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Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 1, 119--129

Persistent URL: http://dml.cz/dmlcz/105981

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

21, 1 (1980)

ON VECTOR-TOPOLOGICAL PROPERTIES OF ZERO-NEIGHBOURHOODS OF TOPOLOGICAL VECTOR SPACES

Thomas RIEDRICH

Abstract: The paper gives a summary of topological (vector-topological) properties of neighbourhoods of zero (nz) of a real, separated topological vector space (tvs). Among other things there is shown that every nz in the space $L_0[0,1]$ of (classes) of real-valued measurable functions (with the topology of the convergence in measure) contains a nz W such that each two points in $L_0[0,1] \setminus W$ can be joined by a five-gon in $L_0[0,1] \setminus W$. This is a partial answer to a question proposed by V. Klee [5].

Key words: Topological linear spaces, connected sets, measurable functions.

Classification: 46A15, 28A 20, 46E30

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1. Introduction. This paper gives a summary of topological resp. vector-topological properties of neighbourhoods of zero (nz) of a (real, separated) topological vector space (tvs) which are important in connection with nonlinear operational equations (see [12]). These properties concern homeomorphisms, retraction properties, boundedness and compactness, product- and trace-properties and the connectedness of the complementary set of a neighbourhood of zero - with a new result about the space $S[0,1]$ (= $L_0[0,1]$) of all real (Lebesgue-) measurable functions on [0,1] with

the topology corresponding to the convergence in measure. No assumption is made about the convexity of the neighbourhoods considered. If (E,τ) is a tvs and $U \subseteq E$ a nz of E, then $p_U(.)$ denotes the Minkowski-functional of U, defined by $p_U(x) = \inf(t>0 | t^{-1}x \in U)$ $(x \in E)$. Some of the results have been announced in [13].

2. Basic definitions. If (E,τ) is a tvs and $U \subseteq E$ is a nz then U is called radially bounded, if each line through zero (denoted by 0) intersects U in a relatively compact set. If there is at least one radially bounded nz, then (E,τ) is called a locally radially bounded space (this notion was introduced by M. Landsberg [8]). U is called shrinkable, if $x \in \bar{U}$ and $0 \leq t < 1$ implies that the element tx belongs to int U (interior of U). This notion was introduced by R.T. Ives (see [2]) and especially investigated by V. Klee [3]. Every tvs possesses a basis of shrinkable nz's (V. Klee [3]). The map

$$r_U : E \to E, \ r_U(x) = \begin{cases} (p_U(x))^{-1}x & (x \in E \setminus U) \\ x & (x \in U) \end{cases}$$

is called the radial retraction with respect to U, and we call the map $\sigma_U : E \to E$, defined by the equation $\sigma_U(x) = (1 + p_U(x))^{-1}x$ $(x \in E)$ the bounding transformation for U. If V is another nz of E, then we call the map $\varphi_{U,V} : E \to E$,

$$\varphi_{U,V}(x) = \begin{cases} p_U(x)(p_V(x))^{-1}x & (p_V(x) \neq 0) \\ 0 & (p_V(x) = 0) \end{cases}$$

the associated radial map of U and V.

3. Homeomorphisms; retractions

Theorem 1. (V. Klee [3].) If (E,τ) is a tvs and U is an open, shrinkable nz, then σ_U is a homeomorphism from E onto U.

Theorem 2. (V. Klee [3].) If (E,τ) is a tvs and U is a closed, shrinkable nz, then r_U is a retraction from E onto U.

Theorem 3. Let (E,τ) be a locally radially bounded tvs and U and V two closed, shrinkable radially bounded nz. If there are real numbers $\alpha > 0$, $\beta > 0$ with $U \subseteq \alpha V$ and $V \subseteq \beta U$ (U and V absorb each other), then $\varphi_{U,V}|U$ is a homeomorphism from U onto V.

Proof. Let U, V and $\alpha > 0$, $\beta > 0$ as in the assumption be given. The inclusions $U \subseteq \alpha V$ and $V \subseteq \beta U$ imply the inequalities $p_V(x) \leqq \alpha p_U(x)$ and $p_U(x) \leqq \beta p_V(x)$ for all $x \in E$ respectively. From the shrinkability of U and V follows the continuity of $p_U(.)$ and $p_V(.)$ (see [3]). The radial boundedness of U and of V implies that $p_U(x) \neq 0$, $p_V(x) \neq 0$ for $x \neq o$. We denote the map $\varphi_{U,V}|U$ by φ. Then, by elementary calculations, φ is injective and $\varphi(U) = V$; the inverse mapping is given by $\varphi^{-1}(z) = \varphi_{V,U}(z)$ $(z \in V \setminus \{o\})$ and $\varphi^{-1}(o) = o$. From the above mentioned properties of $p_U(.)$, $p_V(.)$ follows easily the continuity of φ for $x \in U \setminus \{o\}$. To show the continuity of φ in the point $x = o$, let an arbitrary nz W be given. Without loss of generality, W is closed and shrinkable. Then, for $x \in \frac{1}{\beta} W$ we have for $x \neq 0$

$$p_W(\varphi(x)) = p_U(x)(p_V(x))^{-1} p_W(x) \leqq \beta p_W(x) = p_W(\beta x) \leqq 1$$

i.e. $\varphi(x) \in W$. The continuity of φ^{-1} follows analogously.

