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REMOVING THE INTERIOR OF THE SPECTRUM

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Abstract: The problem under consideration is the following one. Let x be an element of a unital Banach algebra and denote by $\sigma(x)$ the spectrum of x with its holes filled in. Does there exist a commutative, unital Banach algebra in which x (and hence the algebra which it generates) can be isometrically embedded having the property that the spectrum of x in this new algebra is minimal in the sense that it is the boundary of $\sigma(x)$?

Key words: Interior of the spectrum, Banach algebra extension.

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If x is an element of a unital Banach algebra A, $\sigma_A(x)$ will denote the spectrum of x in A. The subscript is important as we shall be considering the spectrum of an element relative to different algebras which contain it. If A(x) denotes the closed unital subalgebra of A generated by x we shall write $\sigma_{A(x)}(x)$ simply as $\sigma(x)$; this set is well known to be $\sigma_A(x)$ with its holes filled in. As the spectral radius does not change relative to the algebra concerned we shall simply use r(x) to denote the spectral radius of x.

If B is an isometric Banach algebra extension of A and $x \in A$ it is well known that $\sigma_A(x) \supseteq \sigma_B(x)$ while $\partial\sigma_A(x) \subseteq$

$\subseteq \partial\sigma_B(x)$ where ∂ denotes the topological boundary. If A is commutative $\Sigma(A)$ will denote the Gelfand space of A.

An element z is called an approximate divisor of zero (ADZ) in a commutative Banach algebra A if $z \in A$ and there exists a sequence $y_n \in A$ such that $\|y_n\| = 1$ for each n and $y_n z \to o$ $(n \to \infty)$. It is well known that if $\lambda - z$ is an ADZ in A then $\lambda \in \sigma_A(z)$ and that for each $\lambda \in \partial\sigma_A(z)$, $\lambda - z$ is an ADZ in A. It has been proved by Šilov [7], in case A is singly generated and extended by Arens [1], to the case of a general commutative, unital A that an element z of A is invertible in some extension of A if, and only if, z is not an ADZ in A.

If x is an element of a unital Banach algebra A the <u>permanent</u> spectrum of x written $\sigma_{per}(x)$ is the subset of $\sigma(x)$ which is contained in $\sigma_B(x)$ for every commutative, isometric extension B of A(x). $\sigma_{per}(x)$ is the intersection of closed sets and is therefore closed. It follows from the Šilov-Arens result above that $\lambda \in \sigma_{per}(x)$ if, and only if, $\lambda - z$ is an ADZ in A(x). As we shall show for many elements x in many familiar Banach algebras $\sigma_{per}(x) = \partial\sigma(x)$ and if this is the case we say that $\sigma(x)$ has a <u>removable interior</u>. The question naturally arises whether all spectra have removable interiors. Šilov [7] has answered this question negatively, it has also been considered by Arens [1], and a simplified version of Šilov's example is given here. The paper concludes with a theorem of Zemánek on the cortex.

<u>Positive results</u>. Let Ω be a compact Hausdorff space. $C(\Omega)$ denotes the algebra of continuous functions on Ω in

the supremum norm and if A is a closed unital subalgebra of $C(\Omega)$ which separates the points of Ω , A is a <u>uniform algebra</u> on Ω .

<u>Proposition 1.</u> <u>If</u> x <u>is an element of a uniform algebra</u> $\sigma(x)$ <u>has a removable interior</u>.

<u>Proof.</u> Let x be an element of a uniform algebra A. Then $\Sigma(A(x))$ can be identified with $\sigma(x)$ and the algebra A(x) with the Gelfand transform algebra of complex-valued functions on $\sigma(x)$. Each element in this algebra is continuous on $\sigma(x)$ and analytic in its interior. Thus if $A(\sigma(X))$ denotes the algebra of all complex-valued functions continuous on $\sigma(x)$ and analytic on its interior, we have the isometric embeddings

$$A(x) \subseteq A(\sigma(x)) \subseteq C(\partial\sigma(x))$$

where the final extension is achieved by restricting each function to the boundary $\partial\sigma(x)$. The spectrum of x in $C(\partial\sigma(x))$ is just $\partial\sigma(x)$.

Note that it is impossible to remove the interiors of the spectra of all elements in a uniform algebra A by means of a single extension. Take $A = C(\mathbb{D})$ where $\mathbb{D}$ is the closed unit disc. It is easy to see that an element $x \in A$ is an ADZ in A if, and only if, x vanishes somewhere on $\mathbb{D}$. Thus the spectrum of every element in A is preserved in any extension of A.

