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GENERALIZED FLATNESS AND COHERENCE

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Abstract: In this paper flatness and coherence relative to a cohereditary idempotent radical s is studied. Results here obtained are applied to the M-flatness with respect to a pseudoprojective module M.

Key words: Relatively flat modules, relative coherence, preradicals.

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Secondary 18E40

In what follows, R stands for an associative ring with a unit element and R-mod (mod-R) denotes the category of all unitary left (right) R-modules.

First of all, we shall list several basic definitions from the theory of preradicals.

Recall that a preradical r for R-mod is a subfunctor of the identity functor, i.e. r assigns to each module M its submodule r(M) in such a way that every homomorphism of M into N induces a homomorphism of r(M) into r(N) by restriction.

A module M is r-torsion if r(M)=M and r-torsionfree if r(M) = 0. The class of all r-torsion (r-torsionfree) modules will be denoted by $\mathcal{T}_r$ ($\mathcal{F}_r$).

A preradical r is said to be

- idempotent if $r(M) \in \mathcal{T}_r$ for every module M,
- a radical if $M/r(M) \in \mathcal{F}_r$ for every module M,
- hereditary if for every module M and every monomorphism $A \longrightarrow r(M)$ $A \in \mathcal{T}_r$,
- cohereditary if for every module M and every epimorphism $M/r(M) \longrightarrow A$ $A \in \mathcal{F}_r$,
- superhereditary if it is hereditary and $\mathcal{T}_r$ is closed under direct products,
- centrally splitting if it is cohereditary and r(R) is a ring direct summand of R.

If r and s are preradicals then we write $r \leq s$ if $r(M) \subseteq s(M)$ for all $M \in R\text{-mod}$.

The idempotent core $\bar{r}$ of a preradical r is defined by $\bar{r}(M) = \sum K$, where K runs through all r-torsion submodules K of M and the radical closure $\tilde{r}$ is defined by $\tilde{r}(M) = \bigcap L$, where L runs through all submodules L of M with M/L r-torsionfree. Further, the hereditary closure h(r) is defined by $h(r)(M) = M \cap r(E(M))$, where E(M) is an injective hull of a module M and the cohereditary core ch(r) by $ch(r)(M) = r(R) \cdot M$.

A module P is called pseudoprojective if for any epimorphism $f: B \longrightarrow A$ and any homomorphism $0 \neq g: P \longrightarrow A$, there exist homomorphisms $h: P \longrightarrow B$ and $k: P \longrightarrow P$ such that $0 \neq g \circ k = f \circ h$.

For a module M let us define $p_{\{M\}}(N) = \sum \operatorname{Im} f$, f ranging over all $f \in \operatorname{Hom}_R(M,N)$. It is easy to see that $p_{\{M\}}$ is an idempotent preradical. Moreover $p_{\{M\}}$ is cohereditary if and only if M is pseudoprojective.

Let r be a preradical. We say that a submodule A of a module B is

- $(r,1)$-dense in B if there is a module C such that $A \subseteq B \subseteq C$ and $B/A \subseteq r(C/A)$,
- $(r,2)$-dense in B if $B/A \in \mathcal{T}_r$

Let r be a preradical and $i \in \{1,2\}$. A module Q is said to be (r,i)-injective ((i,r)-injective) if for every monomorphism $f:A \longrightarrow B$ and every homomorphism $g:A \longrightarrow Q$ with Im f is (r,i)-dense in B ($f(\mathrm{Ker}\, g)$ is (r,i)-dense in B) there exists a homomorphism $h:B \longrightarrow Q$ such that $h \circ f = g$.

Definition 1. Let s be a preradical for mod-R. A module $_RQ$ is called s-flat if $\mathrm{Tor}_1^R(N,Q) = 0$ for every $N \in \mathcal{T}_s$.

As it is easy to see, a module $_RQ$ is s-flat if and only if its character module Q_R^* is $(s,2)$-injective. Since a module is $(\tilde{\tilde{s}},2)$-injective if and only if it is $(s,2)$-injective, we obtain immediately the following proposition.

Proposition 2. If s is a preradical for mod-R, then a module $_RQ$ is s-flat if and only if it is $\tilde{\tilde{s}}$-flat.

The first part of the following proposition is essentially due to R.W. Miller and M.L. Teply [16].

Proposition 3. Let s be a preradical for mod-R and $Q \in$ $\in$ R-mod. Consider the following conditions.

