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Antonín Sochor; Petr Vopěnka

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REVEALMENTS

A. SOCHOR, P. VOPĚNKA

Abstract: In this paper the notion of revealment is defined. We investigate properties of revealments, especially it is shown that every class has a revealment. The obtained results are applied to a very important case, namely we deal with properties of a revealment of the codable class of all set-theoretically definable classes.

Key words: Alternative set theory, non-standard methods, endomorphic universe, standard extension, codable class, fully revealed, set-theoretically definable class, revealment.

Classification: Primary 02K10, 02K99
Secondary 02H20

Endomorphic universes are copies of the universal class conveniently put in the universal class. In many cases there are natural (called "standard") extensions of all subclasses of the endomorphic universe in question. A lot of properties is transferred from a class to its standard extension, however, all standard extensions have some additional convenient properties (e.g. they are fully revealed). These results described in [S-V] correspond in some aspect to the approach of Robinson's non-standard methods.

A standard extension of a class is a superclass of the original class but the standard extensions can be defined only for subclasses of the investigated endomorphic universe. In many cases it is convenient to associate with <u>every</u> class a fully revealed class fulfilling the analogical properties as the original class. Such a class is called a revealment of the original class. By this approach we have, of course, to get over the loss of the assumption that the original class is a subclass of its revealment. On the other hand, it is very advantageous that the notion of revealment does not depend on the choice of an endomorphic universe.

This article is devoted to the investigation of the notion of revealment. It is useful to conceive the method a little more generally and to deal with revealments of codable classes.

The first two sections deal with the study of various properties of revealments, in particular, we show that every class has a revealment. In the third section we give a full classification of codable classes with respect to the fact how many different revealments they have.

In the last section, the results of previous sections are applied to the codable class of all set-theoretically definable classes. It is shown that revealments of this codable class remove the disadvantage of this codable class which consists in the fact that for set-theoretically definable classes no analogue of the prolongation axiom holds. This fact seems to justify the expectation that using revealments of the codable class of all set-theoretically definable classes, we will be able to extend the results obtained for sets even to set-theoretically definable classes.

This article is a continuation of the book [V] and it uses the results of the paper [S-V]. However, in accordance with the aim of this article the results of that paper are used only in proofs and in auxiliary statements and they are not used in the main theorems.

The article has arisen in the Prague seminar of alternative set theory on the basis of discussions held between both authors.

§ 1. Fully revealed codable classes. Let us recall that a class X is called revealed iff for every countable $Y \subseteq X$ there is a set u with $Y \subseteq u \subseteq X$. Further let us remind that a class X is called fully revealed iff for every normal formula $\varphi(z,Z)$ of the language FL the class $\{x; \varphi(x,X)\}$ is revealed.

A codable class $\mathfrak{M}$ is called fully revealed iff there is its coding pair $\langle K,S \rangle$ which is fully revealed (more precisely we require that the class $K \times \{0\} \cup S \times \{1\}$ is fully revealed).

Thus a class Y is fully revealed iff the codable class $\{X;\ X = Y\} = \{Y\}$ is fully revealed.

If φ is a formula of the language FL_V and if $\mathfrak{M}$ is a codable class then $\varphi^{(\mathfrak{M})}$ denotes the formula resulting from φ by restriction of all quantifiers binding class variables to the elements of $\mathfrak{M}$ (quantifiers binding set variables are let without change). Thus e.g. the symbol $((\exists X)(\forall y)(y \in X))^{(\mathfrak{M})}$ denotes the formula $(\exists X \in \mathfrak{M})(\forall y)(y \in X)$.

Let us assume that a coding pair $\langle K,S \rangle$ code a codable class $\mathfrak{M}$. Thus the formulas $(\exists X \in \mathfrak{M})\ \varphi(X,Z_1,\dots,Z_k)$ and

$(\exists x \in K)\varphi(S"\{x\},Z_1,\ldots,Z_k)$ are equivalent. Hence to every formula $\varphi(Z_1,\ldots,Z_k)$ of a language FL_C we are able to construct a normal formula $\tilde{\varphi}(Z_1,\ldots,Z_{k+2})$ of the same language by induction so that the equivalence $\varphi^{(\mathfrak{M})}(Z_1,\ldots Z_1) \equiv$ $\equiv \tilde{\varphi}(Z_1,\ldots,Z_k,K,S)$ holds.

