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TRILINEAR CONSTRUCTIONS OF QUASIMODULES

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Abstract: A general trilinear construction of quasimodules, i.e. commutative Moufang loops with operators from a ring, is investigated and the associators are computed.

Key words: Commutative Moufang loop, quasimodule.

Classification: 20N05

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1. Introduction. The method of constructing algebraic structures by means of trilinear mappings is very important, it was used e.g. in many classical examples of non-associative commutative Moufang loops. It is also known that all finite commutative Moufang loops nilpotent of class 2 can be obtained in such a way (see [4]). In this paper we present a trilinear construction of quasimodules under relatively weak assumptions on the trilinear mapping.

Let R be an associative ring with unit. A left R-quasimodule Q is an algebra Q(+,rx) with one binary operation + and a set of unary operations $x \longrightarrow rx$, $r \in R$, satisfying the identities $(x + x) + (y + z) = (x + y) + (x + z)$, $(-1)x + (x + y) = y$, $r(x + y) = rx + ry$, $(r + s)x = rx + sx$, $(rs)x = r(sx)$, $1x = x$, $0x = 0y$. Then Q(+) is a commutative Moufang loop, the inverse operation - of Q(+) is given by

$x - y = x + (-1)y$, the element $0 = 0x$ is the neutral element of $Q(+)$ and the unary operations $x \longrightarrow rx$ are endomorphisms of $Q(+)$. Obviously, every commutative Moufang loop is a Z-quasimodule, Z being the ring of integers.

Q is a left R-module iff $Q(+)$ is associative, i.e. $[a,b,c] = ((a + b) + c) - (a + (b + c)) = 0$ for all $a,b,c \in Q$. In this case, a mapping $T:Q^3 \longrightarrow Q$ is called trilinear if $T(a+b,c,d) = T(a,c,d)+T(b,c,d)$, $T(c,a+b,d) = T(c,a,d)+T(c,b,d)$, $T(c,d,a+b) = T(c,d,a)+T(c,d,b)$ and $T(ra,b,c) = T(a,rb,c) = T(amb,rc) = rT(a,b,c)$ for all $a,b,c,d \in Q$, $r \in R$.

The importance of quasimodules stems also from the fact that the theory of trimedial quasigroups is equivalent to the theory of quasimodules over a commutative noetherian ring (see [2]). Basic facts on quasimodules can be found e.g. in [3]. For the material concerning commutative Moufang loops, [1] is a standard reference.

2. <u>Ternary rings</u>. Throughout the paper, let $G = G(+,T)$ be a ternary ring, i.e. $G(+)$ is an abelian group and T is a trilinear mapping of G^3 into G assigning to each ordered triple $\langle a,b,c \rangle$ of elements of G an element $T(a,b,c) = Tabc$ of G (this notation cannot lead to any confusion, e.g. $TTabcde = T(T(a,b,c),d,e)$, $TaTbcde = T(a,T(b,c,d),e)$ etc.) Consider the following identitites for G:

