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COMMUTATIVE MOUFANG LOOPS AND DISTRIBUTIVE STEINER QUASIGROUPS NILPOTENT OF CLASS 3

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Abstract: Free commutative Moufang loops and distributive Steiner quasigroups nilpotent of class 3 are constructed.

Key words: Loop, commutative, Moufang, quasigroup, distributive, Steiner.

Classification: 20N05

Recently, L. Bénéteau has described free 3-elementary commutative Moufang loops nilpotent of class at most 3. The description is based on a trilinear construction using anticommutative graded rings. In the present paper, we give another, more direct construction. The results are also applied to distributive Steiner quasigroups.

1. CM-loops and DS-quasigroups. A loop Q is said to be a CM-loop if it satisfies the identity xx.yz = xy.xz. A quasigroup Q is said to be a DS-quasigroup if it satisfies the identities xy = yx, x.xy = y, x.yz = xy.xz. The reader is referred to [1],...,[9] for basic and some further properties of these structures.

Let Q be a CM-loop. We denote by $C_1(Q)$ the centre of Q and by $C_2(Q)$ the subloop containing $C_1(Q)$ such that

$C_2(Q) / C_1(Q) = \mathcal{C}_1(Q / C_1(Q))$. Further, we put $[a,b,c] = (ab.c)(a.bc)^{-1}$ for all $a,b,c \in Q$. The subloop generated by all $[a,b,c]$ is denoted by $A_1(Q)$ and the subloop generated by all $[[a,b,c],d,e]$ by $A_2(Q)$. We shall say that Q is nilpotent of class at most 3 if $A_2(Q) \subseteq C_1(Q)$. The loop Q is said to be 3-elementary if $D(Q) = 1$, where $D(Q) = \{x^3 \mid x \in Q\}$. Obviously, Q is 3-elementary iff $B(Q) = Q$, where $B(Q) = \{x \mid x \in Q, x^3 = 1\}$.

1.1. <u>Lemma</u>. Let Q be a CM-loop. Then:

(i) $[a,b,c] = [b,a,c]^{-1} = [a,c,b]^{-1}$ for all $a,b,c \in Q$.

(ii) $[[a,b,c],c,d] = [[b,d,c],c,a]$ for all $a,b,c,d \in Q$.

(iii) $[[a,b,c],d,e] = [[a,d,e],b,c][[b,d,e],c,a][[c,d,e],a,b]$ for all $a,b,c,d,e \in Q$.

(iv) $[[a,b,c],d,e] = [[d,b,c],a,e][[a,d,c],b,e][[a,b,d],c,e]$ for all $a,b,c,d,e \in Q$.

(v) $[[[a,b,c],c,d],d,e] = [[[b,e,c],c,d],d,a]$ for all $a,b,c,d,e \in Q$.

(vi) $[[a,b,cd],cd,e] = [[a,b,c],c,e][[a,b,d],d,e].f$, $f = [[[a,b,c],c,d],d,e][[a,b,c],d,e][[a,b,d],c,e]$ for all $a,b,c,d,e \in Q$.

(vii) If Q can be generated by 5 elements then $A_1(Q)$ is a group.

(viii) $[[a,b,c],d,e] = [[b,a,d],c,e][[b,e,c],d,a][[b,e,d],c,a]$ for all $a,b,c,d,e \in Q$.

Proof. (i) See [4, § VIII.2].

(ii) See [4, Lemma VIII.3.9].

(iii) See [4, Lemma VIII.6.4].

(iv) See [4, Lemma VIII.6.4].

(v) See [4, Lemma VIII.6.5].

(vi) See [4, Lemma VIII.6.6].

(vii) See [4, Lemma VIII.6.3].

(viii) This is an easy consequence of (i),(ii),(v),(vi) and (vii).

1.2. <u>Lemma</u>. Let Q be a CM-loop generated by a set S. Then Q is nilpotent of class at most 3 iff $[[[a,b,c],d,e],f,g] = 1$ for all $a,b,c,d,e,f,g \in S$.

Proof. See [4, Lemma VIII.3.8].

1.3. <u>Lemma</u>. Let Q be a CM-loop generated by a set S.

(i) If Q is nilpotent of class at most 2 then $A_1(Q)$ is generated by the elements $[a,b,c], a,b,c \in S$.