A large class of operators on Hilbert space have spectra with removable interiors. Let H be a Hilbert space and let B(H) denote the Banach algebra of all bounded linear operators on H.

Proposition 2. Let T be a bounded linear operator on a Hilbert space then $\sigma(T)$ has a removable interior if any of the following conditions hold.

(i) T is a subnormal operator;

(ii) $\sigma_{B(H)}(T)$ is a spectral set for T;

(iii) $\sigma_{B(H)}(T) \supseteq \{\lambda : |\lambda| = \|T\|\}$;

(iv) $\sigma(T^*)$ has a removable interior.

Proof. In case of categories (i), (ii) and (iii) this consists of showing that T generates a uniform subalgebra of the algebra of all bounded linear operators on the Hilbert space.

(i) If T is subnormal then $\|p(T)\| = r(p(T))$ for each complex polynomial p ([5] page 106), thus the Gelfand transform is an isometry.

(ii) A subset ω of the complex plane is a spectral set for T if $\omega \supseteq \sigma_{B(H)}(T)$ and if $\|f(T)\| \leq \sup_{z \in \omega} |f(z)|$ for each rational function with poles off ω. If $\sigma_{B(H)}(T)$ is a spectral set for T then

$\|p(T)\| \leq \sup_{z \in \sigma_{B(H)}(T)} |p(z)| = r(p(T))$ for any polynomial p by the spectral mapping theorem and the reverse inequality is obvious.

(iii) A theorem due to von Neumann ([3] page 281) states that for each T the set $\{\lambda : |\lambda| \leq \|T\|\}$ is a spectral set for T. This set contains $\sigma(T)$ and by the hypothesis since $\sigma(T)$ contains no holes the reverse inclusion follows. Thus $\sigma(T)$ is spectral for T, so $\|p(T)\| \leq \sup_{|z| \leq 1} |p(z)| = \sup_{z \in \sigma(T)} |p(z)| = r(p(T))$ as above.

(iv) Suppose $\sigma_{per}(T^*) = \partial\sigma(T^*)$. If $\lambda \in \sigma_{per}(T)$, then $(\lambda - T)\, p_n(T) \longrightarrow 0$ for some polynomials $p_n(z)$ with $\|p_n(T)\| = 1$. If p is the polynomial $p(z) = a_0 + a_1 z + \ldots\ldots + a_n z^n$ let $\bar{p}(z) = = \bar{a}_0 + \bar{a}_1 z + \ldots\ldots \bar{a}_n z^n$. Thus $(\bar{\lambda} - T^*)\bar{p}_n(T^*) \longrightarrow 0$ and $\|\bar{p}_n(T^*)\| = 1$. Hence $\bar{\lambda} \in \sigma_{per}(T^*) = \partial\sigma(T^*)$, so $\lambda \in \partial\sigma(T)$.

More is true. Suppose we can show that there is an isometric extension B of $A(T^*)$ such that $\sigma_B(T^*) = \partial\sigma(T^*)$. Form the conjugate Banach algebra B_1 from the elements of B by redefining scalar multiplication $\lambda \cdot x = \bar{\lambda} x$ where λ is a complex number and $x \in B$. The obvious maps

$$A(T) \longrightarrow A(T^*) \longrightarrow B_1$$

are conjugate linear and isometric, thus their composition is a linear isometry and $\sigma_{B_1}(T) = \partial\sigma(T)$.

2. <u>Šilov's Example</u>

<u>Lemma 3</u>. Let $\{d_n\}_0^\infty$ <u>be a sequence of positive real numbers with</u> $d_0 = 1$ <u>such that for each</u> m, n

(1) $$d_{m+n} \leq d_m d_n.$$

<u>A necessary and sufficient condition that this sequence be extendable to a sequence of positive real numbers</u> $\{d_n\}_{-\infty}^\infty$ <u>satisfying condition</u> (1) <u>is that</u>

(2) $$\sup_{n \geq 0} (d_n/d_{n+1}) < \infty$$

<u>Proof</u>. If the extended sequence exists then for each $n \geq 0$

$$d_n/d_{n+1} \leq d_{-1}$$

hence condition (2) holds.