(a1) $3Txyz = 0$

(a2) $3Txyy = 0$

(a3) $3Txxy = 0$

(b1) $Txxy = 0$

(b2) Txyz + Tyxz = 0

(c1) TTxyyxz = 0

(c2) TTxyyzu + TTzyyxu = 0

(c3) TTxyzxu + TTxzyxu = 0

(c4) TTxyzuv + TTxzyuv + TTuyzxv + TTuzyxv = 0

(d1) TTxyyzx = 0

(d2) TTxyyuz + TTzyyux = 0

(d3) TTxyzux + TTxzyux = 0

(d4) TTxyzuv + TTxzyuv + TTvyzux + TTvzyux = 0

(e1) TTxyyzz = 0

(e2) TTxyyzu + TTxyyuz = 0

(e3) TTxyzuu + TTxzyuu = 0

(e4) TTxyzuv + TTxzyuv + TTxyzvu + TTxzyvu = 0

(f1) TTxyyuv = 0

(f2) TTxyzuv + TTxzyuv = 0

(f3) TTxyzxu = 0

(f4) TTxyzuv + TTuyzxv = 0

(f5) TTxyzux = 0

(f6) TTxyzuv + TTvyzux = 0

(f7) TTxyzuu = 0

(f8) TTxyzuv + TTxyzvu = 0

(f9) TTxyzuv = 0

(g1) TxyTyzz = 0

(g2) TxyTzuu + TxzTyuu = 0

(g3) TxyTyzu + TxyTyuz = 0

(g4) TxyTzuv + TxzTyuv + TxyTzvu + TxzTyvu = 0

(g5) TxyTzuu = 0

(g6) TxyTzuv + TxyTzvu = 0

(g7) TxyTyzu = 0

(g8) $TxyTzuv + Txz\ Tyuv = 0$

(g9) $TxyTzuv = 0$

Further, if M is a subset of G, we shall say that (a1) holds for M if $3Tabc = 0$ for all $a,b,c \in M$ and similarly for other identities.

2.1. Lemma. (i) (a) implies (a2) and (a3).

(ii) (a2) implies $3Txyz + 3Txzy = 0$ for all $x,y,z \in G$.

(iii) (a3) implies $3Txyz + 3Tyxz = 0$ for all $x,y,z \in G$.

(iv) G satisfies (b1) iff G satisfies (b2) and (a3).

Proof. Obvious.

2.2. Lemma. Suppose that G satisfies (b2). Then:

(i) $T(x,yx-y) = T(y,x,y-x)$ for all $x,y \in G$.

(ii) (a2) implies $3Txyz = 3T(x,y,x-y) = 3T(x,y,x+y) = 0$ for all $x,y \in G$.

(iii) (a2) implies $3Txyz + 3Tzyx = 0$ for all $x,y,z \in G$.

Proof. (i) and (ii) are obvious.

(iii) Using (ii), we have $0 = 3T(x+z,y,x+z+y) = 3Txyz + 3Tzyx$.

2.3. Lemma. Let M be a generator set of the group $G(+)$. Then:

(i) If (X) is one of (a1),(b2),(c4),(d4),(e4),(f2),(f4),(f6),(f8),(f9),(g4),(g6),(g8),(g9) then G satisfies (X) iff (X) holds for M.

(ii) G satisfies (a2) iff $3Tabc + 3Tacb = 0 = 3Tabb$ for all $a,b,c \in M$.

(iii) G satisfies (a3) iff $3Tabc + 3Tbac = 0 = 3Taab$ for all $a,b,c \in M$.

(iv) G satisfies (b1) iff $Tabc + Tbac = 0 = 3Taab$ for all

$a,b,c \in M$.

Proof. (i) The "only if" assertion is obvious. Conversely, suppose that, e.g., (c4) holds for M. Put $S(x,y,z,u,v) = TTxyzuv + TTxzyuv + TTuyzxv + TTuzyxv$ for all $x,y,z,u,v \in G$. Then S is a multilinear mapping of G^5 into G and $S(M)^5 = 0$, hence $S = 0$ and G satisfies (c4).
The remaining assertions are very easy.

2.4. Lemma. Let y be one of c,d,e,g. Then:
(i) (y1) implies (y2) and (y3), (y2) implies (y4) and (y3) implies (y4).
(ii) If G satisfies (a2) then the identities (y1),(y2),(y3), (y4) are equivalent.

Proof. If G satisfies (c1) then $0 = T(T(x+z,y,y),x+ +z,u) = TTxyyzu + TTzyyxu$ and $0 = T(T(x,y+z,y+z),x,u) = = TTxyzxu + TTxzyxu$, if G satisfies (c2) then $0 = T(T(x,y+z, y+z),uv) + T(T(u,y+z,y+z),x,v) = TTxyzuv + TTxzyuv + + TTuyzxv + TTuzyxv$ and if G satisfies (c3) then $0 = = T(T(x+u,y,z),x+u,v) + T(T(x+u,z,y),x+u,v) = TTxyzuv + + TTxzyuv + TTuyzxv + TTuzyxv$. Finally, if G satisfies (a2) and (c4) then $0 = 4TTxyyxv = T(4T(x,y,y),x,v) = TTxyyxv$. The rest is similar.

