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Otto Vejvoda; Marie Kopáčková

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PERIODIC SOLUTIONS OF ABSTRACT AND PARTIAL DIFFERENTIAL EQUATIONS WITH DEVIATION

O. VEJVODA, M. KOPÁČKOVÁ

Abstract: The existence and the uniqueness of a time-periodic solution of an abstract linear differential equation in a Hilbert space with deviating argument are studied. Certain parabolic and hyperbolic equations are investigated in detail.

Key words: Abstract equation, deviating argument, periodic solution.

Classification: 35B10

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1. Introduction. In this paper time periodic solutions of a certain abstract linear differential equation with unbounded operator and with deviating argument are dealt with. In the second section we prove a general existence theorem whose disadvantage is that it may be sometimes rather difficult to verify its assumptions. That is why the special abstract and partial differential equations of the first and second orders are investigated in the third and fourth sections. In contrast to ordinary differential equations with deviating argument a small attention has been payed till now to existence of periodic solutions of partial differential equations of this type. Previously, C. Monari ([1, 2]) has proved existence of periodic solutions of retarded parabolic

equations with a nonlinear term and V. Comincioli ([3],[4]) has studied linear parabolic equations with periodic deviation.

2. <u>Abstract equation of the order</u> p . Let A be a selfadjoint generally unbounded operator acting in a Hilbert space $H^{o}(\Omega)$ with a domain D(A) which has eigenvalues $\lambda_1 \leqq \lambda_2 \leqq \leqq \dots$ of finite multiplicity and a complete orthonormal system in $H^{o}(\Omega)$ of eigenfunctions $v_1, v_2, \dots$, where Ω is a bounded region in R^n. Let δ be a fixed real number. Putting $P_o(\lambda) = 1$, denote by $P_m(\lambda)$ (m = 1,2,...,p), $Q_m(\lambda)$ (m = = 1,2,...,q) polynomials with constant coefficients of the order p_m, q_m, respectively and set $H^k_{\omega}(R) = \{u \in H^k(R); u(t+\omega) = u(t),\ t \in R\}$ (and similarly for vector-functions). We discuss the operator $L: H^o_{\omega}(R; H^o(\Omega)) \longrightarrow H^o_{\omega}(R; H^o(\Omega))$ given by the following expression

$$Lu(t) = \sum_{m=0}^{p} P_m(A) \frac{d^{p-m}u}{dt^{p-m}}(t) + \sum_{m=0}^{q} Q_m(A) \frac{d^{q-m}u}{dt^{q-m}}(t-\delta)$$

with the domain

$$D(L) = \mathcal{U} = \Big(\bigcap_{m=0}^{p} H^{p-m}_{\omega}(R; D(A^{p_m}))\Big) \cap \Big(\bigcap_{m=0}^{q} H^{q-m}_{\omega}(R; D(A^{q_m}))\Big).$$

Clearly, $u \in \mathcal{U}$ iff $\sum_{\substack{j\in Z\\ k\in N}} \Big[\sum_{m=0}^{p} (j^{p-m}\lambda_k^{p_m})^2 + \sum_{m=0}^{q} (j^{q-m}\lambda_k^{q_m})^2\Big] |u_{jk}|^2 < +\infty$, where

$$u(t) = \sum_{\substack{j\in Z\\ k\in N}} u_{jk} \exp(ij\nu t) v_k, \tag{2.1}$$

Z is the set of integers, N is the set of positive integers and $\nu = 2\pi/\omega$.

We look for a function $u \in \mathcal{U}$ satisfying the equation

$$Lu(t) = g(t) \tag{2.2}$$

almost everywhere. The solution u is sought in the form (2.1). Writing

$$g(t) = \sum_{j\in Z, k\in N} g_{jk} \exp(ij\nu t)v_k \tag{2.3}$$

and inserting (2.1) and (2.3) into (2.2), we get the equations for u_{jk}

$$T_{jk}u_{jk} = g_{jk},\ j\in Z,\ k\in N, \tag{2.4}$$

where $T_{jk} = \sum_{m=0}^{p} P_m(\lambda_k)(ij\nu)^{p-m} + \sum_{m=0}^{q} Q_m(\lambda_k)(ij\nu)^{q-m}\exp(ij\nu\delta)$.

