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ON MINIMAL POINTS

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Abstract: We extend the notion of minimal point with respect to a set in a normed linear space X studied by B. Beauzany and B. Maurey. Using this new notion we obtain a necessary and sufficient condition for the existence of a norm one linear projection of a smooth space X onto a closed subspace $Y \subset X$, as well as a characterization of a strictly convex space.

Key words: Minimal point, strictly convex space, smooth space, norm one linear projection.

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Let X be a real normed linear space and Y a linear subspace of X. We assign to each nonempty subset M of Y, a subset $M_{Y,X}$ of X in the following way: $x \in M_{Y,X}$ if $x \in X$ and there exists no $y \in Y$, $y \neq x$ such that

$$\|y-m\| \leq \|x-m\| \quad \text{for all } m \in M$$

When X is a normed linear space, for $x_0 \in X$ and $r \geq 0$, we denote

$$B_X(x_0,r) = \{x \in X: \|x-x_0\| \leq r\}$$

Then clearly $x \in M_{Y,X}$ if and only if the set

$$\bigcap_{m \in M} B_X(m, \|x-m\|) \cap Y$$

is either empty or the singleton x.

If $X = Y$, $M \subset X$, then the set $M_{X,X}$ is nothing else than the <u>set of minimal points with respect to</u> M studied by B. Beauzamy and B. Maurey in [1],[2], and denoted there by min M.

In Remark 1 below we extend for $M_{Y,X}$ some elementary properties of min M given in [2], the proofs being similar and simple.

<u>Remark 1</u>. a) For each $M \subset Y$ and each real number λ we have $(\lambda M)_{Y,X} = \lambda M_{Y,X}$.

b) For each $M \subset Y$ and each $y \in Y$ we have $(M+y)_{Y,X} = M_{Y,X}+y$.

c) If $M \subset L \subset Y$ then we have $M \subset M_{Y,X} \subset L_{Y,X}$. If M is a dense subset of L, then $M_{Y,X} = L_{Y,X}$.

d) If M is a bounded subset of Y, then $M_{Y,X}$ is a bounded subset of X.

Some simple connections between $M_{Y,Y}$ and $M_{Y,X}$, or $M_{X,X}$ and $M_{Y,X}$ are collected in the next remark, the proofs being straightforward.

<u>Remark 2</u>. We have for each $M \subset Y$:

(1) $$M_{Y,Y} = M_{Y,X} \cap Y$$

(2) $$M_{X,X} \subset M_{Y,X}$$

The inclusions $M_{Y,Y} \subset M_{Y,X}$ and $M_{X,X} \subset M_{Y,X}$ are strictly in general as the following example shows.

<u>Example</u>. Let $X = \ell^{\infty}$, the Banach space of all real bounded sequences endowed with the usual norm and $Y = c_0$, the closed linear subspace of X, of all sequences converging to zero. For each n=1,2,..., let $y_n = (\eta_{1n}, \eta_{2n}, \ldots$

$\ldots, \eta_{nn}, \ldots) \in Y$, where $\eta_{in} = 0$ for $i \neq n$ and $\eta_{nn} = 2$. Let $M = \{y_{2n}: n=1,2,\ldots\} \subset Y$. Then $x=(0,1,0,1,\ldots) \in M_{Y,X}$. Indeed, let $y=(\eta_1,\ldots,\eta_n,\ldots) \in Y$ and let n_0 be such that $|\eta_n| < 1$ for $n > n_0$. Then for $n > \frac{1}{2} n_0$ we have $\| y-y_{2n} \| \geq |2 - \eta_{2n}| > 1 = \| x-y_{2n} \|$, whence $x \in M_{Y,X}$. Since $x \notin Y$, $M_{Y,Y}$ is strictly included in $M_{Y,X}$. Now, $x \notin M_{X,X}$ since for $\bar{x}=(1,1,1,\ldots) \in X$, we have $\| \bar{x}-y_{2n} \| = 1 = \| x-y_{2n} \|$ for each $n=1,2,\ldots$. Therefore $M_{X,X}$ is strictly included in $M_{Y,X}$.