The following three lemmas are obvious.

2.5. Lemma. (i) (c2) and (d1) imply (e1).
(ii) (c2) and (e1) imply (d1).
(iii) (d2) and (e1) imply (c1).

2.6. Lemma. (i) (b1) and (c2) imply $TTxyyyz = 0$ for all $x,y,z \in G$.
(ii) (b2) and (c3) imply $TTxyzyu + TTzyxyu = 0$ for all

$x,y,z \in G$.

(iii) (b1) and (d2) imply $TTxyyzy = 0$ for all $x,y,z \in G$.

(iv) (b2) and (d3) imply $TTxyzuy + TTzyxuy = 0$ for all $x,y,z \in G$.

(v) (b2) and (e3) imply $TTxyzuu + TTzyxuu = 0$ for all $x,y,z \in G$.

(vi) (b1) and (g2) imply $TxyTzyy = 0 = TxyTxzz$ for all $x,y,z \in G$.

2.7. Lemma. (i) (f1) implies (c1),(d1),(e1) and (f2).

(ii) (a2) and (f2) imply (f1).

(iii) (f3) implies (c1) and (f4).

(iv) (a1) and (f4) imply (f3).

(v) (f5) implies (d1) and (f6).

(vi) (a1) and (f6) imply (f5).

(vii) (f7) implies (e1) and (f8).

(viii) (a1) and (f8) imply (f7).

(ix) (f9) implies (f1),(f3),(f5) and (f7).

(x) (g5) implies (g1) and (g6).

(xi) (a2) and (g6) imply (g5).

(xii) (g7) implies (g1) and (g8).

(xiii) (a1) and (g8) imply (g7).

(xiv) (g9) implies (g1,),(g5) and (g7).

2.8. Lemma. If G satisfies (b1),(c1),(d1) and $x,y,z,u,v \in G$ are such that $|\{x,y,z,u,v\}| \leq 3$ then:

(i) $TTxyyuv = 0$.

(ii) $TTxyzuv + TTxzyuv = 0$.

(iii) If $|\{x,y,z\}| \leq 2$ then $TTxyzuv = 0$.

Proof. (i) This is clear from (b1),(c1),(d1), 2.5(i) and 2.6(i),(iii).

(iii) An immediate consequence of (i) and (b1).

(ii) With respect to (iii), we can assume that the elements x,y,z are pair-wise different. As G satisfies (c3),(d3) and (e3), we can also assume that $u \neq x \neq v$, $u \neq v$, and hence either y = u, z = v or y = v, z = u. If y = u, z = v (the other case is symmetric) then TTxyzyz + TTxzyyz = -TTyxzyz - - TTzxyyz = TTyzxyz + TTzyxyz = 0.

2.9. Lemma. If G satisfies (b1),(g1) and $x,y,z,u,v \in G$ are such that $|\{x,y,z,u,v\}| \leq 3$ then:

(i) TxyTzuu = 0.

(ii) TxyTzuv + TxyTzvu = 0.

(iii) If $|\{z,u,v\}| \leq 2$ then TxyTzuv = 0.

Proof. Similar to that of 2.8.

2.10. Lemma. If G satisfies (b1),(c2) and (g1) then TxTyzzTyzz = 0 = TxTyzzTzyy for all $x,y,z \in G$.

Proof. Using (b2) and (c2), we have TxTyzzTzyy = = -TTyzzxTzyy = TTxzzyTzyy = 0 by 2.6(vi). The rest is similar.