(i) Q is s-flat.

(ii) Given any exact sequence

$$0 \longrightarrow K \hookrightarrow P \longrightarrow Q \longrightarrow 0$$

with P projective, there is for every $x \in s(R)\cdot K$ a homomorphism $f_x:P \longrightarrow K$ such that $f_x(x) = x$.

(iii) Given any exact sequence

$$0 \longrightarrow K \hookrightarrow P \longrightarrow Q \longrightarrow 0$$

with P projective, there is for each finite subset $\{x_1, x_2, \dots, x_n\}$ of $s(R)\cdot K$ a homomorphism $f: P \longrightarrow K$ such that $f(x_i) = x_i$ for every $i \in \{1,2,\dots,n\}$.

(iv) Given any $t_p \in s(R)$, $q_j \in Q$, $r_{i,j} \in R$, $i \in \{1,2,\dots,m\}$, $j \in \{1,2,\dots,n\}$, $p \in \{1,2,\dots,q\}$, with $\sum_{j=1}^{n} r_{i,j}\, q_j = 0$ for each $i \in \{1,2,\dots,m\}$, there is $u_k \in Q$ and $b_{j,k} \in R$, $j \in \{1,2,\dots,n\}$, $k \in \{1,2,\dots,t\}$, such that $q_j = \sum_{k=1}^{t} b_{j,k}\cdot u_k$ for $j \in \{1, 2,\dots,n\}$ and $t_p(\sum_{j=1}^{n} r_{i,j}\cdot b_{j,k}) = 0$ for $i \in \{1,2,\dots,m\}$, $k \in \{1,2,\dots,t\}$, $p \in \{1,2,\dots,q\}$.

(v) Every diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & N & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle g} & & \\ & & & & B & \underset{f}{\longrightarrow} & Q & \longrightarrow & 0 \end{array}$$

with exact rows, F free, K, F finitely generated and $K = s(R)\,K$ can be completed by a homomorphism $h: N \longrightarrow B$ to a commutative one.

(vi) For every module N for which there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$$

with F free, K, F finitely generated and $K = s(R)\cdot K$, the natural homomorphism

$\varphi = \varphi_{N,Q}: \mathrm{Hom}_R(N,R) \otimes {}_R Q \longrightarrow \mathrm{Hom}_R(N,Q)$ defined via

$\varphi(f \otimes q)(n) = f(n)\cdot q$, $f \in \mathrm{Hom}_R(N,R)$, $q \in Q$, $n \in N$

is an epimorphism.

(vii) Every diagram

$$\begin{array}{ccccccccc} & & & & & & N & & \\ & & & & & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & K & \longrightarrow & B & \underset{f}{\longrightarrow} & Q & \longrightarrow & 0 \end{array}$$

with exact row, $K = s(R)\cdot K$ and N finitely presented can be completed by a homomorphism $h:N \longrightarrow B$ to a commutative one.

(viii) $Q/(0:s(R))_r Q$ is flat in $R/(0:s(R))_r$-mod.

Then (ii) is equivalent to (iii), (iii) is equivalent to (iv) and (v) is equivalent to (vi). If s is idempotent then (i) implies (ii). Conversely, if s is cohereditary then (ii) implies (i). Further,

- if $s(R)$ is finitely generated as a left ideal then (iv) implies (v),
- if $s(R)$ is finitely generated as a left ideal and $s(R)$ is idempotent then (v) implies (iv),
- if s is a cohereditary idempotent radical and $s(R)$ is finitely generated as a left ideal then (i) is equivalent to (viii),
- if $s(R)$ is finitely generated as a left ideal and $R/s(R)$ is flat as a right R-module then (iv) implies (vii),
- if $s(R)$ is a ring direct summand in R then (vii) implies (iv).

<u>Proof</u>: (ii) is equivalent (iii), (iii) is equivalent to (iv), (i) implies (ii) for s idempotent and (ii) implies (i) for s cohereditary. The proof can be led along the same line as in Theorem 2.1 in [16].