In the course of the first two sections we shall see that a codable class $\mathfrak{M}$ is fully revealed iff $\mathfrak{M}$ satisfies the following two conditions:

Rv_1 If $\varphi(z,Z_1,\ldots,Z_k)$ is a formula of the language FL_V and if $X_1,\ldots,X_k$ are elements of $\mathfrak{M}$ then the class $\{x;\varphi^{(\mathfrak{M})}(x,X_1,\ldots,X_k))\}$ is fully revealed.

Rv_2 If $\{\varphi_n(Z,Z_1,\ldots,Z_{k_n});n \in FN\}$ is a sequence of formulas of the language FL_V and if $\{X_n;n \in FN\}$ is a subclass of $\mathfrak{M}$ then we have

$$(\forall n)(\exists X \in \mathfrak{M})(\varphi_0^{(\mathfrak{M})}(X,X_1,\ldots,X_{k_0}) \& \ldots \& \ \varphi_n^{(\mathfrak{M})}(X,X_1,\ldots,X_{k_n}))\rightarrow$$
$$\rightarrow(\exists X \in \mathfrak{M})(\forall n)\varphi_n^{(\mathfrak{M})}(X,X_1,\ldots,X_{k_n}).$$

Let us realize that if a codable class satisfies the condition Rv_1 then all its elements are fully revealed. Further let us note that according to § 2 [S-V] the condition Rv_1 is equivalent to an illusorily weaker condition - namely to the condition Rv_1 in which only formulas of the language FL are taken into account and in which the words "fully revealed" are replaced by the word "revealed".

Theorem. Every codable fully revealed class satisfies the conditions Rv_1 and Rv_2.

Proof. Let a fully revealed coding pair $\langle K,S\rangle$ code a codable class $\mathfrak{M}$. Assuming that $\varphi(z,Z_1,\ldots,Z_k)$ is a formula of the language FL_V and that $X_1,\ldots,X_k$ are elements of

$\mathfrak{M}$, we can choose $x_1,\dots,x_k \in K$ so that for the normal formula $\tilde{\varphi}$ described above we have $\{x; \varphi^{(\mathfrak{M})}(x,X_1,\dots,X_k)\} = \{x; \tilde{\varphi}(x,S``\{x_1\},\dots,S``\{x_k\},K,S)\}$. Hence the investigated class is fully revealed because the coding pair $\langle K,S\rangle$ is fully revealed and because $\tilde{\varphi}$ is a normal formula. We proved just now the condition Rv_1.

Let $\{X_n; n \in FN\} \subseteq \mathfrak{M}$ and let $\{\varphi_n(Z,Z_1,\dots,Z_{k_n}); n \in FN\}$ be a sequence of formulas of the language FL_V such that for every $n \in FN$ the formula $(\exists X \in \mathfrak{M})(\varphi_0^{(\mathfrak{M})}(X,X_1,\dots,X_{k_0}) \& \dots \dots \& \varphi_n^{(\mathfrak{M})}(X,X_1,\dots,X_{k_n}))$ holds. For every $n \in FN$ we define the class Y_n by

$$Y_n = \{x \in K; \varphi_0^{(\mathfrak{M})}(S``\{x\},X_1,\dots,X_{k_0}) \& \dots \dots \& \varphi_n^{(\mathfrak{M})}(S``\{x\},X_1,\dots,X_{k_n})\} = \{x \in K; \tilde{\varphi}_0(S``\{x\},X_1,\dots \dots,X_{k_0},K,S) \& \dots \& \tilde{\varphi}_n(S``\{x\},X_1,\dots,X_{k_n},K,S)\}.$$

Thus $\{Y_n; n \in FN\}$ is a descending sequence of non-empty revealed classes and therefore there is $x \in \bigcap\{Y_n; n \in FN\}$ by § 5 ch. II [V]. This finishes our proof since $S``\{x\} \in \mathfrak{M}$ and for every $n \in FN$ we have $\varphi_n^{(\mathfrak{M})}(S``\{x\},X_1,\dots,X_{k_n})$ according to the definition of Y_n.

We say that codable classes $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same restrictions of formulas of the language FL_C iff for every closed formula φ of the language FL_C the equivalence $\varphi^{(\mathfrak{M})} \equiv \varphi^{(\mathfrak{N})}$ holds.

If F is a function and if $\mathfrak{M}$ is a codable class then the codable class $\{F``X; X \in \mathfrak{M}\}$ is called the F-range of $\mathfrak{M}$ and denoted by $F``\mathfrak{M}$.

__Theorem__. If codable classes $\mathfrak{M}$ and $\mathfrak{N}$ fulfil the conditions Rv_1 and Rv_2 and if $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same restrictions of formulas of the language FL then there is an automorphism F so that $\mathfrak{M} = F"\mathfrak{N}$.