2.11. Corollary. If G satisfies (b1),(c2) and (g1), then T(x,T(y,z,y-z),T(y,z,y-z)) = 0 for all $x,y,z \in G$.

3. Trilinear constructions of quasimodules. If G = = G(+,T) is a ternary ring, we put $x * y = x + y + T(x,y,x-y) =$ $= x + y + T(x,y,x) - T(x,y,y)$ for all $x,y \in G$. Thus we obtain a groupoid $G(*)$.

3.1. Lemma. (i) 0 is a unit element of $G(*)$.

(ii) If G satisfies (b2) then $G(*)$ is commutative.

Proof. (i) is obvious and (ii) follows from 2.2(i).

3.2. <u>Lemma</u>. If G satisfies (a2),(b1),(c1),(d1),(g1), then $(x*x)*(y*z) = (x*y)*(x*z)$ for all $x,y,z \in G$.

Proof. Using (a2) and (b2), we have $(x*x)*(y*z) = 2x + y + z + u_1 + u_2 + u_3 + u_4$, where $u_1 = -Tyxx - Tzxx - 2Txyy - 2Txyz - 2Txzy - Tzyy - Tyzz - 2Txzz$, $u_2 = -T(T(y,z,y-z),x,x) + 2T(T(y,z,y-z),x,y) + 2T(T(y,z,y-z),x,z)$, $u_3 = -2T(x,y,T(y,z,y-z)) - 2T(x,z,T(y,z,y-z))$ and $u_4 = T(x,T(y,z,y-z),T(y,z,y-z))$. Similarly, by (b1) and (b2) we have $(x*y)*(x*z) = x + y + x + z + v_1 + v_2 + v_3 + v_4 + v_5 + v_6$, where $v_1 = -Tyxx - Txyy - Tzxx - Txzz - Txyy + Txzy - Tzyy + Txyz - Txzz - Tyzz$, $v_2 = T(T(x,y,x-y),x,y) - T(T(x,y,x-y),x,z) + T(T(x,y,x-y),z,y) - T(T(x,y,x-y),z,z) - T(T(x,z,x-z),x,y) + T(T(x,z,x-z),x,z) - T(T(x,z,x-z),y,y) + T(T(x,z,x-z),y,z)$, $v_3 = T(x,z,T(x,y,x-y)) - T(x,z,T(x,z,x-z)) + T(y,x,T(x,y,x-y)) - T(y,x,T(x,z,x-z)) + T(y,z,T(x,y,x-y)) - T(y,z,T(x,z,x-z))$, $v_4 = T(T(x,y,x-y), T(x,z,x-z),y) + T(T(x,z,x-z), T(x,y,x-y),z)$, $v_5 = T(x,T(x,z,x-z), T(x,y,x-y)) - T(x,T(x,z,x-z), T(x,z,x-z)) + T(y,T(x,z,x-z), T(x,y,x-y)) - T(y,T(x,z,x-z), T(x,z,x-z)) - T(x,T(x,y,x-y), T(x,y,x-y)) + T(x,T(x,y,x-y), T(x,z,x-z)) - T(z,T(x,y,x-y), T(x,y,x-y)) + T(z,T(x,y,x-y), T(x,z,x-z))$ and $v_6 = -T(T(x,z,x-z), T(x,y,x-y), T(x,y,x-y)) - T(T(x,y,x-y), T(x,z,x-z), T(x,z,x-z))$. Now $u_2 = v_2 = v_4 = 0$ by 2.8(iii) and $u_3 = 0 = v_3$ by 2.9(iii). Using 2.11, we have $u_4 = 0 = v_6$ and $v_5 = T(x,T(x,z,x-z), T(x,y,x-y)) + T(y,T(x,z,x-z), T(x,y,x-y)) + T(x,T(x,y,x-y), T(x,z,x-z)) + T(z,T(x,y,x-y), T(x,z,x-z)) = 0$ by 2.9(iii). Finally, $v_1 - u_1 = 3Txyz + 3Txzy = 0$ by 2.1(ii).

3.3. <u>Lemma</u>. If G satisfies (a2),(b1),(c1) and (g1) then $(-x)*(x*y) = y = x*((-x)*y)$ for all $x,y \in G$.