Conversely suppose condition (2) holds and put

$d_{-1} = \sup_{n \geq 0} (d_n/d_{n+1})$. Then, if $r, n \geq 0$

$$d_n \leq d_{-1} d_{n+1} \leq d_{-1}^2 \leq d_{n+2} \leq \cdots\cdots \leq d_{-1}^r d_{n+r}$$

thus

$$d_n/d_{r+n} \leq d_{-1}^r.$$

Now if we put $d_{-r} = \sup_{n \geq 0} (d_n/d_{r+n})$ for $r \geq 1$, we have $d_{-r} \leq d_{-1}^r < \infty$.

It remains to check condition (1) holds for all integers m, n. This is straightforward if m or n is positive, so assume $m, n < 0$. Then, for any $r \geq 0$,

$$\begin{aligned} d_{r-n-m} d_n d_m &\geq d_{r-m} d_m \quad \text{as } r-n-m \geq 0 \\ &\geq d_r \qquad \text{as } r-m \geq 0 . \end{aligned}$$

Thus $d_n d_m \geq \sup_{n \geq 0} (d_r/d_{r-n-m}) = d_{n+m}$.

Let $\{d_n\}_0^\infty$ be a sequence of positive reals with $d_0=1$ satisfying condition (1). If a_n $(n \geq 0)$ are complex numbers we define

$$A = \{\sum_0^\infty a_n x^n : \|\sum_0^\infty a_n x^n\| = \sum_0^\infty |a_n| d_n < \infty\}.$$

A is a Banach algebra generated by x and $\sigma_A(x) = \{\lambda : |\lambda| \leq r\}$ where

$$r = r(x) = \lim_{n \to \infty} \|x^n\|^{1/n} = \lim_{n \to \infty} d_n^{1/n}$$

Assume further that condition (2) holds and extend the sequence of weights to $\{d_n\}_{-\infty}^\infty$. Define B by

$$B = \{\sum_{-\infty}^\infty a_n x^n : \|\sum_{-\infty}^\infty a_n x^n\| = \sum_{-\infty}^\infty |a_n| d_n < \infty\}$$

then B is a Banach algebra which extends A and B is generated by x and x^{-1}. Put

$$s^{-1} = r(x^{-1}) = \lim_{n \to \infty} \|x^{-n}\|^{1/n} = \lim_{n \to \infty} d_{-n}^{1/n}$$

Then it is easily seen that $\sigma_B(x) = \{\lambda : s \leq |\lambda| \leq r\}$.

We now show that this annulus bounded by circles of radius s and r is in the spectrum of x relative to any extension of the algebra A. Lemma 4 is valid if condition (2) does not hold and in this case we put $s = 0$.

<u>Lemma 4</u>. $\lambda \in \sigma_{per}(x)$ <u>if, and only if</u>, λ <u>is in the annulus bounded by the circles of radius</u> s <u>and</u> r.

<u>Proof</u>. $\sigma_B(x)$ is precisely the annulus hence no point outside it can be in $\sigma_{per}(x)$. Since $\sigma_{per}(x)$ is closed we need only show that the set $\{\lambda : s < |\lambda| < r\} \subseteq \sigma_{per}(x)$.

Suppose that $\lambda \notin \sigma_{per}(x)$ then there exists $M > 0$ such that for each $y \in A$

$$M \| (\lambda - x)y \| \geq \| y \|$$

Take

$$y = \lambda^p + \lambda^{p-1}x + \ldots\ldots + x^p$$

then $(\lambda - x)y = x^{p+1} - x^{p+1}$, hence

$$M \| \lambda^{p+1} - \lambda^{p+1} \| \geq \| y \|$$

that is

$$M(|\lambda|^{p+1} + d_{p+1}) \geq |\lambda^p| + |\lambda|^{p-1}d_1 + \ldots\ldots + d_p \geq |\lambda|^k d_{p-k}$$

for $0 \leq k \leq p$.

If $|\lambda| < r = \inf_{p \geq 0} d_p^{1/p}$ then $|\lambda|^p < d_p$ for $p \geq 0$

giving

$$2M\, d_{p+1} \geq |\lambda|^k d_{p-k}$$

hence, putting $q = p - k$,

$$2M\, d_{q+k+1} \geq |\lambda|^k d_q.$$

Thus

$$2M\, |\lambda|^{-k} \geq d_q / d_{q+k+1} \quad (q, k \geq 0)$$

so

$$2M\,|\lambda|^{-k} \geq d_{-k-1}.$$

Taking the k+1 -th root of both sides and letting $k \longrightarrow \infty$ given $|\lambda|^{-1} \geq s^{-1}$ i.e. $|\lambda| \leq s$.