A counterexample, if the inclusion $V \subsetneq \beta U$ does not hold for any $\beta > 0$ is given by the space $E = C[0,1]$ of all real-valued continuous functions on the closed interval $[0,1]$ with the usual sup-norm topology, defined by $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$, if we choose the nz's

$$U = \{x \in E \mid \|x\| \leqq 1\} \text{ and } V = \{x \in E \mid \int_0^1 (x(t))^2\, dt \leqq 1\}.$$

Then, the radial map $\varphi_{U,V}|U$ is <u>no</u> homeomorphism, because its inverse mapping is discontinuous at $x = o$ (consider a sequence $x_n \in E$ with

$$p_U(x_n) = \|x_n\| = \frac{1}{\sqrt{n}} \text{ and } p_V(x_n) = [\int_0^1 (x_n(t))^2\, dt]^{\frac{1}{2}} = \frac{1}{n};$$

$n = 1,2,\ldots$). Let us additionally mention that U and V are in fact homeomorphic (but not under the radial map), because they are closed convex bodies in an infinite dimensional Banach space (see [6]).

4. Boundedness and compactness

<u>Theorem 4</u>. Let (E,τ) be a locally radially bounded tvs and U a closed, radially bounded nz. If the boundary ∂U is bounded (in the vector topological sense), then U is also bounded.

Proof. From the radially boundedness of U we get the inclusion $U \subsetneq [0,1]\, \partial U$. If ∂U is bounded, then $[0,1]\, \partial U$ is also bounded. It follows that U is bounded.

<u>Theorem 5</u>. Let (E,τ) be a finite-dimensional tvs. Then any closed, radially bounded and starshaped nz is compact.

For a proof see [10].

Theorem 6. Let (E,τ) be an infinite-dimensional tvs and $U \neq E$ a closed nz. Then ∂U is not compact.

Proof. For any nonempty set $A \subseteq E$ we define

$K(A) = \{y \in E | y = tx;\ x \in A,\ t \geqq 0\} = [0,\infty)A$

and denote $K(\{x\})$ simply by $K(x)$. Now we assume that ∂U is compact. Then the set $K(x) \cap \partial U$ is compact for all $x \neq o$ from E. From a theorem about the properties of $K(.)$ it follows that $K(\partial U)$ is locally compact (see [11], Satz 4). Therefore is $K(\partial U) \neq E$ and there is an $x_o \neq o$ from E with $K(x_o) \cap K(\partial U) = \{o\}$ (otherwise we would have $K(x) \subseteq K(\partial U)$ for every $x \neq o$, this would imply $K(\partial U) = E$ which is excluded). There is a $\delta > 0$ with $tx_o \in U$ for $0 \leqq t \leqq \delta$. Therefore is $K(x_o) \subseteq \operatorname{int} U$, otherwise we would have $K(x_o) \cap K(\partial U) \neq \{o\}$. For every $y \in \partial U$ we set $t(y) = \sup\{t > 0 | ty \in \partial U\}$. From the compactness of $K(y) \cap \partial U$ we have $t(y) < +\infty$ $(y \in \partial U)$. In addition we have the relations $t(y)y \in \partial U$ and $ty \notin U$ $(t > t(y))$. U is closed and therefore the relation $\partial U \subseteq K(-x_o)$ does not hold. It follows the existence of an $y_o \in \partial U$ with $y_o \in \partial U$ and $y_o \notin K(-x_o)$. We denote the linear subspace of E spanned by x_o and y_o by E_o and define $U_o = U \cap E_o$. It is easy to show that the relations $K(x_o) \subseteq \operatorname{int}_o U_o$; $\partial_o U_o \subseteq \partial U$; $t(y_o)y_o \in \partial_o U_o$ hold , here is $\operatorname{int}_o U_o$ resp. $\partial_o U_o$ the interior and the boundary of U_o with respect to the space E_o. Since $\dim E_o = 2$, there is a compact nz of E_o with $\partial_o U_o \subseteq W_o$, for which $E_o \setminus W_o$ is connected. From $t > t(y_o)$ follows the relation $ty_o \notin U_o$ and we have $K(x_o) \subseteq \operatorname{int}_o U_o$. Therefore follow the relations

$$(E_o \setminus W_o) \cap (E_o \setminus U_o) \neq \emptyset \quad \text{and} \quad (E_o \setminus W_o) \cap \operatorname{int}_o U_o \neq \emptyset \ ,$$

and in addition follows from $\partial_o U_o \subseteq W_o$ the equality

$$((E_o \setminus W_o) \cap \mathrm{int}_o U_o) \cup ((E_o \setminus W_o) \cap (E_o \setminus U_o)) = E_o \setminus W_o ,$$

which contradicts to the connectedness of $E_o \setminus W_o$.