To prove the theorem we shall use the following auxiliary definition.

Let $\mathcal{F}$ be a mapping of a subclass of $\mathfrak{N}$ into $\mathfrak{M}$. We say that a function F is a similarity regarding $\mathcal{F}$ iff for every formula $\varphi(z_1,\dots,z_k,Z_1,\dots,Z_m)$ of the language FL, for every $x_1,\dots,x_k \in dom(F)$ and for every $X_1,\dots,X_m$ elements of the domain of $\mathcal{F}$ we have

$$\varphi^{(\mathfrak{N})}(x_1,\dots,x_k,X_1,\dots,X_m) \equiv \varphi^{(\mathfrak{M})}(F(x_1),\dots,F(x_k), \mathcal{F}(X_1),\dots,\mathcal{F}(X_m)).$$

Claim. If $\mathcal{F}$ is a mapping of $\mathfrak{N}$ onto $\mathfrak{M}$ and if F is an automorphism regarding $\mathcal{F}$ then $\mathfrak{M} = F"\mathfrak{N}$.

Proof. According to our auxiliary definition we have $x \in X \equiv F(x) \in \mathcal{F}(X)$ for every $X \in \mathfrak{N}$ and hence we get even $F"X = \mathcal{F}(X)$. Therefore the equality $\mathfrak{M} = \{\mathcal{F}(X); X \in \mathfrak{N}\} = \{F"X; X \in \mathfrak{N}\}$ holds.

Claim. Let $\mathcal{F}$ be a mapping of a subclass of $\mathfrak{N}$ into $\mathfrak{M}$ and let F be a similarity regarding $\mathcal{F}$. Let us suppose that $\mathcal{F}$ and F are at most countable and that $\mathfrak{N}$ and $\mathfrak{M}$ satisfy the condition Rv_1. Then for every u there are v and w so that $F \cup \{\langle v,u\rangle\}$ and $F \cup \{\langle u,w\rangle\}$ are similarities regarding $\mathcal{F}$.

Proof. Let u be given. We are going to prove the first statement, the second one can be proved quite analogically. Let $\mathfrak{A}$ be the codable class consisting of all classes of the form

$\{x;\ \varphi^{(\mathfrak{M})}(x,F(x_1),\dots,F(x_k),\mathcal{F}(X_1),\dots,\mathcal{F}(X_m))\}$ where $\varphi(z,z_1,\dots,z_k,Z_1,\dots,Z_m)$ is a formula of the language FL, $x_1,\dots,x_k\in\mathrm{dom}(F)$ and $X_1,\dots,X_m$ are elements of the domain of $\mathcal{F}$ such that the formula $\varphi^{(\mathfrak{N})}(u,x_1,\dots,x_k,X_1,\dots,X_m)$ holds. Every element of $\mathcal{O}\!\mathcal{L}$ is revealed by the condition Rv_1. Thus $\mathcal{O}\!\mathcal{L}$ is a system of non-empty revealed classes which is at most countable and which is dually directed (w.r.t. inclusion) and hence there is $v\in\bigcap\{X;X\in\mathcal{O}\!\mathcal{L}\}$ according to § 5 ch. II [V]. Such a v fulfils our requirement.

Claim. Let $\mathcal{F}$ be a mapping of a subclass of $\mathfrak{N}$ into $\mathfrak{M}$ and let F be a similarity regarding $\mathcal{F}$. Let us suppose that $\mathcal{F}$ and F are at most countable and that $\mathfrak{N}$ and $\mathfrak{M}$ satisfy the condition Rv_2. Then for every $Y\in\mathfrak{N}$ ($Z\in\mathfrak{M}$ respectively) there is $Z\in\mathfrak{M}$ ($Y\in\mathfrak{N}$ respectively) such that F is a similarity regarding $\mathcal{F}\cup\{\langle Z,Y\rangle\}$.