Thus each point in $\{\lambda : s < |\lambda| < r\} \subseteq \sigma_{per}(x)$ whence the result.

To complete Šilov's example it remains to construct an appropriate sequence of weights $\{d_n\}_0^\infty$. To do so we choose a sequence of real numbers $\{\mu(n)\}_0^\infty$ such that $\mu(o) = 0$ and $\mu(m+n) \leq \mu(m) + \mu(n)$ for each m, n. Put $d_n = \exp \mu(n)$. Then

$$(3) \qquad r = \lim_{n\to\infty} \exp \ (\mu(n)/n).$$

Our example has s = 0 which is equivalent to

$$\sup_{n\geq 0} (d_n/d_{n+1}) = \infty$$

that is

$$(4) \qquad \sup_{n\geq 0} (\mu(n) - \mu(n+1)) = \infty \,.$$

Take $\mu(n) = \sum_{k=n'}^{\infty} k^{-2}$ where

$n' = p+1$ whenever $p^3+1 \leq n \leq (p+1)^3$ for positive integer p. Then $\mu(m+n) \leq \mu(m) + \mu(n)$ since n' increases with n. Further $\mu(n)/n \longrightarrow 0$ as $n \longrightarrow \infty$ hence $r = 1$ by (3).

To check condition (4)

$$\mu(n^3) - \mu(n^3+1) = n^3 \sum_{k=n}^{\infty} k^{-2} - (n^3+1) \sum_{k=n+1}^{\infty} k^{-2}$$

$$= n - \sum_{n+1}^{\infty} k^{-2} \geq n - \pi^2/6$$

Thus in this case $\sigma_{per}(x)$ is the disk of radius one.

An element x of a Banach algebra A is said to have independent powers if

$$\left\| \sum_0^k a_n x^n \right\| = \sum_0^k |a_n| \, \|x^n\| \quad \text{(for any complex numbers } a_0, \ldots\ldots a_k).$$

Spectral properties of independent power elements have been discussed by Williamson [9] and by Bailey, Brown and Moran [2]. It is known [2] that $\sigma_A(x) \supseteq \{\lambda : |\lambda| = r(x)\}$ and conditions are given in [2] that $\sigma_A(x)$ should be a disc where x is an independent power measure and A the algebra of Borel measures on a locally compact abelian group.

The generating element in Šilov's example has independent powers and the analysis given here applies to any independent power element. Thus, by Lemma 4, the interior of $\sigma(x)$ of an independent power element x is removable if, and only if, s = r. This result may be stated as follows.

Proposition 5. *If* x *is an independent power element the interior of* $\sigma(x)$ *is removable if, and only if*

$$\lim_{n\to\infty} \left[\sup_{r \geq 0} \frac{\|x^r\|}{\|x^{r+n}\|} \right]^{-1/n} = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} .$$

If x is an ADZ in a Banach algebra A with isometric extension B, then as x is singular on B the Gelfand transform $\hat{x}$ vanishes on $\Sigma(B)$. The set of characters in $\Sigma(A)$ which extend to characters on each extension B is called the cortex of A. The cortex contains the Šilov boundary and Šilov's example shows that this containment may be proper. The authors conjectured that x is an ADZ if, and only if, $\hat{x}$ vanishes on the cortex. This conjecture was confirmed by J. Zemánek.

Proposition 6 (Zemánek). An element x in a commutative Banach algebra A is an ADZ in A if, and only if, its Gelfand transform vanishes on the cortex of A.

Proof. If $\hat{x}$ vanishes on the cortex of A then x is singular in every extension of A and is hence an ADZ in A.

Conversely, let x be an ADZ in A. There exists a sequence $\{y_n\}$ in A of norm one such that $xy_n \to 0$ as $n \to \infty$. Consider the ideal

$$I = \{ z \in A : zy_n \to 0 \ (n \to \infty) \}.$$

According to a deep result of Słodkowski [8] there exists a maximal ideal M of A which contains I and consists of joint approximate divizors of zero. By a theorem of Želazko [10], M belongs to the cortex of A and clearly $\hat{x}(M) = 0$.

Alternative examples of elements in Banach algebras whose spectra have non-removable interiors seem difficult to construct. In particular, is there an operator T in a Hilbert space whose spectrum has a non-removable interior?

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