(ii) If Q is nilpotent of class at most 3 then $A_2(Q)$ is generated by the elements $[[a,b,c,]d,e]$, $a,b,c,d,e \in S$.

Proof. This is an easy consequence of [4, Lemma VIII. 3.8].

1.4. <u>Proposition</u>. Let $0 \leq n$ be an integer and Q a 3-elementary CM-loop generated by n elements.

(i) If Q is nilpotent of class at most 2 then card $Q \leq 3^{n+m}$ and card $A_1(Q) \leq 3^m$, $m = \binom{m}{3}$.

(ii) If Q is nilpotent of class at most 3 then card $Q \leq$ $\leq 3^{n+m+p}$, card $A_1(Q) \leq 3^{m+p}$ and card $A_2(Q) \leq 3^p$, $4\binom{n}{4} +$ $+ 4\binom{n}{5} = p$.

Proof. Use 1.1 and 1.3.

1.5. <u>Proposition</u>. The following conditions are equivalent for a quasigroup Q:

(i) Q is a DS-quasigroup.

(ii) There exists a 3-elementary CM-loop $Q(\circ)$ such that

$xy = x^{-1} \circ y^{-1}$ for all $x,y \in Q$.

Proof. Easy and well known (see e.g. [7, Satz 1.4]).

2. Ternary rings. Let $G = G(+,F)$ be a ternary ring (i.e., $G(+)$ is an abelian group and F is a triadditive mapping of G^3 into G). Consider the following identities:

(a) $3F(x,y,z) = 0$ for all $x,y,z \in G$.

(b) $F(x,x,y) = 0$ for all $x,y \in G$.

(c) $F(F(x,y,z),u,v) = 0$ for all $x,y,z,u,v \in G$.

(d) $F(x,y,F(y,z,z)) = 0$ for all $x,y,z \in G$.

(e) $F(x,y,F(z,u,F(w,r,s))) = 0$ for all $x,y,z,u,v,w,r,s \in G$.

The following three lemmas are easy observations.

2.1. Lemma. (i) If G satisfies (b) then $F(x,y,z) = -F(y,x,z)$ for all $x,y,z \in G$.

(ii) If G satisfies (b) and (d) then $F(x,y,F(z,y,y)) = 0$ for all $x,y,z \in G$.

2.2. Lemma. Let S be a generator set of the group $G(+)$.

(i) If G satisfies (a) then G satisfies (b) iff $F(a,b,c) = -F(b,a,c)$ for all $a,b,c \in S$.

(ii) G satisfies (c) iff $F(F(a,b,c),d,e) = 0$ for all $a,b,c,d,e \in S$.

(iii) If G satisfies (a) then G satisfies (d) iff $F(a,b,F(c,d,e)) + F(a,c,F(b,d,e)) + F(a,b,F(c,e,d)) + F(a,c,F(b,e,d)) = 0$ for all $a,b,c,d,e \in S$.

(iv) G satisfies (e) iff $F(a,b,F(c,d,F(e,f,g))) = 0$ for all $a,b,c,d,e,f,g \in S$.

2.3. Lemma. Let $G(+)$ be an abelian 3-group with a basis S and E a mapping of S^3 into $B(G(+))$. Then E can be extended in a unique way to a triadditive mapping of G^3 into G.

Put $\bar{F}(x,y,z) = F(x,y,z) + F(y,z,x) + F(z,x,y)$ for all $x,y,z \in G$.

2.4. <u>Proposition</u>. Let G(+,F) be a ternary ring satisfying the identities (a),(b),(c),(d),(e). Put $x \circ y = x + y + F(x,y,x-y)$ for all $x,y \in G$. Then:

(i) $G(\circ)$ is a CM-loop nilpotent of class at most 3.

(ii) $G(\circ)$ is 3-elementary iff $G(+)$ is so.

(iii) $[a,b,c] = \bar{F}(a,b,c)$ and $[[a,b,c],d,e] = F(d,e,\bar{F}(a,b,c))$ for all $a,b,c,d,e \in G$.

(iv) $C_1(G(\circ)) = \{a \in G \mid \bar{F}(a,x,y) = 0$ for all $x,y \in G\}$.

