Let $\mathcal{A} \in \xi$. Then there exists $t \in \bigcap c\ell_Y \mathcal{A}$. If $t \in X$ then $\mathcal{A} \in \xi(\mathcal{S})$. Otherwise $t \in Y-X$ and $\mathcal{F}_t \in \mathcal{S}$. Now let $A \in \mathcal{A}$. Since $t \in c\ell_Y A$, we have that $(Y-c\ell_Y A) \cap X$ does not belong to $\mathcal{O}_t = \mathcal{O}(\mathcal{F}_t)$. Hence $c\ell_Y A \cap X \in \mathcal{F}_t$, and $\xi \subset \xi(\mathcal{S})$.

To show that $\xi(\mathcal{S}) \subset \xi$ it suffices to show that $\mathcal{F}_y \in \xi$ for each $y \in Y-X$. But, since $\mathcal{O}_y = \mathcal{O}(\mathcal{F}_y)$ it follows that $y \in$ $\in \bigcap c\ell_Y \mathcal{F}_y$ and hence $\mathcal{F}_y \in \xi$. (Note: $\mathcal{G}(\mathcal{F}_y)$ now belongs to ξ by axiom N2.) Hence $\xi(\mathcal{S}) = \xi$.

The following theorem gives a slightly stronger result for strict T_1 extensions.

<u>Theorem 3.3.</u> Let Y be a T_1 strict extension of X. Let ξ be the induced nearness structure on X. Then the following conditions are equivalent:

(1) Y is a prime extension of X.

(2) ξ is ultrafilter generated.

<u>Proof</u>. (1) implies (2) by theorem 3.2. (2) implies (1). Let $y \in Y-X$. We must show that $\mathcal{O}_y$ is a prime open filter. Let $\mathcal{A}_y = \{A \subset X : y \in c\ell_Y A\}$. Then $\mathcal{A}_y \in \xi$. Since Y is a strict T_1 extension of X it follows that $\{y\} = \bigcap c\ell_Y \mathcal{A}_y$. Hence $\bigcap c\ell_X \mathcal{A}_y =$ $= \emptyset$, and since ξ is ultrafilter generated there exists a free near ultrafilter $\mathcal{F}$ with $\mathcal{A}_y \subset \mathcal{G}(\mathcal{F})$. Now $\mathcal{F} \in \xi$, the nearness structure induced by Y, and hence we have $\emptyset \neq \bigcap c\ell_Y \mathcal{F} \subset \bigcap c\ell_Y \mathcal{A}_y =$ $= \{y\}$. Thus $\mathcal{O}_y \subset \mathcal{O}(\mathcal{F})$.

Let $O \in \mathcal{O}(\mathcal{F})$. Then $O^* = O \cup \{z \in Y-X : O \in \mathcal{O}_z\}$ is open in Y. Now $X-O \notin \mathcal{F}$ and hence $X-O \notin \mathcal{A}_y$. Hence there exists Q, open in Y and containing y, such that $Q \cap (X-O) = \emptyset$. Hence $Q \cap X \subset O$, and thus $O \in \mathcal{O}_y$. Therefore $\mathcal{O}_y = \mathcal{O}(\mathcal{F})$, a prime open filter.

Theorem 3.4. For any T_1 nearness space (X,ξ) the following conditions are equivalent:

(1) ξ is induced on X by a prime strict extension.

(2) The completion (X^*, ξ^*) of (X,ξ) is topological and X^* is a prime extension of X.

(3) ξ is concrete and ultrafilter generated.

Proof. The proof follows immediately from theorem A and theorem 3.3.

Two extensions Y and Z of X are called equivalent if there exists a homeomorphism $f:Y \to Z$ such that f restricted to X is the identity map. The following theorem will be useful for the work in the next section.

Theorem 3.5. Let (Y,t) be a prime strict T_1 extension of X. For each $y \in Y-X$ choose a free ultrafilter $\mathcal{F}_y$ on X such that $\mathcal{O}(\mathcal{F}_y) = \mathcal{O}_y$. Set $\mathcal{G} = \{\mathcal{F}_y : y \in Y-X\}$ and $\xi = \xi(\mathcal{G})$. Then (Y,t) is equivalent to $(X^*, t(\xi^*))$.