Clearly, when $M = \{m\}$, $m \in Y$, we always have $M_{Y,Y} = M_{X,X} = M_{Y,X} = M$. When $M = \{m_1,m_2\}$, $m_1,m_2 \in Y$, $m_1 \neq m_2$ these equalities do not all hold generally, as the next result shows (see also Remark 3 below).

We recall (see e.g., [3]) that a normed linear space X is called strictly convex if for each $x_1,x_2 \in X$, $x_1 \neq x_2$, $\| x_1 \| = \| x_2 \| = 1$ we have $\left\| \frac{x_1+x_2}{2} \right\| < 1$.

In Proposition 3 of [2] it was proved that the normed linear space X is strictly convex if and only if for each $m_1,m_2 \in X$, $m_1 \neq m_2$, the points of the segment $[m_1,m_2] = \{\lambda m_1+(1-\lambda)m_2 : 0 \leq \lambda \leq 1\}$ are minimal with respect to $M = \{m_1,m_2\}$. The following result gives also informations on $\min\{m_1,m_2\}$ in arbitrary normed linear spaces.

Let us denote by $\operatorname{ex} B_Y(0,1)$ the set of the extreme points of $B_Y(0,1)$ (i.e., $y \in \operatorname{ex} B_Y(0,1)$, if $y \in Y$, $\| y \| = 1$ and the relations $y = \frac{y_1+y_2}{2}$, $y_1,y_2 \in B_Y(0,1)$ imply $y_1=y_2=y$).

<u>Theorem 1</u>. Let X be a normed linear space, Y a linear subspace of X and $M = \{m_1,m_2\}$, $m_1,m_2 \in Y$, $m_1 \neq m_2$. Then

(3) $$M_{Y,Y} = M_{Y,X} = M \text{ or } [m_1,m_2]$$

Moreover, $M_{Y,X} = [m_1,m_2]$ if and only if $\frac{m_1-m_2}{\|m_1-m_2\|} \in$ $\in \mathrm{ex}\, B_Y(0,1)$.

Proof. Let $M = \{m_1,m_2\}$, $m_1,m_2 \in Y$, $m_1 \neq m_2$. We show first that $M_{Y,Y} = M_{Y,X}$. Let $x \in X \setminus Y$ and for $y \in Y$ defined by

$$(4)\qquad y = \frac{\|x-m_1\|}{\|x-m_1\| + \|x-m_2\|} m_2 + \frac{\|x-m_2\|}{\|x-m_1\| + \|x-m_2\|} m_1$$

it is easy to show that

$$\|y-m_i\| \leq \|x-m_i\| \qquad (i=1,2)$$

and so $x \notin M_{Y,X}$. By Remark 2, formula (1) we obtain the first equality in (3).

We claim now that $M_{Y,X}$ is either M or the segment $[m_1,m_2]$. Since $M \subset M_{Y,X}$, assuming $M_{Y,X} \neq M$, there exists $x_o \in$ $\in M_{Y,X}$, $x_o \neq m_i$, $i=1,2$. Let y be defined as in (4) replacing x by x_o. Then $\|y-m_i\| \leq \|x_o-m_i\|$, $i=1,2$, and since $x_o \in M_{Y,X}$ it follows $x_o = y$ and so

$$(5)\qquad x_o = \lambda_o m_1 + (1-\lambda_o)m_2$$

for $\lambda_o = \frac{\|x_o-m_2\|}{\|x_o-m_1\| + \|x_o-m_2\|}$, and we have $0 < \lambda_o < 1$. This proves the inclusion $M_{Y,X} \subset [m_1,m_2]$. Let