5. Products and traces

Theorem 7. Let (E_1, τ_1) and (E_2, τ_2) be two tvs; U_1 resp. U_2 shrinkable nz in E_1 resp. E_2. Then $U_1 \times U_2$ is a shrinkable nz of $E_1 \times E_2$.

Theorem 8. Let (E, τ) be a tvs and E_o a closed linear subspace of E; U a closed shrinkable nz. Then $U_o = U \cap E_o$ is a closed shrinkable nz of E_o (with the induced topology) and we have

$$\partial_o U_o = \partial U \cap E_o \quad \text{and} \quad p_{U_o} = p_U | E_o$$

(∂_o: boundary in E_o).

The (simple) proofs of th. 7 and th. 8 are omitted.

6. Connectedness properties

We consider a question, proposed by V. Klee in [5], about the connectedness properties of the complement of nz's in general tvs.

From the results of Klee follows the Proposition 1. (see [5]).

Proposition 1. (see [5], Theorem A). Let (E, τ) be a tvs with $\dim E \geqq 2$. Then every neighbourhood U of zero contains a nz W such that $E \setminus W$ is connected. Indeed, W can be chosen so that each pair of points of $E \setminus W$ is joined by an 8-gon in $E \setminus W$. (Here by an n-gon is meant an arc composed of n or fewer line segments.)

Klee directs the attention to the fact that if (E,τ) is locally convex, the 8-gons of Proposition 1 are replaced by 2-gons if W is closed and 3-gons if W is open. When E is locally bounded, W may be chosen so as to be bounded and starshaped, whence the 8-gons are replaced by 3-gons if W is closed and 4-gons if W is open. And (one of his questions) he asks: "Can the number 8 in Theorem A ($\Longrightarrow$ Proposition 1) be reduced for general topological linear spaces?" (see [5]).
In this direction we prove the following theorem about the space S(0,1) that is neither locally convex nor radially bounded.

<u>Theorem 9</u>. Let (E,τ) be the tvs of all real-valued (Lebesgue-) measurable functions (more exactly: classes of functions) on the closed unit interval [0,1] with the topology corresponding to the convergence in measure, i.e. $E = S[0,1]$ ($= L_0[0,1]$). Every nz of S[0,1] contains a nz W such that each pair of points of $S[0,1]\setminus W$ is joined by a 5-gon in $E\setminus W$.

Proof. The topology in S[0,1] is given by the metric

$$d(f,g) = \int_0^1 \frac{|f(t) - g(t)|}{1+|f(t) - g(t)|}\,dt \quad (f,g\in S[0,1]) = \int_0^1 \varphi(|f(t) - g(t)|)\,dt$$

with the function $\varphi : [0,\infty) \longrightarrow [0,1)$ given by $\varphi(t) = \dfrac{t}{1 + t}$ $(0 \leq t < +\infty)$. The function φ is strictly monotone increasing and concave and $\lim_{t\to\infty}\varphi(t) = 1$. Let U be an arbitrary nz of S[0,1]. Then U contains all balls $W(\delta) =$

$= \{f \in S[0,1] \mid d(f,o) < \delta\}$ for $0 < \delta \leq \delta_0$. We consider at first the following case

I) Let $f, g \in S[0,1] \setminus W(\delta)$ $(0 < \delta \leq \delta_0)$ be so that

$$f(t)g(t) \geqq 0) \qquad \text{(a.e. in } [0,1]).$$

Then also the line segment

$[f,g] = \{h \in S[0,1] \mid h = \lambda f + (1-\lambda)g,\ 0 \leqq \lambda \leq 1$ is contained in the complement $S[0,1] \setminus W(\delta)$. Indeed we have under these assumptions the equality

$$d(\lambda f + (1-\lambda)g, o) = \int_0^1 \varphi(|\lambda f(t) + (1-\lambda)g(t)|)dt =$$

$$= \int_0^1 \varphi(\lambda|f(t)| + (1-\lambda)|g(t)|)dt \quad (0 \leqq \lambda \leqq 1)$$

and by the concavity of $\varphi(.)$ the inequality

$$\int_0^1 \varphi(\lambda|f(t)| + (1-\lambda)|g(t)|)dt \geqq \lambda \int_0^1 \varphi(|f(t)|)dt +$$

$$+ (1-\lambda) \int_0^1 \varphi(|g(t)|)dt = \lambda d(f,o) + (1-\lambda)d(g,o)$$

$(0 \leq \lambda \leq 1)$.