(v) $A_1(G(\circ))$ is an ideal of the ternary ring and $a \circ x = a + x$ for all $a \in A_1(G(\circ))$ and $x \in G$.

Proof. Easy.

2.5. <u>Corollary</u>. Let G(+,F) be a ternary ring satisfying the identities (b),(c),(d),(e) such that G(+) is 3-elementary. Put $x * y = -x - y + F(x,y,y-x)$ for all $x,y \in G$. Then $G(*)$ is a DS-quasigroup nilpotent of class at most 3.

3. <u>Auxiliary results I</u>. In this section, let K denote the set of all ordered 5-tuples (ijkpq) with $\{i,j,k,p,q\} = \{1,2,3,4,5\}$. Let L be the set of all (ijkpq) $\in$ K such that $i<j$, $k<p$ and either $j<k$ or $p<q$. Obviously, card K = 120 and card L = 14.

Consider a vector space V over the three-element field having K as a basis and define eight endomorphisms of V by $a(x) = (jikpq)$, $b(x) = (ikjpq)$, $c(x) = (ijpkq)$, $d(x) =$

$= (ijkqp)$, $f(x) = x + a(x)$, $g(x) = x + c(x)$, $e(x) = x + b(x) + d(x) + db(x)$, $r(x) = x + cd(x) + dc(x)$ for every $x = (ijkpq) \in K$.

3.1. Lemma. $a^2 = b^2 = c^2 = d^2 = 1$, $f^2 = -f$, $g^2 = -g$, $e^2 = e$, $r^2 = 0$, $aba = bab$, $ac = ca$, $ad = da$, $af = f = fa$, $ag = ga$, $ar = ra$, $bcb = cbc$, $bd = db$, $be = e = eb$, $cdc = dcd$, $cf = fc$, $cg = g = gc$, $cr = rd$, $df = fd$, $de = e = ed$, $dr = rc$, $cdr = dcr = rcd = rdc = r$, $fg = gf$.

Proof. Easy.

Denote by W the subspace of V generated by L and put $U = f(V) + g(V) + e(V)$. Let t be the natural homomorphism of V onto V/U.

3.2. Lemma. $\dim (f(V) + g(V)) \le 90$.

Proof. Define a relation w on K by $(x,y) \in w$ iff either $x = y$ or $x = a(y)$ or $x = c(y)$ or $x = ac(y)$. Then w is an equivalence and has exactly 30 blocks. Let S be a set of representants of w and $R = \{f(x), g(x), x-ac(x) \mid x \in S\}$. It is easy to check that R generates $f(V) + g(V)$.

3.3. Lemma. Let Z be a subspace of V containing $f(V)$ and let $x \in K$ be such that $e(x), ea(x) \in Z$. Then $eab(x) \in Z$.

Proof. We have $e(x) = x + b(x) + d(x) + db(x) \in Z$, $ea(x) = a(x) + ba(x) + da(x) + dba(x) \in Z$, $x + a(x) \in Z$, $da(x) = ad(x)$ and $d(x) + da(x) \in Z$. Hence $y = -x + ba(x) - d(x) + dba(x) \in Z$ and $e(x) + y = b(x) + ba(x) + db(x) + dba(x) \in Z$. However, $a(Z) \in Z$, $aba = bab$, $ad = da$, and therefore $ae(x) + a(y) = eab(x) \in Z$.

3.4. Lemma. Let Z be a subspace of V containing $g(V)$ and let $x \in K$ be such that $e(x), ec(x), ecb(x), ecd(x), ecdb(x) \in Z$.

Then $ecdbc(x) \in Z$.