Proof. Easy.

3.4. <u>Proposition</u>. If G satisfies (a2),(b1),(c1),(d1) and (g1) then $G(*)$ is a commutative Moufang loop.

Proof. By 3.1, 3.2 and 3.3.

3.5. <u>Lemma</u>. If G satisfies (b1),(c1),(d1) and (g1) then $a*(b*c) = a + b + c - Tcbb - Tbcc - Tbaa - Tabb - Tcaa - Tacc - Tabc - Tacb$ and $(a*b)*c = a + b + c - Tbaa - Tabb - Tcaa - Tacc - Tcbb - Tbcc - Tcab - Tcba = a*(b*c) + Tabc + Tbca - 2Tcab$.

Proof. Use (b2), 2.8(iii) and 2.9(iii).

3.6. <u>Proposition</u>. If G satisfies (a2),(b1),(c1),(d1) and (g1) then $[a,b,c] = Tabc + Tbca - 2Tcab$ for all $a,b,c \in G$.

Proof. Denote $u = a*(b*c)$, $v = Tabc + Tbca - 2Tcab$. With respect to 3.5, we have $[a,b,c] = (u + v)*(-u) = v + T(u+v,-u,u+v+u) = v - 2T(v,u,u) - T(v,u,v) = v + T(v,u,u-v)$. Obviously, $u-v = a + b + c - Tabb - Tbaa - Tacc - Tcaa - Tbcc - Tcbb - 2Tabc - Tbca + 3Tcab$. In the rest of the proof, which will be divided into several lemmas, we shall show that $T(v,u,u-v) = 0$. In Corollaries 3.8, 3.10, 3.12, 3.14, 3.16, 3.19 and 3.22, the assumptions are those of Proposition 3.6.

3.7. <u>Lemma</u>. If G satisfies (b2),(c3) and (d3) then $TTabcab = TTbcaab = TTcabab$, $TTabcba = TTbcaba = TTcabba$ and $TTabcaa = TTbcaaa = TTcabaa$ for all $a,b,c \in G$.

Proof. Easy (use the fact that (c4) and (d3) imply (e3).

3.8. Corollary. T(v,a+b+c,a+b+c) = 0.

3.9. Lemma. If G satisfies (b1),(c3) and (g1) then TTabcaTbcc = TTbcaaTbcc = TTcabaTbcc and TTabcaTcbb = = TTbcaaTcbb = TTcabaTcbb for all $a,b,c \in G$.

Proof. We have TTcabaTbcc = -TTacbaTbcc = TTabcaTbcc by (b2) and (c3). Further, with respect to (b2) and (c4), TTcabaTbcc - TTbcaaTbcc = TTcabaTbcc + TTcbaaTbcc = = -TTaabcTbcc - TTabacTbcc = 0 by 2.9(iii). The rest is similar.

3.10. Corollary. T(v,a+b+c, -Tabb-Tbaa-Tacc-Tcaa-Tbcc--Tcbb) = 0.

Proof. Use (g1), 2.6(vi) and 3.9.

3.11. Lemma. If G satisfies (b1) and (g3) then TdaTabc = TdaTcab = TdaTbca - TdbTcaa for all $a,b,c,d \in G$.

Proof. By (g3) and (b2), TdaTabc = -TdaTacb = TdaTcab. Further, using (b2) and (g4), we have TdaTabc = -TdaTbac = = TdaTbca + TdbTaac + TdbTaca = TdaTbca - TdbTcaa by (b1).

3.12. Corollary. T(v,a+b+c, -2Tabc-Tbca+3Tcab) = 0.

Proof. By 3.11, TvaTabc = TvaTcab = TvaTbca - TvbTcaa, however, TvbTcaa = 0 by 3.9. The rest is similar.

3.13. Lemma. If G satisfies (b1),(c1) and (d1) then TTabcTabbd = TTbcaTabbd = TTcabTabbd and TTabcTbaad = = TTbcaTbaad = TTcabTbaad for all $a,b,c,d \in G$.