Evidently, one necessary condition for the existence of a solution $u \in \mathcal{U}$ is

$$g_{jk} = 0 \text{ for } (j,k)\in S_1 = \{(j,k)\in Z\times N;\ T_{jk} = 0\}. \tag{2.5}$$

Using the wellknown facts we get the following

Theorem 2.1. Let $g\in H_\omega^0(R;H^0(\Omega))$ satisfy (2.5). Then (2.2) has a solution $u\in\mathcal{U}$ iff

$$\sum_{(j,k)\in(Z\times N)\setminus S_1} \left[\sum_{m=0}^{p}(j^{p-m}\lambda_k^{p_m})^2 + \sum_{m=0}^{q}(j^{q-m}\lambda_k^{q_m})^2\right] |T_{jk}|^{-2}|g_{jk}|^2 < +\infty . \tag{2.6}$$

If (2.6) is satisfied, then every solution of (2.2) is the sum of the general solution $u_0 \in \mathcal{U}$ of the equation $Lu_0 = 0$, $u_0(t) = \sum_{(j,k)\in S_1} u_{jk}\exp(ij\nu t)v_k$, and of particular solution

$$u_1(t) = \sum_{(j,k)\in(Z\times N)-S_1} g_{jk}T_{jk}^{-1}\exp(ij\nu t)v_k .$$

The solution is unique iff $S_1 = \emptyset$.

Remark 2.1. In this paper we do not deal with the weakly nonlinear problem

(2.7) $$Lu = \varepsilon f(u).$$

Let us note briefly how to solve this problem. Let $S_1 = \emptyset$, then assuming f to be a continuous and Lipschitzian mapping from $\mathcal{U}$ into the space of g's (for which there exists a solution of the corresponding linear problem) we can prove easily the existence of a solution to (2.7) for sufficiently small ε (making use of a fixed point theorem). Let $S_1 \neq \emptyset$. Then the problem is equivalent to a system consisting of the auxiliary equation and of the bifurcation equations

$$\int_0^{\omega} (f(u(t)), v_k)_{H^0(\Omega)} \exp(-ij\nu t)dt = 0, \quad (j,k) \in S_1$$

and this may be solved, for instance, by an implicit function theorem.

3. <u>The first order equation</u>. Let the equation

(3.1) $$Lu \equiv u_t(t) + Au(t) + \alpha Au(t-\delta) + \gamma u(t-\delta) = g(t)$$

be given. Assume that the operator A is selfadjoint and bounded from below, α, γ are real constants and $\mathcal{U} \equiv H^0_{\omega}(R;D(A)) \cap H^1_{\omega}(R;H^0(\Omega))$. The coefficients u_{jk} of an ω-periodic solution $u \in \mathcal{U}$ of (3.1) must satisfy (2.4), where

(3.2) $$T_{jk} = ij\nu + \lambda_k + (\alpha\lambda_k + \gamma)e^{-ij\nu\delta}.$$

<u>Theorem 3.1.</u>

(a) If $|\alpha| < 1$, then S_1 is finite and the solution $u \in \mathcal{U}$ of the equation (3.1) exists for every $g \in H^0_{\omega}(R;H^0(\Omega))$ satisfying (2.5).

(b) If $\alpha = 1$ and $\gamma = 0$, then S_1 is finite. If $\alpha = -1$ and $\gamma = 0$, then $S_1 = \{(0,k);\ k = 1,2,\dots\}$ is infinite.

In both cases the solution $u \in \mathcal{U}$ of (3.1) exists for $g \in H^0_\omega(R;D(A^2))$ satisfying (2.5).

(c) If $|\alpha| \neq 1$ and $\alpha\gamma < 0$, then S_1 is finite and the solution $u \in \mathcal{U}$ of (3.1) exists for $g \in H^0_\omega(R;D(A))$ satisfying (2.5).