Proof. We have $e(x) = x + b(x) + d(x) + bd(x)$, $ec(x) = c(x) + bc(x) + dc(x) + bdc(x)$, $ecb(x) = cb(x) + bcb(x) + dcb(x) + bdcb(x) \in Z$. Consequently, $y = e(x) - x - c(x) + ec(x) - cecb(x) + dcb(x) + cdcb(x) + dbcb(x) + cdbcb(x) = d(x) + db(x) + dc(x) + bdc(x) + dcb(x) + dbcb(x) \in Z$. Further, $ecd(x) = cd(x) + bcd(x) + dcd(x) + bdcd(x) \in Z$ and $ecdb(x) = edcdcb(x) = ecdcb(x) = cdcb(x) + bcdcb(x) + dcdcb(x) + bdcdcb(x) \in Z$. From this, $z = ecd(x) - d(x) - cd(x) - dc(x) - cdc(x) + ecdcb(x) - dcb(x) - cdcb(x) - db(x) + bdcdcb(x) \in Z$. On the other hand, $bcd = bcdbb = bcbdb = cbcdb = cbdcdcb$, and hence $u = y + z - gbdcdcb(x) = y + z - bcd(x) - bdcdcb(x) = bdc(x) + dbcb(x) + bdcd(x) + bcdcb(x) = bdc(x) + dcbc(x) + bcdc(x) + bcdcb(x) \in Z$. But $c(u) = cbdc(x) + dcdbc(x) + bcbdc(x) + bdcbdc(x) = ecdbc(x) \in Z$, since $cbcdcb = bdcbdc$.

3.5. <u>Lemma</u>. $\dim U \leq 106$.

Proof. Define a relation v on K by $(x,y) \in v$ iff either $x = y$ or $x = b(y)$ or $x = d(y)$ or $x = bd(y)$. Then v is an equivalence and has exactly 30 blocks. Denote by s the natural mapping of K onto K/v. Clearly, $s(x) \neq s(y)$, provided $x = (i....)$ and $y = (j....)$ are from K such that $i \neq j$. Moreover, it is easy to verify that for each $x \in K$, the elements $s(x)$, $sc(x)$, $scb(x)$, $scd(x)$, $scdb(x)$ and $scdbc(x)$ are pairwise different. Now, put $x_1 = (12345)$, $x_2 = a(x_1)$, $x_3 = ab(x_1)$, $x_4 = abc(x_1)$, $x_5 = abcd(x_1)$. For $1 \leq i \leq 5$, let $x_{i1} = x_i$, $x_{i2} = c(x_i)$, $x_{i3} = cb(x_i)$, $x_{i4} = cd(x_i)$, $x_{i5} = cdb(x_i)$ and $x_{i6} = cdbc(x_i)$. Put $J = \{x_{ij} \mid 1 \leq i \leq 5, 1 \leq j \leq 6\}$. Then

$s(J) = s(K)$, and therefore $e(V)$ is generated by $e(J)$. Further, let $M = \{x_{ij} \mid 1 \leq i, j \leq 5\}$. According to 3.4, U is generated by $f(V) \cup g(V) \cup e(M)$. On the other hand, we have $a(x_{11}) = x_{21}$, $ab(x_{11}) = x_{31}$, $a(x_{12}) = x_{22}$, $ab(x_{12}) = x_{41}$, $a(x_{13}) = x_{32}$, $ab(x_{13}) = x_{42}$, $a(x_{14}) = x_{24}$, $ab(x_{14}) = x_{51}$, $a(x_{15}) = x_{34}$, $ab(x_{15}) = x_{52}$, $a(x_{16}) = x_{44}$, $ab(x_{16}) = x_{54}$, $a(x_{23}) = x_{33}$, $ab(x_{23}) = x_{43}$, $a(x_{25}) = x_{35}$, $ab(x_{25}) = x_{53}$, $a(x_{26}) = x_{45}$ and $ab(x_{26}) = x_{55}$. Using 3.3, it is easy to show that U is generated by $f(V) \cup g(V) \cup e(N)$, where $N = M \setminus \{x_{31}, x_{41}, x_{42}, x_{43}, x_{51}, x_{52}, x_{53}, x_{54}, x_{55}\}$. However, card $N = 16$ and 3.2 yields the result.