(iv) implies (v). Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{k} & N & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & B & \xrightarrow[f]{} & Q & \xrightarrow{g} & 0 \end{array}$$

with exact rows, where F is finitely generated free with a free basis $\{x_1,x_2,\dots,x_n\}$, $K = \sum_{i=1}^{m} Rk_1$, $K = s(R)\cdot K$ and

$s(R) = \sum_{p=1}^{q} Rt_p$. Set $q_j = (g \circ k)(x_j)$, $j \in \{1,2,\dots,n\}$. Then $k_i = \sum_{j=1}^{n} r_{i,j} \cdot x_j$, $i \in \{1,2,\dots,m\}$, and hence $0 = (g \circ k)(k_i) = \sum_{j=1}^{n} r_{i,j} \cdot q_j$. By (iv) there is $u_k \in Q$ and $b_{j,k} \in R$, $k \in \{1,2,\dots,t\}$, $j \in \{1,2,\dots,n\}$ such that $q_j = \sum_{k=1}^{t} b_{j,k} \cdot u_k$ for $j \in \{1,2,\dots,n\}$ and $t_p(\sum_{j=1}^{n} r_{i,j} \cdot b_{j,k}) = 0$ for $i \in \{1,2,\dots,m\}$, $k \in \{1,2,\dots,t\}$ and $p \in \{1,2,\dots,q\}$. For $k \in \{1,2,\dots,t\}$ choose $e_k \in B$ such that $f(e_k) = u_k$ and define $h: F \longrightarrow B$ by $h(x_j) = \sum_{k=1}^{t} b_{j,k}\, e_k$. Then $(f \circ h)(x_j) = f(\sum_{k=1}^{t} b_{j,k} \cdot e_k) = \sum_{k=1}^{t} b_{j,k} \cdot u_k = q_j = (g \circ k)(x_j)$, $j \in \{1,2,\dots,n\}$ and consequently $f \circ h = g \circ k$. Further, if $i \in \{1,2,\dots,m\}$, $p \in \{1,2,\dots,q\}$ then $h(t_p k_i) = h(\sum_{j=1}^{n} t_p r_{i,j} x_j) = \sum_{j=1}^{n} t_p r_{i,j} (\sum_{k=1}^{t} b_{j,k}\, e_k) = \sum_{k=1}^{t} (t_p \sum_{j=1}^{n} r_{i,j}\, b_{j,k}) \cdot e_k = 0$. Thus $h(K) = 0$ and h induces a homomorphism $\bar{h}: N \longrightarrow B$ such that $f \circ \bar{h} = g$.

(v) implies (ii). Let $0 \longrightarrow K \hookrightarrow F \xrightarrow{f} Q \longrightarrow 0$ be an exact sequence, where F is free with a free basis $\{x_\alpha, \alpha \in A\}$. If $k \in K$ then $k = \sum_{i=1}^{n} r_i x_{\alpha_i}$, $r_i \in R$, $\alpha_i \in A$. Set $F_n = \sum_{i=1}^{n} Rx_{\alpha_i}$ and define a homomorphism $g: F_n \longrightarrow Q$ by $g(x_{\alpha_i}) = f(x_{\alpha_i})$ for $i \in \{1,2,\dots,n\}$. It is easy to see that $g(s(R)k) = 0$, hence g induces a homomorphism $\bar{g}: F_n/s(R)k \longrightarrow Q$. Now $F_n/s(R)k$ is finitely presented since $s(R)$ is finitely generated as a left ideal and $s(R)^2 = s(R)$ yields $s(R)(s(R)k) = s(R)k$. By (v) there exists a homomorphism $h: F_n/s(R)k \longrightarrow F$ such that $f \circ h = \bar{g}$. Setting $h(x_{\alpha_i} + s(R)k) = e_i$ for $i \in \{1,2,\dots,n\}$, we have $f(e_i) = (f \circ h)(x_{\alpha_i} + s(R)k) = \bar{g}(x_{\alpha_i} + s(R)k) = f(x_{\alpha_i})$, hence $x_{\alpha_i} - e_i \in K$ for $i \in \{1,2,\dots,n\}$. Let us define $\varphi: F \longrightarrow K$ by

$\varphi(x_{\alpha_i}) = x_{\alpha_i} - e_i$ for $i \in \{1,2,\dots,n\}$ and $\varphi(x_\alpha) = 0$ if $\alpha \notin \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. For $t \in s(R)$ we have $t \cdot \sum_{i=1}^{m} r_i e_i =$ $= t \cdot \sum_{i=1}^{m} r_i h(x_{\alpha_i} + s(R)k) = h(t \cdot \sum_{i=1}^{m} r_i x_{\alpha_i} + s(R)k) = h(tk +$ $+ s(R)k) = 0$. Thus $\varphi(tk) = \varphi(\sum_{i=1}^{m} tr_i x_{\alpha_i}) = \sum_{i=1}^{m} tr_i(x_{\alpha_i} -$ $- e_i) = t \cdot \sum_{i=1}^{m} r_i x_{\alpha_i} = tk$.