$$x = \lambda m_1 + (1-\lambda)m_2, \qquad 0 < \lambda < \lambda_o$$

and we show that $x \in M_{Y,X}$. (The case $\lambda_o < \lambda < 1$ is similar.) Let $y \in Y$ be such that

$$(6)\qquad \|y-m_1\| \leq \|x-m_1\| = (1-\lambda)\|m_1-m_2\|$$

$$(7)\qquad \|y-m_2\| \leq \|x-m_2\| = \lambda\|m_1-m_2\|$$

Then in both (6) and (7) we have equality, since otherwise $\|y-m_1\| + \|y-m_2\| < \|m_1-m_2\|$ which is impossible. Therefore

(8) $\|y-m_1\| = (1-\lambda)\|m_1-m_2\|$

(9) $\|y-m_2\| = \lambda\|m_1-m_2\|$

Let

(10) $$u = \frac{1-\lambda_o}{1-\lambda}\, y + \left(1 - \frac{1-\lambda_o}{1-\lambda}\right) m_1$$

Note that by our assumptions on λ we have $0 < \frac{1-\lambda_o}{1-\lambda} < 1$.
Hence using (10),(8),(9) and (5) we obtain:

$$\|u-m_1\| = \|x_o-m_1\|$$
$$\|u-m_2\| \leq \|x_o-m_2\|$$

Since $x_o \in M_{Y,X}$ we have $u = x_o$, whence by (10) and (5) we obtain $y = \lambda m_1 + (1-\lambda)m_2 = x$, that is $x \in M_{Y,X}$. This completes the proof of (3).

We show now that $M_{Y,X} = [m_1,m_2]$ if and only if $\frac{m_1-m_2}{\|m_1-m_2\|} \in \text{ex}\, B_Y(0,1)$. In order to show the "if" part, by Remark 1 a) we can suppose $\|m_1-m_2\| = 1$, and by the above claim, it is enough to show that $x = \frac{m_1+m_2}{2} \in M_{Y,X}$. Let $y \in Y$ be such that $\|y-m_i\| \leq \|x-m_i\| = \frac{1}{2}$, i=1,2. Then as in the proof of the claim $\|y-m_i\| = \frac{1}{2}$, i=1,2. Let $y_1 = 2(m_1-y) \in Y$ and $y_2 = 2(y-m_2) \in Y$. We have $\|y_i\| = 1$, i=1,2, and $m_1-m_2 = \frac{y_1+y_2}{2}$. Since $m_1-m_2 \in \text{ex}\, B_Y(0,1)$, it follows $y_i = m_1-m_2$, and so $y = \frac{m_1+m_2}{2} = x$, that is $x \in M_{Y,X}$. Conversely, suppose that for $m_1,m_2 \in Y$ with $\|m_1-m_2\| = 1$ we have $M_{Y,X} = [m_1,m_2]$ and let $m_1-m_2 = \frac{y_1+y_2}{2}$, $y_i \in Y$, $\|y_i\| = 1$, i=1,2. Let $y = \frac{y_1}{2} + m_2$. We have $\|y-m_i\| = \frac{1}{2} = \left\|\frac{m_1+m_2}{2} - m_i\right\|$, i=1,2. By hypothesis, $\frac{m_1+m_2}{2} \in M_{Y,X}$, whence

$y = \frac{m_1+m_2}{2}$, which implies $y_1 = y_2 = m_1-m_2$, and so $m_1-m_2 \in$ $\in \text{ex } B_Y(0,1)$. This completes the proof of the theorem.

Remark 3. When X is not strictly convex, there exists a closed linear subspace $Y \subset X$ such that $M_{X,X} \neq M_{Y,X}$ for some $M = \{m_1,m_2\}$, $m_1,m_2 \in Y$, $m_1 \neq m_2$. Indeed, when X is not strictly convex, there exists $m \in X$, $\|m\| = 1$, $m \notin \text{ex } B_X(0,1)$. Let $Y = \text{sp}\{m\}$ and $M = \{0,m\}$. By Theorem 1 we have $M_{X,X} = \{0,m\}$ and $M_{Y,X} = [0,m]$.