Proof. Let ξ_0 be the nearness structure induced on X by Y. Then, as shown in the proof of theorem 3.2, $\xi_0 = \xi = \xi(\mathcal{G})$. Hence X^* and Y are strict T_1 extensions of X, generating the same nearness structure. By corollary 2.12 in [4] it follows that (Y,t) and $(X^*, t(\xi^*))$ are equivalent extensions.

We call an extension Y of X maximal if the trace filters corresponding to points in $Y-X$ are open ultrafilters. Hence each maximal extension is a prime extension.

Lemma 3.6. Let Y and S be extensions of X with $f:Y \to Z$ satisfying:

(1) f restricted to X is the identity map.

(2) f is one-to-one and onto.

(3) f is continuous

(4) Z is a maximal extension of X.

Then:

(A) Y and Z generate the same collection of trace filters on X.

(B) If Y and Z are both strict (simple) extensions of X then f is a homeomorphism.

Proof. f establishes a one-to-one correspondence between the trace filters generated by Y and those generated by Z. Suppose $f(y) = z$, the $\mathcal{O}_z \subset \mathcal{O}_y$ since f is continuous. Since Z is a maximal extension of X, $\mathcal{O}_z$ is an open ultrafilter and $\mathcal{O}_z = \mathcal{O}_y$. Part (B) then follows at once.

If $\xi = \xi(\mathcal{G})$, where $\mathcal{O}(\mathcal{F})$ is an open ultrafilter for each $\mathcal{F} \in \mathcal{G}$, then by corollary 2.4 ξ is a concrete nearness structure. Thus, results similar to those obtained for prime extensions can be combined as the following theorem indicates. The proof, essentially the same as the proofs for the corresponding results already presented is omitted.

Theorem 3.7. For any T_1 nearness space (X, ξ) the following conditions are equivalent:

(1) ξ is induced on X by a maximal extension.

(2) The ultrafilter completion (X', ξ') of (X, ξ) is topological and X' is a maximal extension of X.

(3) $\xi = \xi(\mathcal{G})$ for some collection $\mathcal{G}$, of free ultrafilters, such that $\mathcal{F} \in \mathcal{G}$ implies that $\mathcal{O}(\mathcal{F})$ is an open ultrafilter.

(4) The completion (X^*, ξ^*) of (X, ξ) is topological and X^* is a maximal extension of X.

(5) ξ is induced by a strict maximal extension.

4. Applications. The results of the previous sections are now applied to several special types of spaces.

Theorem 4.1. Let X be a Hausdorff topological space. Let $\mathcal{G}$ be the collection of all ultrafilters $\mathcal{F}$ on X such that $\mathcal{O}(\mathcal{F})$ is a free open ultrafilter on X. Set $\xi = \xi(\mathcal{G})$. Then $(X^*, t(\xi^*))$ is homeomorphic to the Fomin H-closed extension of X.

(Note: This theorem appears in slightly different form in [8] and [6]. It is included here to note that the Fomin H-closed extension can be constructed as the completion of an ultrafilter generated nearness space.)

For a completely regular space, Herrlich [11] has shown that there exists a compatible nearness structure on X such that the completion X^* with respect to this nearness structure is homeomorphic to βX, the Stone-Čech compactification of X. The following theorem characterizes the completely regular space X for which there exists an ultrafilter generated nearness structure such that the completion with respect to this structure is homeomorphic to βX.

Theorem 4.2. Let (X,t) be a completely regular topological space. The following statements are equivalent:

(1) Every maximal completely regular filter is prime.

(2) There exists a compatible ultrafilter generated nearness structure ξ on X such that $(X^*, t(\xi^*))$ is homeomorphic to βX.

Proof. A completely regular filter $\mathcal{F}$ is an open filter with a base $\mathcal{B}$ such that for each $V \in \mathcal{B}$ there exists $U \in \mathcal{B}$

and an $f: X \to [0,1]$, a continuous map, such that $f(U) = \{0\}$ and $f(X-V) = \{1\}$.

βX can be constructed as the strict extension on the set of all maximal completely regular open filters. (See Bourbaki [2].)

(2) implies (1). By theorem 3.3, X^* and hence βX is a prime extension of X. Thus the trace filters are prime.