3.6. Lemma. $V = U + W$.

Proof. Put $Z = U + W$. We are going to show that $K \subseteq Z$. For, let $x = (ijkpq) \in K$. Taking into account that $x \in Z$ iff $a(x) \in Z$ iff $c(x) \in Z$, we can assume that $i < j$ and $k < p$. Further, we can restrict ourselves to the case $x \notin L$. Then $k < j$ and $q < p$. If $i < k$ and $j < p$ then $b(x) \in L$. If $k < i$ and $j < p$ then $ab(x) \in L$, and hence $b(x) \in Z$. If $i < k$ and $p < j$ then $cb(x) \in L$, and hence $b(x) \in Z$. If $k < i < p < j$ then $acb(x) \in L$, and so $b(x) \in Z$. If $p < i$ then $bacb(x)$, $cdacb(x)$, $cdbacb(x) \in L$, hence $bacb(x)$, $dacb(x)$, $dbacb(x) \in Z$, $acb(x) \in Z$ and $b(x) \in Z$. We have proved that $b(x) \in Z$ and it remains to show that $d(x)$, $db(x) \in Z$. If $k < q$ then $d(x) \in L$. If $q < k$ then $cd(x) \in L$, and hence $d(x) \in L$. Now, we are going to prove that $db(x) \in Z$. As one may check easily, we can assume that $q < j$. It suffices to show that $y = cdb(x) \in Z$. If $i < k$ and $j < p$ then $y \in L$. If $k < i$ and $j < p$ then $a(y) \in L$ and $y \in Z$. Suppose $p < j$. If $i < k < q$ then $y \in L$. If $k < i < q$ then $a(y) \in L$ and $y \in Z$. Further, it is easy to see that $d(y) \in Z$ and $db(y) \in Z$. Hence, it is enough to show that $ab(y) \in Z$. We can assume that $k < i$ and $q < i$. Then $bab(y)$, $dab(y) \in Z$. If $i < p$ then $dbab(y) \in Z$. If $p < i$ then $cdbab(y) \in Z$.

3.7. Lemma. V is the direct sum of the subspace U and W.

Proof. By 3.6, $V = U + W$. Hence $\dim (U \cap W) = \dim U + \dim W - \dim V \leq 106 + 14 - 120 = 0$. Consequently, $U \cap W = 0$.

3.8. Lemma. $4 \leq \dim tr(V)$.

Proof. Put $y_1 = (12345)$, $y_2 = (12354)$, $y_3 = (12453)$, $y_4 = (13452)$, $y_5 = (23451)$, $y_6 = (13245)$, $y_7 = (14235)$, $y_8 = (23145)$. Then $y_i \in L$ and there are uniquely determined $z_i \in W$ such that $t(z_i) = tr(y_i)$. One may check easily that $z_1 = y_1 - y_2 + y_3$, $z_4 = y_1 + y_2 + y_4 - y_6$, $z_5 = -y_1 - y_2 + y_5 - y_8$, $z_7 = -y_1 + y_3 - y_4 + y_6$. Put $P = \{z_1, z_4, z_5, z_7\}$. It is an easy exercise to show that P is an independent subset of W. However, by 3.7, $t|W$ is injective and the rest is clear.

3.9. Lemma. Let $x \in K$. Then $r(x) \notin U$.

Proof. Suppose, on the contrary, that $r(x) \in U$ for some $x = (ijkpq) \in K$. We have $ra(x) = ar(x)$, $r(x) + ar(x) \in U$, and so $ra(x) \in U$. Similarly, $cr = rd$, $r(x) + cr(x) \in U$, $rd(x) \in U$. Finally, $dr(x) = d(x) + cdc(x) + c(x) = d(x) + cd(x) - r(x) + x + c(x) + dc(x) + cdc(x) \in U$. But $dr(x) = rc(x)$. Using this information, we can assume $i < j$ and $k < p < q$. Then $x = y_i$ for some $i \in \{1,6,7,8,9,10,11,12,13,14\}$, where $y_1, \ldots, y_8$ are defined in the same way as in 3.8 and $y_9 = (45123)$, $y_{10} = (24135)$, $y_{11} = (25134)$, $y_{12} = (34125)$, $y_{13} = (35124)$ and $y_{14} = (15234)$. There are uniquely determined $z_i \in W$ with

$t(z_i) = tr(y_i)$. We have $t(z_i) \neq 0$ and $z_i \notin U$, a contradiction.

4. Auxiliary results II. In this section, let K be the set of all ordered 5-tuples (ijkpq) such that $\{i,j,k,p,q\} = \{1,2,3,4\}$ and $i + j + k + p + q = 11$. Obviously, card K = = 60. Put w = (12341).