By Theorem 1 we know that in general for a set $M \subset Y$ we have not $\overline{\text{co}}\, M \subset M_{Y,X}$, where $\overline{\text{co}}\, M$ denotes the closed convex hull of M. However, for some special subsets $M \subset Y$ we have the above inclusion. This will be a consequence of Remark 4 below and Remark 2. Note that if X is a Hilbert space, then this is always true as follows by Proposition 4 of [2] and Remark 2.

Remark 4. Let X be a normed linear space and let M be the boundary of a bounded, closed, convex body of X. Then $\overline{\text{co}}\, M \subset M_{X,X}$. Indeed, since $M \subset M_{X,X}$, let $x \in (\overline{\text{co}}\, M) \setminus M$, and suppose there exists $y \in X$, $y \neq x$, such that $\|y-m\| \leq \|x-m\|$ for each $m \in M$. Since $x \in \text{Int } (\overline{\text{co}}\, M)$, there exists $\lambda > 1$ such that $m = \lambda x + (1-\lambda)y \in M$. Then $\|y-m\| = \lambda \|x-y\| \leq \|x-m\| =$ $= (\lambda-1)\|x-y\|$, which is impossible. Therefore $x \in M_{X,X}$. In particular, for each normed linear space X, we have $B_X(0,1) \subset$ $\subset (\text{bd } B_X(0,1))_{X,X}$, where $\text{bd } B_X(0,1) = \{x \in X: \|x\| = 1\}$. One can also show that if X is a normed linear space, then for each bounded convex body $M \subset X$ we have $X = (X \setminus M)_{X,X}$.

Let X^* be the dual space of X. We recall (see e.g.,

[4]) that in a normed linear space X a point $x \in X$, $\|x\| = 1$ is called a smooth point of $B_X(0,1)$, if there exists a unique $x_x^* \in X^*$, $\|x_x^*\| = 1$ such that $x_x^*(x) = \|x\|$. We denote by sm $B_X(0,1)$ the set of all smooth points of $B_X(0,1)$. The normed linear space X is called smooth if each $x \in X$, $\|x\| = 1$ is a smooth point of $B_X(0,1)$.

In Proposition 5 of [2], B. Beauzamy and B. Maurey proved the following result: Let X be a reflexive, strictly convex and smooth Banach space and Y a closed linear subspace of X. If $Y_{X,X} = Y$ then there exists a (unique) linear projection $P: X \longrightarrow Y$, $\|P\| = 1$. They also noted that the existence of a norm one linear projection P of X onto Y implies $Y_{X,X} = Y$. We shall also give a necessary and sufficient condition for the existence of a norm one projection of X onto Y, weakening the conditions on X (requiring only the smoothness of X) but strengthening the condition $Y_{X,X} = Y$. To prove our result we need Lemma 2 of [2]. Since the proof of this lemma does not use the completeness of the space X we state it in a normed linear space.

Lemma ([2], Lemma 2). Let X be a normed linear space, Y a closed linear subspace of X and $x_1, x_2 \in X$, such that $\|x_1 - y\| \leq \|x_2 - y\|$ for each $y \in Y$. Then $x_y^*(x_1 - x_2) = 0$ for each $y \in Y \setminus \{0\}$, $\frac{y}{\|y\|} \in$ sm $B_X(0,1)$.

Theorem 2. Let X be a normed linear space and Y a closed linear subspace of X. A necessary, and if bd $B_Y(0,1) \subset$ sm $B_X(0,1)$ also sufficient condition for the existence of a norm one linear projection P of X onto Y, is that $Y_{Y,X} = Y$. If bd $B_Y(0,1) \subset$ sm $B_X(0,1)$, then there exists at most one norm one linear projection of X onto Y.