(v) is equivalent to (vi) is routine.

(i) is equivalent to (viii). It follows immediately from [16] Corollary 3.4.

(iv) implies (vii). Consider the following diagram

$$\begin{array}{ccccccccc} & & & & F/s(R)K & & \\ & & & & \downarrow \pi & & \\ & & & & F/K & & \\ & & & & \downarrow g & & \\ 0 & \longrightarrow & L & \longrightarrow & B \xrightarrow{f} Q & \longrightarrow & 0 \end{array}$$

with exact row , where $L = s(R)L$, F is finitely generated free with a free basis $\{x_1, x_2, \dots, x_n\}$, $K = \sum_{i=1}^{m} Rk_i$, $s(R) = \sum_{p=1}^{q} Rt_p$ and π is a natural epimorphism. By the same fashion as in (iv) implies (v) we can show that there exists a homomorphism $h: F/s(R)K \longrightarrow B$ such that $f \circ h = g \circ \pi$. Let r be a cohereditary radical in R-mod corresponding to $s(R)$ (i.e. $r(A) =$ $= s(R)A$ for all $A \in$ R-mod). By assumption $L \in \mathcal{T}_r$. Further $R/s(R)$ is flat as a right R-module, hence r is hereditary. Since $h(K/s(R)K) \subseteq L$, we have $h(K/s(R)K) \in \mathcal{T}_r \cap \mathcal{F}_r = 0$. Thus h induces a homomorphism $\bar{h}: F/K \longrightarrow B$ such that $f \circ \bar{h} = g$.

(vii) implies (ii). Let $0 \longrightarrow K \hookrightarrow F \xrightarrow{f} Q \longrightarrow 0$ be an exact sequence, where F is free with a free basis $\{x_\alpha, \alpha \in A\}$. By assumption $s(R)$ is a ring direct summand in R. Thus $R = s(R) \dotplus I$ for some ideal I. Consider the exact sequence

$0 \longrightarrow K/IK \longrightarrow F/IK \xrightarrow{\bar{f}} Q \longrightarrow 0$, where $\bar{f}$ is induced by f. As it is easy to see $s(R)(K/IK) = K/IK$. Now, if $k \in K$ then $k = \sum_{i=1}^{m} r_i x_{\alpha_i}$, $r_i \in R$, $\alpha_i \in A$. Set $F_n = \sum_{i=1}^{m} R x_{\alpha_i}$ and define a homomorphism $g: F_n \longrightarrow Q$ via $g(x_{\alpha_i}) = f(x_{\alpha_i})$ for $i \in \{1,2,\dots,n\}$. Then $g(Rk) = 0$ and g induces a homomorphism $\bar{g}: F_n/Rk \longrightarrow Q$. Further, F_n/Rk is finitely presented hence $\bar{f} \circ h = \bar{g}$ for some homomorphism $h: F_n/Rk \longrightarrow F/IK$ by (vii). Put $h(x_{\alpha_i} + Rk) = e_i + IK = \bar{e}_i$. As it is easy to see $x_{\alpha_i} - e_i \in K$ and we can define $\varphi: F \longrightarrow K$ by $\varphi(x_{\alpha_i}) = x_{\alpha_i} - e_i$ for $i \in \{1,2,\dots,n\}$ and $\varphi(x_\alpha) = 0$ if $\alpha \notin \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. We have $\sum_{i=1}^{m} r_i e_i \in IK$ since $\sum_{i=1}^{m} r_i \bar{e}_i = \sum_{i=1}^{m} r_i \cdot h(x_{\alpha_i} + Rk) = h(\sum_{i=1}^{m} r_i x_{\alpha_i} + Rk) = 0$. Now, if $t \in s(R)$ then $t \cdot \sum_{i=1}^{m} r_i e_i \in s(R)IK = 0$, hence $\varphi(tk) = \varphi(\sum_{i=1}^{m} t r_i x_{\alpha_i}) = \sum_{i=1}^{m} t r_i (x_{\alpha_i} - e_i) = t \sum_{i=1}^{m} r_i x_{\alpha_i} = tk$.