Consider a vector space V over the three-element field having K as a basis and define eight endomorphisms of V by $a(x) = (jikpq)$, $b(x) = (ikjpq)$, $c(x) = (ijpkq)$, $d(x) = (ijkqp)$, $f(x) = x + a(x)$, $g(x) = x + c(x)$, $e(x) = x + b(x)$, $r(x) = x + dc(x) + cd(x)$ for every $x = (ijkpq) \in K$. Denote by W the subspace of V generated by w and put $U = f(V) + g(V) + e(V)$. Let t be the natural homomorphism of V onto V/U.

4.1. Lemma. V is the direct sum of the subspaces U and W.

Proof. Define an endomorphism s of V as follows: $s(x) = 0$ if $x = (ijkpq) \in K$ is such that $q \neq 1$; $s(x) = w$ if $x = (ijkpq)$ is such that $q = 1$ and the permutation (ijkp) is even; $s(x) = -w$ if $x = (ijkpq)$ is such that $q = 1$ and the permutation (ijkp) is odd. One may see easily that $f(V) \cup g(V) \cup e(V) \subseteq \operatorname{Ker} s$. Hence $U \subseteq \operatorname{Ker} s$. On the other hand, $\operatorname{Im} s = W$, $W \cap \operatorname{Ker} s = 0$, $W \cap U = 0$ and the rest is clear.

4.2. Lemma. $1 \leq \dim tr(V)$.

Proof. We have $tr(w) = t(w)$. However, $t(w) \neq 0$ by 4.1.

5. Main results. For $4 \leq n$, let $I = I_n$ be the set of all ordered triples (ijk) and $K = K_n$ the set of all ordered 5-tuples (ijkpq) with $1 \leq i,j,k,p,q \leq n$. Denote by $J = J_n$ the set of all $(ijk) \in I$ with $i < j$ and put card x =

$= \operatorname{card}\{i,j,k,p,q\}$ for every $x = (ijkpq) \in K$. Let $L = L_n$ be the set of all $x = (ijkpq) \in K$ such that either card $x = 5$, $i < j$, $k < p$ and either $j < k$ or $p < q$, or card $x = 4$ and $i < j < k < p$. Further, let $S = S_n = \{a_1, \dots, a_n\}$ be a set containing n elements such that the sets S, I and K are pair-wise disjoint.

Consider a vector space $V = V_n$ over the three-element field having $M = M_n = S \cup I \cup K$ as a basis and put $a(x) = (jik)$, $a(y) = (jikpq)$, $b(y) = (ikjpq)$, $c(y) = (ijpkq)$, $d(y) = (ijkqp)$ for all $x = (ijk) \in I$ and $y = (ijkpq) \in K$. Let $U = U_n$ be the subspace generated by $\{x + a(x) \mid x \in I\} \cup \{x \mid x \in K,$ card $x \leq 3\} \cup \{x + a(x) \mid x \in K\} \cup \{x + c(x) \mid x \in K\} \cup \{x + b(x) \mid x \in K,$ card $x \leq 4\} \cup \{x + b(x) + d(x) + bd(x) \mid x \in K\}$. Finally, let $W = W_n$ be the subspace generated by $N = N_n = S \cup J \cup L$.

5.1. <u>Lemma</u>. V is the direct sum of the subspaces U and W.

Proof. This is an easy consequence of 3.7 and 4.1.

Define a mapping $F: M^3 \to V$ as follows: $F(a_i, a_j, a_k) = (ijk)$ for all $1 \leq i,j,k \leq n$; $F(x,u,v) = F(y,u,v) = F(u,y,v) = F(u,v,y) = 0$ for all $x \in I$, $y \in K$, $u,v \in M$; $F(a_i, a_j, (kpq)) = (ijkpq)$ for all $1 \leq i,j,k,p,q \leq n$. By 2.3, F can be extended in a unique way to a trilinear mapping (denoted again by $F = F_n$) of V^3 into V. Thus we obtain a ternary algebra $V(+,F)$.

5.2. <u>Lemma</u>. U is an ideal of $V(+,F)$.

Proof. Easy.

Let $P = P(+,T) = P_n(+,T_n) = V(+,F)/U$. Denote by t the natural homomorphism of $V(+,F)$ onto $P(+,T)$.

5.3. <u>Lemma</u>. $P(+,T)$ satisfies (a),(b),(c),(d),(e).

Proof. Easy (use 2.2).