Proof. By (3.2), $T_{jk} = 0$ only if

$$\lambda_k^2 + j^2\nu^2 = (\alpha\lambda_k + \gamma)^2$$

which implies immediately the assertion of the finiteness of the set S_1 in the cases(a) and (c). In (b) the equality $T_{jk} = 0$ is of the form

$$\lambda_k[1 \pm \cos(j\nu\sigma)] = 0; \; j\nu \mp \lambda_k \sin(j\nu\sigma) = 0$$

from which the form of the set S_1 is evident. The other assertions of Theorem 3.1 follow from the estimate

$$|T_{jk}|^2 = \lambda_k^2 + j^2\nu^2 + (\alpha\lambda_k + \gamma)^2 + 2(\alpha\lambda_k + \gamma)$$
$$[\lambda_k \cos(j\nu\sigma) - j\nu \sin(j\nu\sigma)] \geq \lambda_k^2 + j^2\nu^2 +$$
$$+ (\alpha\lambda_k + \gamma)^2 - 2|\alpha\lambda_k + \gamma|(\lambda_k^2 + j^2\nu^2)^{1/2} =$$
$$= [(\lambda_k^2 + j^2\nu^2)^{1/2} - |\alpha\lambda_k + \gamma|]^2 =$$
$$= [(1-\alpha^2)\lambda_k^2 - 2\alpha\gamma\lambda_k + j^2\nu^2 - \gamma^2]^2 [(\lambda_k^2 +$$
$$+ j^2\nu^2)^{1/2} - |\alpha\lambda_k - \gamma|]^{-2}$$

and from (2.6).

Remark 3.1. If $|\alpha| = 1$, $\alpha\gamma > 0$, the set S_1 is determined by the equations

$$\lambda_k^2 + j^2\nu^2 = (\alpha\lambda_k + \gamma)^2, \; \mathrm{tg}(j\nu\sigma) = -\frac{j\nu}{\lambda_k} \; (\lambda_k > 0).$$

Hence S_1 is finite for $\nu\delta\pi^{-1}$ rational. For irrational $\nu\delta\pi^{-1}$ we have not been able to decide whether S_1 is finite or not. If $|\alpha| > 1$, then S_1 is either finite (e.g. $S_1 = \{(0,k);\ \lambda_k = 0\}$ for $Lu(t) \equiv u_t(t) + Au(t) + \alpha Au(t-\delta)$, $\nu\delta = 2\pi$) or infinite (e.g. $S_1 = \{(\pm j,k),\ j = 12m(m+1) + 3,$ $k = 2(2m+1)\ m = 0,1,2,\dots\}$ for $Lu(t) \equiv u_t(t) - u_{xx}(t) - \sqrt{2}u_{xx}(t-\pi/4)$, $u(t,0) = u(t, 2\pi 3^{-1/2}) = 0$, $\omega = 2\pi$).

4. <u>The second order equation</u>. Another special case of (2.2) which is dealt with here is the equation

(4.1) $u_{tt}(t) + au_t(t) + Au(t) + Au(t-\delta) + \gamma u(t-\delta) = g(t)$,

where A is a selfadjoint operator in $H^0(\Omega)$ with eigenvalues $-\infty < \lambda_1 \leqq \lambda_2 \leqq \dots$ of finite multiplicities and a complete orthonormal system in $H^0(\Omega)$ of eigenfunctions $v_1, v_2, \dots$. Concerning an ω-periodic solution $u \in \mathcal{U} = H^0_\omega(R;D(A)) \cap H^2_\omega(R;H^0(\Omega))$ we will prove three theorems.

<u>Theorem 4.1</u>. Let $\alpha = 0$ and $g \in H^1_\omega(R;H^0(\Omega)) \cup H^0_\omega(R;D(A^{1/2}))$. Then S_1 is finite and the solution $u \in \mathcal{U}$ of (4.1) exists iff $g_{jk} = 0$ for $(j,k) \in S_1$.