Proof. Let $Y\in\mathfrak{N}$ be given and let $\{x_n;n\in Q\}$ and $\{X_n;n\in Q'\}$ be enumerations of the domains of F and $\mathcal{F}$ respectively (Q and Q′ being either finite natural numbers or FN). Let us assume that $\{\varphi_n;n\in FN\}$ is an enumeration of all formulas $\varphi(Z,z_1,\dots,z_k,Z_1,\dots,Z_m)$ of the language FL such that the formula $\varphi^{(\mathfrak{N})}(Y,x_1,\dots,x_k,X_1,\dots,X_m)$ holds. Thus for every $n\in FN$ we have

$$(\exists X\in\mathfrak{N})(\varphi_0^{(\mathfrak{N})}(X,x_1,\dots,x_{k_0},X_1,\dots,X_{m_0})\ \&\dots$$
$$\dots\&\ \varphi_n^{(\mathfrak{N})}(X,x_1,\dots,x_{k_n},X_1,\dots,X_{m_n}))$$

and hence for every $n\in FN$ we get even

$$(\exists X\in\mathfrak{M})(\varphi_0^{(\mathfrak{M})}(X,F(x_1),\dots,F(x_{k_0}),\mathcal{F}(X_1),\dots,\mathcal{F}(X_{m_0}))\ \&\dots$$
$$\dots\&\ \varphi_n^{(\mathfrak{M})}(X,F(x_1),\dots,F(x_{k_n}),\mathcal{F}(X_1),\dots,\mathcal{F}(X_{m_n})))$$ because

F is a similarity regarding $\mathcal{F}$. Hence by the condition Rv_2 there is $Z \in \mathfrak{M}$ with $(\forall n)\, \varphi_n^{(\mathfrak{M})}(Z, F(x_1), \ldots, F(x_{k_n}), \mathcal{F}(X_1), \ldots, \mathcal{F}(X_{m_n}))$. The second statement can be proved in the same way.

To prove our theorem let us suppose that $\{Z_\alpha ; \alpha \in \Omega\}$ and $\{Y_\alpha ; \alpha \in \Omega\}$ are enumerations of codable classes $\mathfrak{M}$ and $\mathfrak{N}$ respectively and that $\{a_\alpha ; \alpha \in \Omega\}$ is an enumeration of the universal class (the case $\mathfrak{M} = \mathfrak{N} = 0$ is trivial). Evidently 0 is a similarity regarding 0 since $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same restrictions of formulas of the language FL. Hence using the previous claims we are able to construct by transfinite induction sequences $\{F_\alpha ; \alpha \in \Omega\}$ and $\{\mathcal{F}_\alpha ; \alpha \in \Omega\}$ so that for every $\alpha \in \Omega$, F_α is a similarity regarding $\mathcal{F}_\alpha$, both F_α and $\mathcal{F}_\alpha$ are at most countable, $\mathcal{F}_\alpha$ is a mapping of a subclass of $\mathfrak{N}$ into $\mathfrak{M}$, $a_\alpha \in \mathrm{dom}(F_\alpha) \cap \mathrm{rng}(F_\alpha)$, Y_α and Z_α are elements of the domain and of the range of $\mathcal{F}_\alpha$ respectively and $\bigcup\{F_\beta ; \beta \in \alpha \cap \Omega\} \subseteq F_\alpha$ & $\bigcup\{\mathcal{F}_\beta ; \beta \in \alpha \cap \Omega\} \subseteq \mathcal{F}_\alpha$. Thus $\bigcup\{F_\alpha ; \alpha \in \Omega\}$ is an automorphism regarding $\bigcup\{\mathcal{F}_\alpha ; \alpha \in \Omega\}$ and $\mathfrak{N}$ and $\mathfrak{M}$ are the domain and the range of $\bigcup\{\mathcal{F}_\alpha ; \alpha \in \Omega\}$ respectively. Therefore the use of the first claim proves our theorem.

§ 2. <u>Revealments of codable classes</u>. A codable class $\mathfrak{M}$ is called a revealment of a codable class $\mathfrak{N}$ iff $\mathfrak{M}$ is a fully revealed codable class satisfying the same restriction of formulas of the language FL as the codable class $\mathfrak{N}$. A class X is called a revealment of a class Y iff the codable class $\{X\}$ is a revealment of the codable class $\{Y\}$.

Let us note that if a codable class $\mathcal{M}$ is a revealment of a codable class $\mathcal{N}$ then $\mathcal{M}$ fulfils the conditions Rv_1 and Rv_2. Moreover let us realize that in this case the formulas $(\exists X \in \mathcal{N})\varphi(X)$ and $(\exists X \in \mathcal{M})\varphi(X)$ are equivalent for every normal formula $\varphi(Z)$ of the language FL.

We say that classes X and Y satisfy the same normal formulas of the language FL_C iff for every normal formula $\varphi(Z)$ of the language FL_C we have $\varphi(X) \equiv \varphi(Y)$.

A class X is a revealment of a class Y iff X is a fully revealed class satisfying the same normal formulas of the language FL as the class Y.