Definition 4. Let s be a preradical for mod-R. A module ${}_RQ$ satisfying one of the equivalent conditions (ii), (iii) and (iv) of Proposition 3 is said to be weakly s-flat.

Let ${}_RQ$ be a flat module. A module N_R is called Q-finitely generated (see [6]) if the natural homomorphism $\psi = \psi_{N,I}: N \otimes_R Q^I \longrightarrow (N \otimes_R Q)^I$ defined via $\psi(n \otimes q)(i) = n \otimes q(i)$ for $n \in N$, $q \in Q^I$, $i \in I$ is an epimorphism for every set I.

Theorem 5. Let s be a preradical for mod-R and ${}_RQ$ be a flat module. Consider the following conditions

(i) Q^I is weakly s-flat for every index set I.

(ii) If $\{Q_\alpha, \alpha \in A\}$ is a family of weakly s-flat modules, where $Q_\alpha \in \mathcal{T}_{P_{\{Q\}}}$ for every $\alpha \in A$ then $\prod_{\alpha \in A} Q_\alpha$ is weakly s-flat.

(iii) $\mathrm{Hom}_R(P,R)$ is Q-finitely generated for every module ${}_RP$ for which there exists an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$$

with F free, K, F finitely generated and $K = s(R)K$.

(iv) For every finitely generated right ideal I in R and an exact sequence $0 \longrightarrow K \longrightarrow F \xrightarrow{f} I \longrightarrow 0$ with F finitely generated free there is a finitely generated submodule K' of F such that $K \otimes_R Q \subseteq K' \otimes_R Q$ and $s(R)f(K') = 0$.

(v) $(Q/(0:s(R))_r Q)^I$ is flat in $R/(0:s(R))_r$-mod for every set I.

Then

- (ii) implies (i), (iv) implies (i),
- if s(R) is finitely generated as a left ideal then (i) implies (iii) and (i) implies (iv),
- if s(R) is finitely generated as a left ideal and s(R) is idempotent then (iii) implies (ii),
- if s is a cohereditary idempotent radical, s(R) is finitely generated as a left ideal and $(0:s(R))_r$ is finitely generated as a right ideal then (i) is equivalent to (v).

<u>Proof</u>: (ii) implies (i) trivially.

(i) implies (iii). Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(P,R) \otimes_R Q^I & \xrightarrow{\varphi_{P,Q^I}} & \mathrm{Hom}_R(P,Q^I) \\ \psi \downarrow & & \downarrow \omega \\ (\mathrm{Hom}_R(P,R) \otimes_R Q)^I & \xrightarrow{(\varphi_{P,Q})^I} & (\mathrm{Hom}_R(P,Q))^I \end{array}$$

where ω is the natural isomorphism and φ is defined as in Proposition 3 (vi). Now $(\varphi_{P,Q})^I$ is an isomorphism (see [14]), since Q is flat and P is finitely presented. Further, φ_{P,Q^I} is an epimorphism by Proposition 3 (vi). Hence ψ is an epimorphism and consequently $Hom_R(P,R)$ is Q-finitely generated.

(iii) implies (ii). For $N \in$ mod-R the class of all $M \in$ R-mod for which N is M-finitely generated is closed under the formation of direct sums of copies of M. Now if $Q_\alpha \in \mathcal{T}_{P_{\{Q\}}}$, $\alpha \in A$ then there exist a set I and epimorphisms $f_\alpha : Q^{(I)} \longrightarrow Q_\alpha$, $\alpha \in A$. Consider the following commutative diagram

$$\begin{array}{ccc} Hom_R(P,R) \otimes_R (Q^{(I)})^A & \xrightarrow{\psi} & (Hom_R(P,R) \otimes_R Q^{(I)})^A \\ \downarrow {\scriptstyle 1 \otimes \prod_{\alpha \in A} f_\alpha} & & \downarrow {\scriptstyle \prod_{\alpha \in A}(1 \otimes f_\alpha)} \\ Hom_R(P,R) \otimes_R \prod_{\alpha \in A} Q_\alpha & \xrightarrow[\psi_1]{} & \prod_{\alpha \in A}(Hom_R(P,R) \otimes_R Q\ ) \end{array}$$