Proof. Clearly, if there exists a linear projection $P:X \longrightarrow Y$, $\|P\| = 1$, then for each $x \in X \setminus Y$ and each $y \in Y$ we have $\|P(x)-y\| = \|P(x-y)\| \leq \|x-y\|$, which shows that $x \notin Y_{Y,X}$. Therefore $Y_{Y,X} \subset Y$ and by Remark 1 c) it follows $Y_{Y,X} = Y$.

Suppose now that bd $B_Y(0,1) \subset$ sm $B_X(0,1)$ and $Y_{Y,X} = Y$. Then for each $x \in X$ we have

$$\bigcap_{y \in Y} B_Y(y, \|x-y\|) \neq \emptyset \tag{11}$$

We claim that the left hand side of (11) contains exactly one element. Indeed, let $y_1, y_2 \in \bigcap_{y \in Y} B_Y(y, \|x-y\|)$ and suppose that $y_0 = y_1 - y_2 \neq 0$. Then for i=1,2 we have

$$\|y_i - y\| \leq \|x-y\| \qquad \text{for all } y \in Y$$

whence by the above Lemma we obtain

$$x^*_{y_0}(y_i - x) = 0 \qquad (i=1,2)$$

Then $\|y_0\| = x^*_{y_0}(y_0) = x^*_{y_0}(y_1 - y_2) = 0$, a contradiction. Therefore for each $x \in X$, $\bigcap_{y \in Y} B_Y(y, \|x-y\|)$ is a singleton and we denote it by $P(x)$. We show now that $P:X \longrightarrow Y$ defined as above is a norm one linear projection. Clearly $P^2 = P$. Let now $\lambda \in R$ and $x \in X$. Since for $\lambda = 0$ we have $P(\lambda x) = \lambda P(x)$, suppose $\lambda \neq 0$. Then $\|P(\lambda x) - y\| \leq \|\lambda x - y\|$ for each $y \in Y$ and so $\left\| \frac{P(\lambda x)}{\lambda} - y \right\| \leq \|x-y\|$ for each $y \in Y$. Therefore $P(x) = \frac{P(\lambda x)}{\lambda}$ whence $P(\lambda x) = \lambda P(x)$. Let now $x_1, x_2 \in X$ and suppose that $y_0 = P(x_1+x_2) - P(x_1) - P(x_2) \neq 0$. We have

$$\|P(x_1+x_2) - y\| \leq \|x_1+x_2-y\| \qquad \text{for all } y \in Y$$

$$\|P(x_i) - y\| \leq \|x_i - y\| \qquad \text{for all } y \in Y,\ i=1,2.$$

By Lemma we obtain

$$x^*_{y_o}(x_1+x_2-P(x_1+x_2)) = 0$$

$$x^*_{y_o}(x_i-P(x_i)) = 0 \qquad (i=1,2)$$

Hence $\|y_o\| = x^*_{y_o}(y_o) = 0$, a contradiction. Finally, $\|P(x)\| \leq \leq \|x\|$ for each $x \in X$ follows by the fact that $P(x)$ belongs to the left hand side of (11). Therefore P is a norm one linear projection of X onto Y.

To complete the proof of the theorem, let us suppose bd $B_Y(0,1) \subset$ sm $B_X(0,1)$. If P is a norm one projection of X onto Y, then by the above, for each $x \in X$, $P(x)$ belongs to the left hand side of (11) which is a singleton. Therefore there exists at most one linear projection $P:X \longrightarrow Y$, $\|P\| = 1$.

Let E be a normed linear space and E^*, E^{**}, E^{***}, and $E^{(4)}$ the successive dual spaces. We shall consider E (respectively E^{**}) as a subspace of E^{**} (respectively of $E^{(4)}$) by the natural embedding of E into E^{**} (respectively E^{**} into $E^{(4)}$). When $X = E^{**}$ and $Y = E$ then for each nonempty subset $M \subset E$ we shall denote MIN $M = M_{Y,X}$ $(= M_{E,E^{**}})$ and min $M = M_{Y,Y}$ $(= M_{E,E})$.