<u>Lemma</u>. Let F be an endomorphism and let Ex denote a standard extension on F"V (cf. [S-V]). If a coding pair $\langle K,S\rangle$ codes a codable class $\mathcal{N}$ and if the coding pair $\langle Ex(F"K), Ex(F"S)\rangle$ codes a codable class $\mathcal{M}$ then $\mathcal{M}$ is a revealment of $\mathcal{N}$. In particular, for every class Y, the class Ex(F"Y) is a revealment of Y.

Proof. The coding pair $\langle Ex(F"K), Ex(F"S)\rangle$ is fully revealed by § 2 [S-V]. Let φ be a closed formula of the language FL and let $\tilde{\varphi}$ be the corresponding normal formula described in the first section. Then

$$\varphi^{(\mathcal{M})} \equiv \tilde{\varphi}(Ex(F"K), Ex(F"S)) \equiv \tilde{\varphi}^{F"V}(F"K, F"S) \equiv \tilde{\varphi}(K,S) \equiv \varphi^{(\mathcal{N})}$$

by the definition of standard extension and by the second theorem of § 1 ch. V [V]. We have proved that $\mathcal{M}$ and $\mathcal{N}$ satisfy the same restrictions of formulas of the language FL and therefore $\mathcal{M}$ is a revealment of $\mathcal{N}$.

The following statement seems to be the most important consequence of the lemma. To prove it it is sufficient to realize that § 3 [S-V] assures the existence of an endomorphism such that there is a standard extension on the endomorphic universe F"V.

Theorem. Every codable class has a revealment.

Lemma. If a coding pair $\langle K,S\rangle$ codes a codable class $\mathfrak{M}$ and if F is an automorphism then the coding pair $\langle F"K,F"S\rangle$ codes the codable class $F"\mathfrak{M}$ and moreover $\mathfrak{M}$ and $F"\mathfrak{M}$ satisfy the same restrictions of formulas of the language FL.

Proof. Assuming F to be an automorphism we have obviously

$\{(F"S)"\{x\};x\in F"K\} = \{(F"S)"\{F(x)\};x\in K\} = \{F"(S"\{x\});\ x\in K\} = \{F"X;X\in\mathfrak{M}\}$. Moreover if φ is a closed formula of the language FL and if $\tilde{\varphi}$ is the corresponding normal formula then we have $\varphi^{(F"\mathfrak{M})}\equiv\tilde{\varphi}(F"K,F"S)\equiv\tilde{\varphi}(K,S)\equiv\varphi^{(\mathfrak{M})}$ according to the second theorem of § 1 ch. V [V].

Theorem. Let a codable class $\mathfrak{M}$ be a revealment of a codable class $\mathfrak{N}$. Then a codable class $\mathfrak{M}'$ is a revealment of $\mathfrak{N}$ iff there is an automorphism F with $\mathfrak{M}' = F"\mathfrak{M}$. In particular, if a class X is a revealment of a class Y then a class Z is a revealment of Y iff there is an automorphism F with Z = F"X.

Proof. The implication from left to right is a trivial consequence of the second theorem of the paper. To prove the converse implication it is sufficient to use the last lemma and to appreciate that automorphisms transfer fully revealed classes onto fully revealed ones.

The following result which is a strong form of the converse of the first lemma has important consequences.

Lemma. Let a codable class $\mathfrak{M}$ fulfil the conditions Rv_1 and Rv_2, let a coding pair $\langle K,S\rangle$ code a codable class $\mathfrak{N}$ satisfying the same restrictions of formulas of the language FL as $\mathfrak{M}$. Then there is an endomorphism F such that there is a standard extension Ex on the endomorphis universe F"V so that the coding pair $\langle Ex(F"K), Ex(F"S)\rangle$ codes $\mathfrak{M}$.

Proof. Let G be an endomorphism such that there is a standard extension Ex′ on the endomorphic universe G"V and let the coding pair $\langle Ex'(G"K), Ex'(G"S)\rangle$ code a codable class $\mathfrak{M}'$. By the first lemma of this section $\mathfrak{M}'$ is a revealment of $\mathfrak{N}$ and hence both $\mathfrak{M}$ and $\mathfrak{M}'$ fulfil the conditions Rv_1 and Rv_2. Moreover $\mathfrak{N}$, $\mathfrak{M}$ and $\mathfrak{M}'$ satisfy the same restricition of formulas of the language FL and therefore by the second theorem of the paper there is an automorphism H with $\mathfrak{M} = H"\mathfrak{M}'$.