where $\psi_1(f \otimes q)(\alpha) = f \otimes q(\alpha)$ for $f \in Hom_R(P,R)$, $q \in \prod_{\beta \in A} Q_\beta$, $\alpha \in A$. Then ψ is an epimorphism since $Hom_R(P,R)$ is $Q^{(I)}$-finitely generated , hence ψ_1 is an epimorphism. Now consider the following commutative diagram

$$\begin{array}{ccc} Hom_R(P,R) \otimes_R \prod_{\alpha \in A} Q_\alpha & \xrightarrow{\varphi_{P,\prod_{\alpha \in A} Q_\alpha}} & Hom_R(P, \prod_{\alpha \in A} Q_\alpha) \\ \downarrow {\scriptstyle \psi_1} & & \downarrow {\scriptstyle \omega} \\ \prod_{\alpha \in A}(Hom_R(P,R) \otimes_R Q_\alpha) & \xrightarrow[\prod_{\alpha \in A}(\varphi_{P,Q_\alpha})]{} & \prod_{\alpha \in A} Hom_R(P,Q_\alpha) \end{array}$$

where ω is the natural isomorphism and φ is defined as in Proposition 3 (vi). Then φ_{P,Q_α} is an epimorphism for every $\alpha \in A$ by Proposition 3 (vi). Hence $\varphi_{P,\prod_{\alpha \in A} Q_\alpha}$ is an epimorphism and consequently $\prod_{\alpha \in A} Q_\alpha$ is weakly s-flat by Proposition 3.

(i) implies (iv). Suppose $I = \sum_{i=1}^{n} a_i R$ and $0 \longrightarrow K \longrightarrow F \xrightarrow{f} I \rightarrow$ $\longrightarrow 0$ is an exact sequence, where F is free with a free basis $\{x_1, x_2, \dots, x_n\}$ and $f(x_i) = a_i$ for $i \in \{1,2,\dots,n\}$. Now, if $k \in K$ then $k = \sum_{i=1}^{n} x_i r_i(k)$ for some $r_i(k) \in R$, $i \in \{1,2,\dots,n\}$. Let us define $q_i \in Q^{K\times Q}$ by $q_i(k,q) = r_i(k)q$ for $q \in Q$, $k \in K$, $i \in \{1,2,\dots,n\}$. Since $0 = \sum_{i=1}^{n} a_i r_i(k)\cdot q$ for every $k \in K$ and $q \in Q$, we have $\sum_{i=1}^{n} a_i q_i = 0$ in $Q^{K\times Q}$. Let $s(R) = \sum_{p=1}^{v} Rt_p$. Then there exist $u_j \in Q^{K\times Q}$ and $b_{i,j} \in R$, $i \in \{1,2,\dots,n\}$, $j \in \{1,2,\dots$ $\dots,m\}$ such that $q_i = \sum_{j=1}^{m} b_{i,j} u_j$ for $i \in \{1,2,\dots,n\}$ and $t_p(\sum_{i=1}^{n} a_i b_{i,j}) = 0$ for $j \in \{1,2,\dots,m\}$, $p \in \{1,2,\dots,v\}$. Set $k'_j = \sum_{i=1}^{n} x_i b_{i,j}$, $j \in \{1,2,\dots,m\}$ and $K' = \sum_{j=1}^{m} k'_j R$. For $k \in K$, $q \in Q$ we have $k \otimes q = \sum_{i=1}^{n} x_i r_i(k) \otimes q = \sum_{i=1}^{n} x_i \otimes r_i(k)q =$ $= \sum_{i=1}^{n} x_i \otimes \sum_{j=1}^{m} b_{i,j}\cdot u_j(k,q) = \sum_{j=1}^{m} (\sum_{i=1}^{n} x_i b_{i,j}) \otimes u_j(k,q) =$ $= \sum_{j=1}^{m} k'_j \otimes u_j(k,q) \in K' \otimes {}_R Q$. Further, $t_p f(k'_j) =$ $= t_p f(\sum_{i=1}^{n} x_i b_{i,j}) = t_p(\sum_{i=1}^{n} a_i b_{i,j}) = 0$ and consequently $s(R)f(K') = 0$.