Since $d(f,o) \geqq \delta$; $d(g,o) \geqq \delta$ it follows that

$d(\lambda f + (1-\lambda)g, o) \geqq \delta (0 \leq \lambda \leqq 1)$ which proves our assertion.

II) Now choose δ with $0 < \delta < \min(\frac{1}{4}, \delta_0)$ and let two arbitrary chosen elements f and g of $(S[0,1]) \setminus W(\delta)$ be given. We define the functions (elements of $S[0,1]$) $\hat{f}$ and $\hat{g}$ by the equations

$$\hat{f}(t) = \begin{cases} f(t) & t \text{ with } f(t) \neq 0 \\ 1 & \text{if } f(t) = 0 \end{cases} \qquad (0 \leqq t \leqq 1)$$

($\hat{g}$ is defined analogously).

We have the relation $\hat{f}(t) \cdot f(t) \geqq 0$ $(t \in [0,1])$ (resp. $\hat{g}(t)g(t) \geqq 0$ for all $t \in [0,1]$). In addition $\hat{f} \in (S[0,1]) \setminus W(\delta)$, because, with $A = \{t \in [0,1] \mid f(t) \neq 0\}$,

$$d(\hat{f},o) = \int_0^1 \varphi(|\hat{f}(t)|)dt \geqq \int_A \varphi(|f(t)|)dt = d(f,o) \geqq \delta.$$

Also $\hat{g} \in (S[0,1]) \setminus W(\delta)$.

Now we define the functions (elements of S[0,1]) $\hat{\hat{f}}$ and $\hat{\hat{g}}$ by the equations

$$\hat{\hat{f}}(t) = \begin{cases} \hat{f}(t) & (0 \leqq t \leqq \frac{1}{2} \\ 0 & (\frac{1}{2} < t \leqq 1) \end{cases}$$

and

$$\tilde{\tilde{g}}(t) = \begin{cases} 0 & (0 \leqq t \leqq \frac{1}{2}) \\ \hat{g}(t) & (\frac{1}{2} < t \leqq 1). \end{cases}$$

We consider the sequences $(n\hat{\hat{f}})$ and $(n\tilde{\tilde{g}})$ $(n = 1,2,\ldots)$. For all $n = 1,2,\ldots$ we have the relations $n\hat{\hat{f}}(t)\hat{f}(t) \geqq 0$ and $n\tilde{\tilde{g}}(t)\hat{g}(t) \geqq 0$ $(t \in [0,1])$ and for all $n = 1,2,\ldots$ and all $m = 1,2,\ldots$ the relation $(n\hat{\hat{f}}(t))(m\tilde{\tilde{g}}(t)) \geqq 0$ $(t \in [0,1])$. The sequences of functions $(\varphi(n|\hat{\hat{f}}(t)|))$ resp. $(\varphi(m|\tilde{\tilde{g}}(t)|))$ $(n,m = 1,2,\ldots)$ converge monotonely increasing to 1 on $[0,\frac{1}{2}]$ and 0 on $(\frac{1}{2},1]$ resp. 0 on $[0,\frac{1}{2}]$ and 1 on $(\frac{1}{2},1]$ because $|\hat{\hat{f}}(t)| > 0$ $(0 \leqq t \leqq \frac{1}{2})$ resp. $|\tilde{\tilde{g}}(t)| > 0$ $(\frac{1}{2} < t \leqq 1)$.

From the Levi's theorem it follows that

$$\lim_{n\to\infty} d(n\hat{\hat{f}},o) = \tfrac{1}{2} \text{ and } \lim_{m\to\infty} d(m\tilde{\tilde{g}},o) = \tfrac{1}{2}.$$

Therefore, we have (see the choice of δ)

$$d(n_0\hat{\hat{f}},o) \geqq \delta \quad \text{and} \quad d(m_0\tilde{\tilde{g}},o) \geqq \delta$$

for sufficiently great n_0, m_0.

From these results and the considerations under case I) it follows that the following pairs of elements of $(S[0,1]) \setminus W(\delta))$ are joinable in this set $(S[0,1]) \setminus W(\delta))$ by the joining line segment. The union of these line segments gives the desired 5-gon:

$$[f,\hat{f}]\cup[\hat{f},n_0\hat{\hat{f}}]\cup[n_0\hat{\hat{f}},m_0\tilde{\tilde{g}}]\cup[m_0\tilde{\tilde{g}},\hat{g}]\cup[\hat{g},g]$$

which joins f and g in the complement of $W(\sigma)$.

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(Oblatum 12.5. 1979)