Proof. Since

(4.2) $$T_{jk} = \lambda_k - \nu^2 j^2 + a\nu ji + \gamma \exp(\nu\delta ji)$$

we can write the estimate

$$(\lambda_k^2 + \nu^4 j^4)|u_{jk}|^2 = (\lambda_k^2 + \nu^4 j^4)|g_{jk}|^2 |T_{jk}|^{-2} \leqq$$

$$\leqq (\lambda_k^2 + \nu^4 j^4)\{[(\lambda_k - \nu^2 j^2)^2 + (a\nu j)^2]^{1/2} - |\gamma|\}^{-2} |g_{jk}|^2 \leqq$$

$$\leqq \text{const.}(\lambda_k^2 + \nu^4 j^4)\,[(\lambda_k - \nu^2 j^2)^2 + (a\nu j)^2]^{-1} \cdot |g_{jk}|^2$$

for j, λ_k being sufficiently large. Now, putting $\nu^2 j^2 = \Theta \lambda_k$ let us investigate the term $(\lambda_k - \nu^2 j^2)^2 + (a\nu j)^2$ for Θ from the intervals $[0,1-\varepsilon]$, $[1-\varepsilon,1+\varepsilon]$, $[1+\varepsilon,+\infty)$, respectively (ε is a fixed positive number). In the first and third intervals it holds evidently

$$(\lambda_k^2 + \nu^4 j^4)|u_{jk}|^2 \leqq c|g_{jk}|^2,$$

whereas in the second the following inequalities

$$(\lambda_k^2 + \nu^4 j^4)|u_{jk}|^2 \leqq cj^2|g_{jk}|^2, \quad (\lambda_k^2 + \nu^4 j^4)|u_{jk}|^2 \leqq c|\lambda_k|\,|g_{jk}|^2$$

hold. Thus the sum $\sum_{j,k} (\lambda_k^2 + \nu^4 j^4)|u_{jk}|^2$ converges if at least one of the sums $\sum_{j,k} |\lambda_k|\,|g_{jk}|^2$, $\sum_{j,k} j^2|g_{jk}|^2$ converges. Since $T_{jk} = 0$ implies

$$(\lambda_k - \nu^2 j^2) + (a\nu j)^2 = \gamma^2,$$

S_1 must be finite.

In the cases $\alpha a \neq 0$, $a = 0$ we cannot say more than in Theorem 2.1. Better results may be obtained for special values of ν, σ and λ_k. Let us state two of them which may be proved easily.

<u>Theorem 4.2</u>. Let $\nu\sigma = \pi$, $|\alpha| < 1$, $a > 0$ and $g \in H^1_\omega(R;H^0(\Omega)) \cup H^0_\omega(R;D(A^{1/2}))$. If $g_{ok} = 0$ for every k: $\lambda_k = 0$, then there exists a solution $u \in \mathcal{U}$.

<u>Theorem 4.3.</u>

(1) If $a = 0$, $\alpha = 0$, $\min_{(j,k)\in Z\times N} |\lambda_k - \nu^2 j^2| > |\gamma|$, then $S_1 = \emptyset$ and the solution $u \in \mathcal{U}$ of (4.1) exists for every $g \in \mathcal{U}$ and is unique.

(2) If $a = 0$, $|\alpha| < 1$, $\lambda_k = k^2$, $\gamma = 0$, $\nu\sigma = \pi$, $(1-\alpha)^{1/2}\nu^{-1}$ is rational, then S_1 is infinite, $S_1 =$

$= \{(j,k);\ jk^{-1} = (1-\alpha)^{1/2}\nu^{-1}$ and the solution $u \in \mathcal{U}$ of (4.1) exists for $g \in H^1_\omega(R;H^0(\Omega)) \cup H^0_\omega(R;D(A^{1/2}))$, $g_{jk} = 0$ for $(j,k) \in S_1$.

(3) If $a = 0$, $|\alpha| < 1$, $\lambda_k = k^2$, $\gamma = 0$, $\nu\delta = \pi$, $(1-\alpha)^{1/2}\nu^{-1}$ is irrational and $|(1-\alpha)^{1/2}\nu^{-1} - jk^{-1}| \geqq \text{const. } k^{-2}$ for $(j,k) \in Z \times N$, then $S_1 = \emptyset$ and the solution $u \in \mathcal{U}$ of (4.1) exists for every $g \in \mathcal{U}$ and is unique.

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Matematický ústav ČSAV

Žitná 25, Praha 2

Československo

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