(1) implies (2). Let $\mathcal{V} = \{\mathcal{O} : \mathcal{O}$ is a free maximal completely regular filter$\}$. Let $\mathcal{V}$ be indexed by Ω. That is, $\mathcal{V} = \{\mathcal{O}_\alpha : \alpha \in \Omega\}$. By hypothesis, each $\mathcal{O}_\alpha$ is a prime open filter. Then by theorem 3.1, there exists an ultrafilter $\mathcal{F}_\alpha$ with $\mathcal{O}(\mathcal{F}_\alpha) = \mathcal{O}_\alpha$, for each $\alpha \in \Omega$. Set $\mathcal{S} = \{\mathcal{F}_\alpha : \alpha \in \Omega\}$. Then $\xi = \xi(\mathcal{S})$ is an ultrafilter generated nearness structure.

Now, by theorem 3.5, the strict extension generated by the $\{\mathcal{O}_\alpha : \alpha \in \Omega\}$ is homeomorphic to $(X^*, t(\xi^*))$. That is: βX is homeomorphic to $(X^*, t(\xi^*))$.

Theorem 4.3. Every uniform ultrafilter generated nearness structure is induced by a paracompactification.

Proof. This follows immediately from the results in section 6 of [4] and the fact that each ultrafilter generated nearness structure is subtopological.

One can construct ultrafilter generated nearness structures by starting with a collection of free closed ultrafilters. This follows from the fact, due to Frolík [10], that if $\mathcal{F}$ is a free closed filter then there exists a free ultrafilter $\mathcal{F}'$ such that $\mathcal{G}(\mathcal{F}) = \mathcal{G}(\mathcal{F}')$. Hence, if $\mathcal{S}$ is a collection of free closed ultrafilters then there exists a collection of free ultrafilters $\mathcal{S}'$ such that $\xi(\mathcal{S}) = \xi(\mathcal{S}')$. Herrlich, in [12], gives a nearness structure for a space for which the corresponding completion is homeomorphic to the Wallman compactification for a Hausdorff space. Using the collection of free closed ultrafilters and the above comment, one can see that the nearness strucţure is ultrafilter generated and hence the Wallman compactification for a Hausdorff space is a prime extension. Similarly, the Stone-Čech compactification of a normal space is a prime extension.

A topological space is compact if and only if each prime open filter converges. Moreover, a Hausdorff topological space is H-closed if and only if each prime open filter clusters.

Let (X,t) be a Hausdorff topological space. Let M be the set of all free open ultrafilters on X. Set $Y = X \cup M$. The set Y with the simple extension topology is the Fomin H-closed extension of X, while the set Y with the strict extension topology is the Katětov H-closed extension of X. Both are prime extensions of X. By enlarging M to the set P, of all non-convergent prime open filters, we are able to construct a compactification of X that is also a prime extension of X. Set $Y = X \cup P$ and let ρX be the set Y with the topology generated by the base $\{0^*:0 \in t\}$ where $0^* = 0 \cup \{\mathcal{O} \in P: 0 \in \mathcal{O}\}$.

<u>Theorem 4.4.</u> Let (X,t) be a topological space. Then ρX is a prime compactification of X.

<u>Proof</u>. Easily ρX is a strict prime extension of X. To see that ρX is compact let $\mathcal{O}'$ be a prime open filter on ρX. Set $\mathcal{O} = \{0 \in t: 0^* \in \mathcal{O}'\}$. Then $\mathcal{O}$ is a prime open filter on X.

Case 1. Suppose $\mathcal{O} \to x \in X$. Then $\mathcal{N}(x) \subset \mathcal{O}$, where $\mathcal{N}(x)$ is the collection of all open sets in X containing x.

Then $\mathcal{B} = \{O^*: O \in \mathcal{N}(x)\}$ is a base for the open sets in ρX containing x. By the definition of $\mathcal{O}$ it follows that $\mathcal{B} \subset \mathcal{O}'$ and hence $\mathcal{O}' \longrightarrow x$ in ρX.