Proof. With respect to (b2) and (c4) we have TTbcaTabbd = -TTcbaTabbd = TTcabTabbd + TTTabbbacd + + TTTabbabcd = TTcabTabbd by 2.8(iii) and TTcabTabbd = = -TTacbTabbd = TTabcTabbd again by (c4) and 2.8(iii). The rest is similar.

3.14. Corollary. T(v,-Tabb-Tbaa-Tacc-Tcaa-Tbcc-Tcbb, u-v) = 0.

3.15. Lemma. If G satisfies (b1) and (d3) then 0 = = TTabcTabcd = TTcabTacbd = TTcabTabca = TTabcTacba = = TTbcaTabcb = TTbcaTacbc, TTbcaTabca + TTbcaTacba = 0, TTcabTabcb = TTabcTacbb = TTbcaTacbb and TTbcaTabcc = = TTcabTabcc = TTabcTacbc for all a,b,c,d $\in$ G.

Proof. Using (b2) and (d3), we have TTbcaTabca = = - TTabcTbcaa = TTacbTbcaa = - TTbcaTacba, TTbcaTacbb = = -TTbacTacbb = TTabcTacbb = -TTacbTabcb = TTcabTabcb, TTbcaTabcc = -TTcbaTabcc = TTcabTabcc = -TTacbTabcc = = TTabcTacbc and TTcabTabca = -TTacbTabca = TTabcTabca = 0 by (d3) and (b1). The rest is similar.

3.16. Corollary. T(v,-Tabc-Tacb,a+b+c) = 0.

3.17. Lemma. If G satisfies (g2) and (e3) then TfTacbTdee = -TfTabcTdee for all a,b,c,d,e,f $\in$ G.

Proof. Using consecutively (g2),(e3) and (g2), we have TfTacbTdee = -TfdTTacbee = TfdTTabcee = -TfTabcTdee.

3.18. Lemma. If G satisfies (b1),(g2) and (e3) then TTabcTabcTdee = TTbcaTabcTdee = TTcabTabcTdee = 0 for all a,b,c,d,e $\in$ G.

Proof. Obviously, TTabcTabcTdee = 0 by (b1). Further, using (b2) and 3.17, we have TTcabTabcTdee = -TTacbTabcTdee= = TTacbTacbTdee = 0. Finally, by 3.17 and (b2), TTbcaTabcTdee = -TTbcaTacbTdee = TTbcaTcabTdee = 0 by the preceding part of the proof.

3.19. Corollary. T(v,-Tabc-Tacb,-Tabb-Tbaa-Tacc-Tcaa-Tbcc-Tcbb) = 0.

3.20. <u>Lemma</u>. If G satisfies (b1) and (e3) then TTabcTabcTabc = TTbcaTabcTabc = TTcabTabcTabc = 0 for all a,b,c $\in$ G.

Proof. By (e3) and (b2) we have TTbcaTabcTabc = = -TTbacTabcTabc = TTabcTabcTabc = -TTacbTabcTabc = = TTcabTabcTabc and it suffices to use (b1).

3.21. <u>Lemma</u>. If G satisfies (b1),(c4),(e3) and (g4) then TTabcTbcaTcab = TTbcaTcabTabc = TTcabTabcTbca for all a,b,c $\in$ G.

Proof. The preceding lemma together with (b2) yield TTcabTabcTcab = 0 and TTcbaTabcTcba = TbacTcbaTcba = 0. Thus, using consecutively (b2),(c4),(g3),(c4) and (b2), we get TTabcTbcaTcab = TTcbaTabcTcab = -TTcabTabcTcab - - TTTabcbacTcab - TTTabcabcTcab = TTTabcbacTcba + + TTTabcabcTcba + TTcbaTabcTcba = -TTcabTabcTcba = = TTcabTabcTbca. Now it suffices to use this equality once more.