F. Sullivan [5] called a Banach space E <u>very smooth</u> if bd $B_E(0,1) \subset$ sm $B_{E^{**}}(0,1)$. Examples of non-reflexive very smooth spaces as well as some properties of very smooth spaces are given in [5]. An immediate consequence of Theorem 2 is:

<u>Corollary</u>. Let E be a very smooth Banach space. There exists a linear projection $P:E^{**} \longrightarrow E$, $\|P\| = 1$, if and only if MIN E = E. Moreover, this projection is unique.

We recall [2] that a set $M \subset E$ is called <u>optimal</u> if $\min M = M$.

<u>Remark 5</u>. If E is a Banach space such that MIN E = E, then E is an optimal subspace of E^{**}. Indeed, this follows by Remark 2, formula (2).

<u>Proposition 1</u>. Let M be a nonempty subset of the normed linear space E, such that MIN M is optimal in E^{**}. Then there exists a unique, maximal closed subset $\tilde{M} \subset E$ such that $M \subset \tilde{M}$ and MIN M = MIN $\tilde{M}$.

<u>Proof</u>. Let $\mathcal{M}$ be the collection of all subsets $A \subset E$ such that MIN A = MIN M. Then $\mathcal{M}$ is nonempty since $M \in \mathcal{M}$. Let $\tilde{M} = \overline{\bigcup_{A \in \mathcal{M}} A}$. Since $M \subset \tilde{M}$, by Remark 1 c) we have MIN $M \subset$ $\subset$ MIN $\tilde{M}$. On the other hand, for each $A \in \mathcal{M}$ we have $A \subset$ MIN A= = MIN M and so $\bigcup_{A \in \mathcal{M}} A \subset$ MIN M. Hence, using Remark 1 c), it follows

$$(12) \qquad \text{MIN } \tilde{M} = \text{MIN } \overline{\bigcup_{A \in \mathcal{M}} A} = \text{MIN } \bigcup_{A \in \mathcal{M}} A \subset \text{MIN (MIN M)}$$

Since E^{**} is a dual space, there exists a linear projection $P: E^{(4)} \longrightarrow E^{**}$, $\|P\| = 1$. By Theorem 2, Remark 1 c), Remark 2 formula (1), and the assumption on MIN M it follows MIN (MIN M) = min (MIN M) = MIN M, whence by (12) we have MIN $\tilde{M} \subset$ MIN M, which completes the proof.

With a similar proof one can show:

<u>Proposition 2</u>. If M is a nonempty subset of a normed linear space E such that min M is optimal, then there exists a unique, maximal, closed subset $\tilde{M} \subset E$ such that $M \subset \tilde{M}$ and $\min M = \min \tilde{M}$.

<u>Remark 6</u>. Let E be a normed linear space and $m_1, m_2 \in E$,

$m_1 \neq m_2$. Then $M = [m_1, m_2]$ is optimal. Indeed, let $x \notin M$ and let $y = \lambda_o m_2 + (1-\lambda_o)m_1$, where $\lambda_o = \frac{\|x-m_1\|}{\|x-m_1\| + \|x-m_2\|}$ (since $0 < \lambda_o < 1$, we have $y \in M$ and so $y \neq x$). Then for $m \in M$, $m =$ $= \lambda m_2 + (1-\lambda)m_1$, $0 \leq \lambda \leq 1$, we have

$$\|y-m\| = |\lambda_o - \lambda| \, \|m_1-m_2\| = \left| \frac{\|x-m_1\| - \lambda\|x-m_1\| - \lambda\|x-m_2\|}{\|x-m_1\| + \|x-m_2\|} \right| \|m_1 -$$
$$- m_2\| \leq \|(1-\lambda)\, \|x-m_1\| - \lambda\|x-m_2\| \, | \leq \|(1-\lambda)(x-m_1) -$$
$$- \lambda(m_2-x)\| = \|x-(\lambda m_2+(1-\lambda)m_1)\| = \|x-m\|$$

and so $x \notin \min M$. As a consequence of this result and Theorem 1, it follows that for $M = \{m_1, m_2\}$ we have always that $\min M$ (respectively MIN M) is optimal in E (respectively in E^{**}). Note that this is obviously true if $m_1 = m_2$.