(iv) implies (i). Suppose I is an arbitrary set, $t_w \in s(R)$, $a_i \in Q^I$, $r_i \in R$, $i \in \{1,2,\dots,n\}$, $w \in \{1,2,\dots,z\}$ and $\sum_{i=1}^{n} r_i a_i =$ $= 0$. Set $J = \sum_{i=1}^{n} r_i R$ and consider an exact sequence $0 \longrightarrow K \longrightarrow$ $\longrightarrow F \xrightarrow{f} J \longrightarrow 0$, where F is free with a free basis $\{x_1, x_2, \dots$ $\dots, x_n\}$ and $f(x_i) = r_i$ for $i \in \{1,2,\dots,n\}$. Then there exists a finitely generated submodule $K' = \sum_{p=1}^{q} k'_p R$ of F such that $K \otimes {}_R Q \subseteq K' \otimes_R Q$ and $s(R)f(K') = 0$. Now Q is flat and $\sum_{i=1}^{n} r_i a_i(\alpha) = 0$ for every $\alpha \in I$, hence there exist $v_j(\alpha) \in Q$ and $b_{i,j}(\alpha) \in R$, $i \in \{1,2,\dots,n\}$, $j \in \{1,2,\dots,m\}$,

$\alpha \in A$ such that $a_i(\alpha) = \sum_{j=1}^{m} b_{i,j}(\alpha)v_j(\alpha)$ for $i \in \{1,2,\dots,n\}$, $\alpha \in A$ and $\sum_{i=1}^{n} r_i b_{i,j}(\alpha) = 0$, $j \in \{1,2,\dots,m\}$. Let us denote $u_j(\alpha) = \sum_{i=1}^{n} x_i b_{i,j}(\alpha)$, $j \in \{1,2,\dots,m\}$. Then $f(u_j(\alpha)) = \sum_{i=1}^{n} r_i b_{i,j}(\alpha) = 0$ for $j \in \{1,2,\dots,m\}$, $\alpha \in A$. Thus $u_j(\alpha) \in K$. Hence $\sum_{j=1}^{m} u_j(\alpha) \otimes v_j(\alpha) = \sum_{p=1}^{q} k_p' \otimes w_p(\alpha)$ for some $w_p(\alpha) \in Q$, $p \in \{1,2,\dots,q\}$, $\alpha \in A$. Further, $k_p' = \sum_{i=1}^{n} x_i d_{i,p}$, $d_{i,p} \in R$, $i \in \{1,2,\dots,n\}$, $p \in \{1,2,\dots,q\}$. Thus $\sum_{i=1}^{n} x_i \otimes a_i(\alpha) = \sum_{i=1}^{n} x_i \otimes \sum_{j=1}^{m} b_{i,j}(\alpha)v_j(\alpha) = \sum_{j=1}^{m} (\sum_{i=1}^{n} x_i b_{i,j}(\alpha)) \otimes v_j(\alpha) = \sum_{j=1}^{m} u_j(\alpha) \otimes v_j(\alpha) = \sum_{p=1}^{q} (\sum_{i=1}^{n} x_i d_{i,p}) \otimes w_p(\alpha) = \sum_{i=1}^{n} x_i \otimes (\sum_{p=1}^{q} d_{i,p} w_p(\alpha))$. Hence $a_i(\alpha) = \sum_{p=1}^{q} d_{i,p} w_p(\alpha)$ for $i \in \{1, 2,\dots,n\}$, $\alpha \in A$ and consequently $a_i = \sum_{p=1}^{q} d_{i,p} w_p$, $i \in \{1,2,\dots \dots,n\}$. We have $t_w(\sum_{i=1}^{n} r_i d_{i,p}) = t_w f(k_p') \in s(R)f(K') = 0$ for $w \in \{1,2,\dots,z\}$, $p \in \{1,2,\dots,q\}$. Hence Q^I is weakly s-flat by Proposition 3.

(i) is equivalent to (v). It immediately follows from Proposition 3 (viii).

Corollary 6. Let s be a preradical for mod-R. Consider the following conditions:

(i) ${}_R R^I$ is weakly s-flat for every set I.

(ii) Weakly s-flat modules are closed under direct products.

(iii) $Hom_R(P,R)$ is finitely generated for every module ${}_R P$ for which there exists an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$ with F free, K, F finitely generated and $K = s(R)K$.

(iv) For every finitely generated right ideal I in R and an exact sequence $0 \longrightarrow K \longrightarrow F \xrightarrow{f} I \longrightarrow 0$ with F finitely generated free there is a finitely generated submodule K' of F such that $K \subseteq K'$ and $s(R)f(K') = 0$.