Put $r(x) = x + (jki) + (kij)$ and $r(y) = y + dc(y) + cd(y)$ for all $x = (ijk) \in I$ and $y \in K$. Let X designate the subspace generated by $\{r(x) \mid x \in I\}$ and Y the subspace generated by $\{r(y) \mid y \in K\}$.

5.4. <u>Lemma</u>. $4\binom{n}{4} + 4\binom{n}{5} \leq \dim t(y)$.

Proof. This follows from 3.8 and 4.2.

5.5. <u>Lemma</u>. $\binom{n}{3} = \dim t(X)$.

Proof. Let $x = (ijk) \in I$. If $\{i,j,k\}$ contains at most two elements then $tr(x) = 0$. Suppose $\{i,j,k\} = \{1,2,3\}$ and put $z = (123) + (231) + (312)$, $v = (123) + (231) - (132)$. Then $v \in W$ and $t(z) = t(v) \neq 0$. The rest is clear.

5.6. <u>Lemma</u>. $t(X) \cap t(Y) = 0$.

Proof. It is easy to see that $X \cap (Y + U) \subseteq U$.

5.7. <u>Lemma</u>. $\binom{n}{3} + 4\binom{n}{4} + 4\binom{n}{5} \leq \dim t(X + Y)$.

Proof. Use 5.4, 5.5 and 5.6.

Now, let $x \circ y = x + y + T(x,y,x-y)$ for all $x,y \in P$. Let $Q(\circ) = Q_n(\circ)$ be the subgroupoid of $P(\circ)$ generated by $t(S)$.

5.8. <u>Lemma</u>. $P(\circ)$ and $Q(\circ)$ are 3-elementary CM-loops nilpotent of class at most 3.

Proof. See 5.3 and 2.4.

5.9. <u>Lemma</u>. $A_2(P(\circ)) = A_2(Q(\circ)) = t(Y)$.

Proof. Put $b_i = t(a_i)$ for $i = 1,2,\dots,n$ and $e(y) = [[b_k,b_p,b_q],b_i,b_j]$ for $y = (ijkpq) \in K$. By 2.4(iii), $e(y) = tr(y)$. According to 1.3(ii), $A_2(Q(\circ))$ is just the subloop generated by $\{e(y) \mid y \in K\}$. On the other hand, $u \circ v = u + v$

for all $u,v \in Z$, where Z is the subspace of $\mathbf{P}$ generated by $t(I \cup K)$. Now, it is clear that $A_2(Q(\circ)) = t(Y)$. Similarly the rest.

5.10. <u>Lemma</u>. $A_1(Q(\circ)) = t(X + Y)$.

Proof. Let $Z = t(X + Y)$ and let g be the natural homomorphism of $Q(\circ)$ onto $Q(\circ)/A_2(Q(\circ)) = G(\circ)$. By 1.3(i), $A_1(G(\circ))$ is generated by all $[gt(a_i), gt(a_j), gt(a_k)]$, $1 \leq i, j, k \leq n$. Further, $u \circ v = u + v$ for all $u,v \in Z$. Hence $Z(\circ)$ is a subloop and $g(Z) = A_1(G(\circ))$. However, Ker $g = A_2(Q(\circ)) \subseteq Z$, and so $Z = A_1(Q(\circ))$.

5.11. <u>Theorem</u>. Let $4 \leq n$ and $Q(\circ) = Q_n(\circ)$. Then:

(i) $Q(\circ)$ is a free loop of rank n in the variety of 3-elementary CM-loops nilpotent of class at most 3.

(ii) card $Q = 3^m$, $m = n + \binom{n}{3} + 4\binom{n+1}{5}$.

(iii) card $A_1(Q(\circ)) = 3^p$ and card $A_2(Q(\circ)) = 3^q$, $q = 4\binom{n+1}{5}$ and $p = \binom{n}{3} + q$.

(iv) $C_1(Q(\circ)) = A_2(Q(\circ))$ and $C_2(Q(\circ)) = A_1(Q(\circ))$.