3.22. <u>Corollary</u>. V = T(v,-Tabc-Tacb,-2Tabc-Tbca+3Tcab)= = 0.

Proof. Taking into account 3.20, we see that V = = TTabcTacbTbca - 3TTbcaTabcTcab + 2TTbcaTacbTabc - - 2TTcabTabcTbca = -TTcabTabcTbca - 3TTbcaTabcTcab - - 3TTbcaTcabTabc + TTbcaTcabTabc = 0 by 2.1(ii) and 3.21.

Summarizing the results of Corollaries 3.8, 3.10, 3.12, 3.14, 3.16, 3.19 and 3.22, we see that T(v,u,u-v) = 0 and the proof of Proposition 3.6 is finished.

3.23. <u>Lemma</u>. If G satisfies (b2) and (e4), a,b,c,d,e $\in$ $\in$ G and v = Tabc + Tbca - 2Tcab then Tvde = Tevd.

Proof. We have Tvde + Tved = TTabcde + TTbcade - - 2TTcabde + TTabced + TTbcaed - 2TTcabed = -TTacbde - - TTacbed - TTbacde - TTbaced - 2TTcabde - 2TTcabed = = TTacbde + TTacbed + TTabcde + TTabced = 0.

Before summarizing our results, we introduce a trilinear mapping T' of G^3 into G via $T'(a,b,c) = T(a,b,c) - T(c,a,b)$ for all $a,b,c \in G$.

3.24. <u>Theorem</u>. Let R be an associative ring with unit, $G(+,rx)$ be a left R-module and T be a trilinear mapping of G^3 into G. Suppose that the following holds: $3T(x,y,y) = 0$, $T(x,x,y) = 0$, $T(T(x,y,y),x,z) = 0$, $T(T(x,y,y),z,x) = 0$, $T(x,y,T(y,z,z)) = 0$ and $(r^3-r)T(x,y,y) = 0$ for all $x,y,z \in G$ and $r \in R$. Put $x * y = x + y + T(x,y,x-y)$ for all $x,y \in G$. Then:

(i) $G(*,rx)$ is a left R-quasimodule.

(ii) $G(*,rx)$ is a module iff $T(x,y,z) + T(y,z,x) = 2T(z,x,y)$ for all $x,y,z \in G$.

(iii) $[a,b,c] = T''(a,b,c)$, $[[a,b,c],d,e] = T'(d,e,T''(a,b,c))$ and $[[\ldots[[a_1,a_2,a_3],a_4,a_5],\ldots],a_{n-1},a_n] = T'(a_{n-1},a_n,T'(\ldots,T'(a_4,a_5,T''(a_1,a_2,a_3))))$ for all $a,b,c, d,e, a_1,\ldots,a_n \in G$.

(iv) $T'''(x,y,z) = 0$ and $x * y = x + y + T'(x,y,x+y)$ for all $x,y,z \in G$.

Proof. (i) and (ii) follow easily from 3.3, 3.4 and 3.6.

(iii) First, $T''(a,b,c) = T'(a,b,c) - T'(c,a,b) = T(a,b,c) - T(c,a,b) - T(c,a,b) + T(b,c,a) = [a,b,c]$ by 3.6. Now, denoting $v = [a,b,c]$, we have by 3.23 $[[a,b,c],d,e] = [d,e,[a,b,c]] = T(d,e,v) + T(e,v,d) - 2T(v,d,e) =$

$= T(d,e,v) - T(v,d,e) = T'(d,e,v) = T'(d,e,T''(a,b,c))$. The rest follows by an easy induction.

(iv) Obviously, $T'''(x,y,z) = T''(x,y,z) - T''(z,x,y) = [x,y,z] - [z,x,y] = 0$ and $T'(x,y,x+y) = T(x,y,x+y) - T(x+y,x,y) = T(x,y,x) + T(x,y,y) - T(y,x,y) = T(x,y,x) - T(x,y,y) = T(x,y,x-y)$ by (b1),(b2) and (a2).

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