(v) $R/(0:s(R))_r$ is a right coherent ring.

Then

- (ii) implies (i), (iv) implies (i),
- if $s(R)$ is finitely generated as a left ideal then (i) implies (iii) and (iv),
- if $s(R)$ is finitely generated as a left ideal and $s(R)$ is idempotent then (iii) implies (ii),
- if s is an idempotent cohereditary radical, $s(R)$ is finitely generated as a left ideal and $(0:s(R))_r$ is finitely generated as a right ideal then (i) is equivalent to (v).

Let $M \in$ mod-R. We recall that a module $_RQ$ is said to be M-flat if $- \otimes_R Q$ is exact on all exact sequences of the form $0 \longrightarrow A \longrightarrow M \longrightarrow C \longrightarrow 0$.

Proposition 7. Let M_R be a pseudoprojective module. Then a module $_RQ$ is M-flat if and only if it is $p_{\{M\}}$-flat.

Proof: First of all, $p_{\{M\}}$ is an idempotent cohereditary radical for M pseudoprojective. Further, Q is M-flat if and only if its character module Q^*_R is M-injective. Now Q^*_R is M-injective iff it is $(1,p_{\{M\}})$-injective. We have Q^*_R is $(1,p_{\{M\}})$-injective iff it is $(p_{\{M\}},2)$-injective since $p_{\{M\}}$ is idempotent cohereditary. Finally Q^*_R is $(p_{\{M\}},2)$-injective iff $_RQ$ is $p_{\{M\}}$-flat.

Now, if we apply Proposition 3, Theorem 5 and Corollary 6 to the $p_{\{M\}}$-flatness with respect to a pseudoprojective module M, we obtain a characterization of M-flat modules and a characterization of rings for which a direct product of M-flat modules is M-flat.

Proposition 8. For a preradical r for R-mod let us define the following classes of modules

$\mathcal{A}_r = \{X \in \text{mod-}R;\ X \otimes_R T = 0 \text{ for each } T \in \mathcal{T}_r\}$,

$\mathcal{B}_r = \{X \in \text{mod-}R;\ X \otimes_R r(A) = 0 \text{ for each } A \in R\text{-mod}\}$,

$\mathcal{C}_r = \{X \in \text{mod-}R;\ X \otimes_R Y = 0 \text{ for each } A \in R\text{-mod and } Y \subseteq r(A)\}$,

$\mathcal{D}_r = \{X \in \text{mod-}R;\ X \otimes_R r(P) = 0 \text{ for each projective } P \in R\text{-mod}\}$,

$\mathcal{E}_r = \{X \in \text{mod-}R;\ X \otimes_R Y = 0 \text{ for each projective } P \in R\text{-mod}$
$\text{and } Y \subseteq r(P)\}$.

It is easy to see that $\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r, \mathcal{D}_r$ and $\mathcal{E}_r$ are torsion classes. Let us denote A_r, B_r, C_r, D_r and E_r idempotent radicals corresponding to them. Then

- $\mathcal{A}_r = \mathcal{B}_{\bar{r}} = \mathcal{B}_{\tilde{\bar{r}}}$, $\mathcal{C}_r = \mathcal{B}_{h(r)} = \mathcal{B}_{\widetilde{h(r)}}$, $\mathcal{D}_r = \mathcal{B}_{ch(r)} =$

$= \{X \in \text{mod-}R;\ X \otimes_R r(R) = 0\}$, $\mathcal{E}_r = \mathcal{B}_{h(ch(r))} = \mathcal{B}_{\widetilde{h(ch(r))}} =$

$= \{X \in \text{mod-}R;\ X \otimes_R Rm = 0 \text{ for each } m \in r(R)\} =$

$= \{X \in \text{mod-}R;\ X = X(0:m)_\ell \text{ for each } m \in r(R)\}$,

- if $\widetilde{h(r)}$ is superhereditary then C_r is cohereditary,
- if $h(ch(r))$ is a superhereditary radical then E_r is cohereditary and $E_r(R) = C_{ch(r)}(R) = (0:\tilde{r}(R))_\ell$.

<u>Proof</u>: Easy.

As consequences of Propositions 3,5,6 and 8 we obtain for $\widetilde{h(r)}$ superhereditary a characterization of C_r-flat modules and of rings for which a direct product of C_r-flat modules is C_r-flat.

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