We conclude this paper with a characterization of a strictly convex space using the notion "MIN". The proof of the "only if" part is essentially the same with the proof of Proposition 2 of [2].

<u>Theorem 3</u>. The normed linear space E is strictly convex if and only if for each nonempty subset $M \subset E$ and each $z^{**} \in E^{**}$ we have

$$(13) \qquad \left(\bigcap_{m \in M} B_{E^{**}}(m, \|z^{**}-m\|)\right) \cap \text{MIN } M \neq \emptyset$$

<u>Proof</u>. Suppose E strictly convex and let M be a nonempty subset of E. For each $z^{**} \in E^{**}$ we define a function on M by

$$f_{z^{**}}(m) = \|z^{**}-m\| \qquad (m \in M)$$

As in the proof of [2], Proposition 2 one can show that the set $\{f_{z^{**}}\}_{z^{**} \in E^{**}}$ is inductive (for the usual ordering), and if $z^{**} \in E^{**}$ is given, by Zorn's Lemma, there exists $x^{**} \in E^{**}$ such that $f_{x^{**}}$ is minimal (for the ordering), and $f_{x^{**}} \leq f_{z^{**}}$. Therefore

(14) $\quad \| x^{**} - m \| \leq \| z^{**} - m \| \quad$ for each $m \in M$

If $x^{**} \in$ MIN M then by (14) it follows (13). If $x^{**} \notin$ MIN M, there exists $x \in E$, $x \neq x^{**}$ such that $\| x-m \| \leq \| x^{**} - m \|$ for each $m \in M$. Since $f_{x^{**}}$ is minimal we must have

(15) $\quad \| x-m \| = \| x^{**} - m \| \quad$ for each $m \in M$.

We show that $x \in$ min M. If not, there exists $y \in E$, $y \neq x$ such that $\| y-m \| \leq \| x-m \|$, for each $m \in M$, whence by (15) and the fact that $f_{x^{**}}$ is minimal, it follows

(16) $\quad \| y-m \| = \| x-m \| \quad$ for each $m \in M$.

Since E is strictly convex, by (16) and (15) we have for $m \in M$

$$\left\| \frac{x+y}{2} - m \right\| < \left\| \frac{x-m}{2} \right\| + \left\| \frac{y-m}{2} \right\| = \| x-m \| = \| x^{**} - m \|$$

which contradicts the minimality of $f_{x^{**}}$. Therefore $x \in$ min M and by Remark 2, formula (1) we have $x \in$ MIN M, whence by (15) and (14) we get (13).

Conversely, suppose that for each $M \subset E$ and each $z^{**} \in E^{**}$ (13) holds and E is not strictly convex. Then, by Theorem 1 there exist $m_1, m_2 \in E$, $m_1 \neq m_2$ such that MIN $\{m_1, m_2\} = \{m_1, m_2\}$. Let $z^{**} = (m_1+m_2)/2$. By hypothesis there exists $x^{**} \in$ MIN$\{m_1, m_2\}$ such that $\| x^{**} - m_i \| \leq \| z^{**} - m_i \|$, $i=1,2$. Suppose $x^{**} = m_1$. Then $\| m_1 - m_2 \| \leq \| z^{**} - m_2 \| = \| (m_1-m_2)/2 \|$ which is impossible since $m_1 \neq m_2$. Therefore E is strictly convex, which completes the proof of the theorem.

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