Let F be the composition of H and G. Then F"V is an endomorphic universe and we define an operation Ex for all its subclasses by $Ex(X) = H"Ex'(H^{-1}"X)$. Thus for every normal formula $\varphi(z_1,\dots,z_k,Z_1,\dots,Z_m)$ of the language FL, for every $x_1,\dots,x_k \in F"V$ and for every $X_1,\dots,X_m \subseteq F"V$ we have

$\varphi^{F"V}(x_1,\dots,x_k,X_1,\dots,X_m) \equiv \varphi^{H"G"V}(x_1,\dots,x_k,X_1,\dots,X_m) \equiv$
$\equiv \varphi^{G"V}(H^{-1}(x_1),\dots,H^{-1}(x_k),H^{-1}"X_1,\dots,H^{-1}"X_m) \equiv$
$\equiv \varphi(H^{-1}(x_1),\dots,H^{-1}(x_k),Ex'(H^{-1}"X_1),\dots,Ex'(H^{-1}"X_m)) \equiv$
$\equiv \varphi(x_1,\dots,x_k,Ex(X_1),\dots,Ex(X_m))$ according to the definition of standard extension and to the second theorem of § 1 ch. V [V]. Therefore we have proved that Ex is a standard extension on F"V. Moreover the coding pair $\langle Ex(F"K), Ex(F"S)\rangle = \langle H"Ex'(G"K), H"Ex'(G"S)\rangle$ codes $\mathfrak{M}$ by the last lemma.

Consequence. If X is a fully revealed class satisfying the same normal formulas of the language FL as a class Y then there is an endomorphism F such that there is a standard extension Ex on the endomorphic universe F"V so that $X = Ex(F"Y)$.

Theorem. A codable class is fully revealed iff it fulfils the conditions Rv_1 and Rv_2.

Proof. According to the first theorem of the article we have to prove only the implication from right to left. However, this is an easy consequence of the last lemma (applied to the case $\mathfrak{N} = \mathfrak{M}$) and of § 2 [S-V].

We say that a coding pair $\langle K^*,S^*\rangle$ is a revealment of a coding pair $\langle K,S\rangle$ iff the class $K^*\times\{0\}\cup S^*\times\{1\}$ is a revealment of the class $K\times\{0\}\cup S\times\{1\}$.

Theorem. Let a coding pair $\langle K,S\rangle$ code a codable class $\mathfrak{N}$. Then a codable class $\mathfrak{M}$ is a revealment of $\mathfrak{N}$ iff there is a coding pair which codes $\mathfrak{M}$ and which is a revealment of $\langle K,S\rangle$.

Proof. The first and third lemmas and the consequence of this section obviously imply our statement.

Let us suppose that we have a class and its revealment. The following two theorems enable us to construct some other pairs of classes so that in every pair the second class is a revealment of the first one.

Theorem. Let $\varphi(Z)$ be a formula of the language FL and let a codable class $\mathfrak{M}$ be a revealment of a codable class $\mathfrak{N}$. Then the class $\{x; \varphi^{(\mathfrak{M})}(x)\}$ is a revealment of the class $\{x; \varphi^{(\mathfrak{N})}(x)\}$. In particular, if $\psi(z,Z)$ is a normal formula of the language FL and if a class X is a revealment of a class Y then the class $\{x; \psi(x,X)\}$ is a revealment of the class $\{x; \psi(x,Y)\}$.

Proof. The class $\{x; \varphi^{(\mathfrak{M})}(x)\}$ is fully revealed by the property Rv_1. If $\vartheta(Z)$ is a normal formula of the language FL then according to the definition of $\varphi^{(\mathfrak{M})}$ we have $\vartheta(\{x; \varphi^{(\mathfrak{M})}(x)\}) \equiv (\vartheta(\{x; \varphi(x)\}))^{(\mathfrak{M})} \equiv (\vartheta(\{x; \varphi(x)\}))^{(\mathfrak{N})} \equiv$ $\equiv \vartheta(\{x; \varphi^{(\mathfrak{N})}(x)\})$ since $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same restrictions of formulas of the language FL. Therefore we have shown that $\{x; \varphi^{(\mathfrak{M})}(x)\}$ and $\{x; \varphi^{(\mathfrak{N})}(x)\}$ satisfy the same normal formulas of the language FL.