Proof. (i),(ii) and (iii). Let $G(\circ)$ be a free 3-elementary CM-loop nilpotent of class at most 3 freely generated by the set S. There is a surjective homomorphism g of $G(\circ)$ onto $Q(\circ)$ such that $g(a_i) = t(a_i)$ for every i. We have $g(A_2(G(\circ))) = A_2(Q(\circ))$ and $3^q \leq$ card $A_2(Q(\circ)) \leq$ card $A_2(G(\circ)) \leq 3^q$ by 1.4(ii) and 5.4, 5.9. Hence card $A_2(Q(\circ)) = 3^q$. Similarly, card $A_1(Q(\circ)) = 3^p$. The loop $Q(\circ)$ cannot be generated by $n - 1$ elements (otherwise card $A_2(Q(\circ)) < 3^q$, a contradiction) and consequently $Q(\circ)/A_1(Q(\circ)) = H(\circ)$ cannot be generated by $n - 1$ elements. From this, $3^n =$ card H, $3^{p+n} \leq$ card $Q \leq$ card $G \leq 3^{p+n}$, $3^{p+n} =$ card $Q =$ card G and g is an isomorphism.

(iv) Obviously, $C_1(Q(\circ)) \subseteq A_1(Q(\circ))$. It suffices to show that $u \in U$, whenever $u \in X$ and $F(a_i, a_j, u) \in U$ for all $1 \leq i,j \leq n$. There are $1 \leq s$, $k_1, \dots, k_s \in \{0,1,-1\}$ and $x_1, \dots, x_s \in I$ such that $u = k_1 r(x_1) + \dots + k_s r(x_s)$. Define a relation w on I by $(x,y) \in w$, where $x = (ijk) \in I$ and $y \in I$, iff either $x = y$ or $y = (jki)$ or $y = (kij)$. We can assume that $(x_i, x_j) \notin w$ for all $1 \leq i < j \leq s$. Now, let $x_1 = (kpq)$. If card $\{k,p,q\} \leq 2$ then $r(x_1) \in U$. If card $\{k,p,q\} = 3$ and $5 \leq n$ then there are $1 \leq i$, $j \leq n$ such that card $\{i,j,k,p,q\} = 5$ and the result follows from 3.9. Finally, suppose that $\{k,p,q\} = \{1,2,3\}$ and $n = 4$. We can assume that $s = 8$, $x_1 = (123)$, $x_2 = (213)$, $x_3 = (124)$, $x_4 = (214)$, $x_5 = (134)$, $x_6 = (314)$, $x_7 = (234)$, $x_8 = (324)$. Then $k_1 = k_2$ and $k_1 r(x_1) + k_2 r(x_2) \in U$. The rest is clear.

5.12. Corollary. Let $4 \leq n$ and $x * y = -x - y + T(x,y,y-x)$ for all $x,y \in Q_n$. Then $Q_n(*)$ is a free quasigroup of rank $n + 1$ in the variety of DS-quasigroups nilpotent of class at most 3.

5.13. Lemma. Let $G(\circ)$ be a normal subloop of $Q(\circ)$ such that $G \subseteq A_1(Q(\circ))$. Then G is an ideal of the ternary ring $P(+,T)$.

Proof. It suffices to show that $t(F(a_i,a_j,u)) \in G$, whenever $1 \leq i,j \leq n$ and $u \in X + Y$ is such that $t(u) \in G$. We have $t(F(a_i,a_j,u)) = t(\bar{F}(a_i,a_j,u)) = \bar{T}(t(a_i),t(a_j),t(u)) = [t(a_i),t(a_j),t(u)] \in G$.

5.14. Proposition. Let G be a finite 3-elementary CML-loop nilpotent of class at most 3. Then there exists a finite ternary algebra $H(+,E)$ over the three-element field such that $G \subseteq H$, $H(+,E)$ satisfies the identities (a),(b),(c),(d),(e) and

$xy = x + y + E(x,y,x-y)$ for all $x,y \in G$.

Proof. Assume that G can be generated by n elements but not by n - 1 elements. Then there is a surjective homomorphism g of $Q(\circ)$ onto G such that Ker $g \subseteq A_1(Q(\circ))$ and the rest follows from 5.13.

5.15. Proposition. Let G be a 3-elementary CM-loop nilpotent of class at most 3. Then there exists a ternary algebra $H(+,E)$ over the three-element field such that $G \subseteq H$, $H(+,E)$ satisfies the identities (b),(c),(d),(e) and $xy = x + y + E(x,y,x-y)$ for all $x,y \in G$.

Proof. G is an ultraproduct of its finitely generated subloops and the result follows from 5.14.

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