Case 2. Suppose $\mathcal{O}$ does not converge in X. Then $\mathcal{O} \in \mathbf{P}$. Let Q' be any open set in ρX containing $\mathcal{O}$. Then $Q = Q' \cap X$ is a nonempty open set in X. Since $Q^* \subset Q'$ and $Q \in \mathcal{O}$ it follows that $Q^* \in \mathcal{O}'$ and hence $Q' \in \mathcal{O}'$. Therefore, $\mathcal{O}' \longrightarrow \mathcal{O}$ in ρX.

Hence every prime open filter in X converges and thus it follows that ρX is compact.

5. Ultrafilter generated is bicoreflective in near. Let (X, ξ) be a nearness space and $\mathcal{S} = \{\mathcal{F}: \mathcal{F}$ is a free near ultrafilter on $X\}$. Then $\xi(\mathcal{S}) \subset \xi$ and $\xi(\mathcal{S})$ is compatible with the underlying topology on X.

Theorem 5.1. Let (X, ξ) be a nearness space and let (Y, η) be an ultrafilter generated nearness space. Let $f:(Y, \eta) \longrightarrow$ $\longrightarrow (X, \xi)$ be a nearness map. Then there exists a unique nearness map $\tilde{f}$ such that the following diagram commutes.

$$\begin{tikzcd}
(X, \xi) & (X, \xi(\mathcal{S})) \arrow[l, "i"'] \\
& (Y, \eta) \arrow[ul, "f"] \arrow[u, dashed, "\tilde{f}"']
\end{tikzcd}$$

Proof. Define $\tilde{f}$ by $\tilde{f}(y) = f(y)$ for each $y \in Y$. It suffices to show that $\tilde{f}$ is a near map. Let $\mathcal{B} \in \eta$. If $\bigcap cl_Y \mathcal{B} \neq$ $\neq \emptyset$ then $\bigcap cl_X f(\mathcal{B}) \neq \emptyset$ and $\tilde{f}(\mathcal{B}) \in \xi(\mathcal{S})$.

If $\bigcap cl_Y \mathcal{B} = \emptyset$ then there exists a near ultrafilter $\mathcal{F}$ such that $\mathcal{B} \subset \mathcal{G}(\mathcal{F})$. Let $\mathcal{R} = \{f(F): F \in \mathcal{F}\}$. If $\bigcap cl_X \mathcal{R} \neq \emptyset$, then

$\cap\, c\ell_X \tilde{f}(\mathcal{B}) \neq \emptyset$ and $\tilde{f}(\mathcal{B}) \in \xi(\mathcal{Y})$. If $\cap\, c\ell_X \mathcal{R} = \emptyset$ let $\mathcal{H} = \{A \subset X :$ $:$ there exists $F \in \mathcal{F}$ with $A \supset f(F)\}$. Then $\mathcal{H} \in \xi$ and $\mathcal{H}$ is a filter. To see that $\mathcal{H}$ is an ultrafilter let $A \cup B \in \mathcal{H}$. Then there exists $F \in \mathcal{F}$ with $A \cup B \supset f(F)$. Then $f^{-1}(A \cup B) \in \mathcal{F}$ and hence either $f^{-1}(A)$ or $f^{-1}(B)$ belongs to $\mathcal{F}$. Then, since $A \supset$ $\supset f(f^{-1}(A))$ and $B \supset f(f^{-1}(B))$, eihter A or B belongs to $\mathcal{H}$. Hence $\mathcal{H}$ is a near ultrafilter in (X, ξ) with empty adherence and thus $\mathcal{H} \in \mathcal{Y}$.

Let $B \in \mathcal{B}$. Then $c\ell_Y B \in \mathcal{F}$, and $f(c\ell_Y B) \in \mathcal{H}$. Now $c\ell_X(f(B)) \supset$ $\supset f(c\ell_Y B) \in \mathcal{H}$. Thus $f(\mathcal{B}) \subset G(\mathcal{H})$ and hence $\tilde{f}(\mathcal{B}) \in \xi(\mathcal{Y})$.

Corollary 5.2. The category of ultrafilter generated nearness spaces and nearness maps is bicoreflective in the category NEAR.

Proof. For each nearness space (X,ξ) the coreflection is given by $(X,\xi) \xleftarrow{1} (X, \xi(\mathcal{Y}))$. The result then follows by the above theorem.

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