<u>Theorem</u>. Let $\varphi(Z)$ be a formula of the language FL and let a codable class $\mathfrak{M}$ be a revealment of a codable class $\mathfrak{N}$. Then the codable class $\{X \in \mathfrak{M}; \varphi^{(\mathfrak{M})}(X)\}$ is a revealment of the codable class $\{X \in \mathfrak{N}; \varphi^{(\mathfrak{N})}(X)\}$.

Proof. Let a coding pair $\langle K,S \rangle$ code $\mathfrak{N}$ and let its revealment $\langle K^*,S^* \rangle$ code $\mathfrak{M}$. Thus the class $K_1^* = \{x \in K^*; \tilde{\varphi}(S^{*}``\{x\},K^*,S^*)\}$ is a revealment of the class $K_1 = \{x \in K; \tilde{\varphi}(S``\{x\},K,S)\}$ by the last theorem and moreover $\langle K_1^*,S^* \rangle$ is a revealment of $\langle K_1,S \rangle$. To finish the proof it is sufficient to realize that the coding pairs $\langle K_1^*,S^* \rangle$ and $\langle K_1,S \rangle$ code the classes $\{X \in \mathfrak{M}; \varphi^{(\mathfrak{M})}(X)\}$ and $\{X \in \mathfrak{N}; \varphi^{(\mathfrak{N})}(X)\}$ respectively.

At the end of this section we are going to deal with the indiscernibility equivalence $\doteq$ defined in § 1 ch. V [V]. Let $\overline{X}$ denote the closure of X in this topology. Thus $\overline{X}$ is the intersection of all classes of the form $\{x; \varphi(x)\}$ where $\varphi(z)$ is a set-formula of the language FL with $(\forall x \in X)\varphi(x)$.

<u>Theorem</u>. $\overline{X} = \bigcup\{Y; ``Y$ is a revealment of $X"\}$.

Proof. If $x \notin \overline{X}$ then there is a set-formula $\varphi(z)$ of the language FL with $\neg\varphi(x) \mathbin{\&} (\forall y \in X)\varphi(y)$. Assuming that Y is a revealment of X we have $(\forall y \in Y)\varphi(y)$ since X and Y have to satisfy the same normal formulas of the language FL and therefore we get $x \notin Y$ in this case.

To prove the converse inclusion let us suppose that $x \in \overline{X}$ and that Y is a revealment of X. If $\varphi(z)$ is a set-formula of the language FL such that $\varphi(x)$ holds then there is $y \in X$ with $\varphi(y)$ (otherwise $X \subseteq \{z; \neg\varphi(z)\}$ and this contradicts the assumption $x \in \overline{X}$). Moreover since X and Y satisfy the same normal formulas of the language FL there is $q \in Y$ with $\varphi(q)$. The class $\{z \in Y; \varphi(z)\}$ is revealed because Y is fully revealed. The codable class consisting of all classes of the form $\{z \in Y; \varphi(z)\}$ where φ is a formula of the language FL such that $\varphi(x)$ holds is a dually directed (w.r.t. inclusion) system of non-empty revealed classes and hence there is a set $z \in Y$ so that the equivalence $\varphi(z) \equiv \varphi(x)$ holds for every set-formula φ of the language FL. Thus we have shown that there is $z \in Y$ with $z \overset{o}{=} x$. Therefore by § 1 ch. V [V] there is an automorphism F with $F(z) = x$. The class $F"Y$ is a revealment of X and moreover $x \in F"Y$.

Theorem. For every $X \subseteq u \in Def$ the class $\overline{X}$ equals to the class $\bigcap\{v \in Def; X \subseteq v\}$.

Proof. If $v \in Def$ and $X \subseteq v$ then there is a set-formula $\varphi(z)$ of the language FL so that the statement $\varphi(v)$ & & $(\exists ! q)\varphi(q)$ holds. Let $\psi(z)$ denote the formula $(\exists q)(\varphi(q) \,\&\, \& \, z \in q)$. We have $(\forall x \in X)\psi(x)$ and thence even $(\forall x \in \overline{X})\psi(x)$, Therefore we have proved $\overline{X} \subseteq v$ from which the inclusion $\overline{X} \subseteq \bigcap\{v \in Def; X \subseteq v\}$ follows.

On the other hand, let us assume that $x \notin \overline{X}$ i.e. that there is a set-formula $\varphi(z)$ of the language FL such that

$\neg \varphi(x) \& (\forall y \in X)\varphi(y)$ holds. The set $v = \{y \in u; \varphi(y)\}$ is an element of Def and moreover $X \subseteq v$. Thus to prove the converse of our inclusion it suffices to realize that we have $x \notin v$.

The following result is a trivial consequence of the above mentioned statements (since $\widetilde{Def} = V$ by § 1 ch. V [V]).

Theorem. $\{\alpha; (\forall v \in Def)(FN \subseteq v \rightarrow \alpha \in v\} = \bigcup \{Y; "Y$ is a revealment of $FN"\}$.

§ 3. Codable classes with uniquely determined revealment. If C is an arbitrary class then a class X is called set-theoretically definable with parameters in C iff there is a set-formula $\varphi(z)$ of the language FL_C such that $X = \{x; \varphi(x)\}$. In agreement with [V] we define that a class is set-theoretically definable iff it is set-theoretically definable with parameters in V. The system of all set- theoretically definable classes is a codable class by § 5 ch. II [V]. Hence for every class C, the system of all set-theoretically definable classes with parameters in C is a codable class and we are going to denote them by the symbol Sd_C. We say that a class is set-theoretically definable without parameters iff it is an element of Sd_0.

Let us note that the formula $X \in Sd_V$ is equivalent to Vopěnka's predicate Sd(X).

We say that a codable class is Sd_0-codable (set-theoretically codable without parameters) iff there is its coding pair $\langle K,S \rangle$ such that both K and S are elements of Sd_0.

Thus $X \in Sd_0$ iff the codable class $\{X\}$ is Sd_0-codable. Moreover if X is an element of a Sd_0-codable class then there is $y \in V$ and a set-formula $\varphi(z_1,z_2)$ of the language FL so that $X = \{x; \varphi(x,y)\}$ and hence X is set-theoretically definable. Thence we have proved that every Sd_0-codable class is a subclass of Sd_V.

Theorem. If $\mathfrak{M}$ is a Sd_0-codable class then

a) there is a normal formula $\varphi(Z)$ of the language FL such that $\mathfrak{M} = \{X; \varphi(X)\}$;

b) for every codable class $\mathfrak{N}$ we have $\mathfrak{N} = \mathfrak{M}$ iff $\mathfrak{N}$ and $\mathfrak{M}$ satisfy the same restrictions of formulas of the language FL.

Proof. Let $K,S \in Sd_0$ and let the coding pair $\langle K,S\rangle$ code $\mathfrak{M}$. Then there are set-formulas $\psi(z)$ and $\vartheta(z)$ of the language FL with $K = \{x; \psi(x)\}$ and $S = \{x; \vartheta(x)\}$.

a) We have evidently $\mathfrak{M} = \{X; (\exists x)(\psi(x) \& (\forall y)(y \in X \equiv \equiv \vartheta(\langle y,x\rangle)))\}$.

b) Let $\mathfrak{N}$ and $\mathfrak{M}$ satisfy the same restrictions of formulas of the language FL and let $\varphi(Z)$ be the normal formula guaranteed by the first statement. Thus the formula $((\forall X)\varphi(X))^{(\mathfrak{M})}$ holds and hence we get $((\forall X)\varphi(X))^{(\mathfrak{N})}$ from which the inclusion $\mathfrak{N} \subseteq \mathfrak{M}$ follows. Further we have $((\forall x)(\exists X)(\psi(x) \longrightarrow (\forall y)(y \in X \equiv \vartheta(\langle y,x\rangle))))^{(\mathfrak{M})}$ and therefore we obtain even $((\forall x)(\exists X)(\psi(x) \longrightarrow (\forall y)(y \in X \equiv \vartheta(\langle y,x\rangle))))^{(\mathfrak{N})}$. However, the last statement means that $(\forall x \in K)S''\{x\} \in \mathfrak{N}$ and thence we have proved the equality $\mathfrak{N} = \mathfrak{M}$.

Theorem. If $\mathfrak{M}$ is a Sd_0-codable class then $\mathfrak{M}$ itself is its sole revealment.

Proof. Let $K,S \in Sd_0$ and let the coding pair $\langle K,S\rangle$ code $\mathfrak{M}$. Then this coding pair is fully revealed and hence $\mathfrak{M}$ itself is its revealment. Thus the use of the last theorem finishes the proof.

Theorem. A codable class has exactly one revealment iff it is Sd_0-codable. In particular, a class has exactly one revealment iff it is an element of Sd_0.

Proof. The last theorem assures the implication from right to left. To prove the converse one let us suppose that a codable class $\mathfrak{M}$ is the sole revealment of a codable class $\mathfrak{N}$. Moreover let us assume that a fully revealed coding pair $\langle K,S\rangle$ codes $\mathfrak{M}$,

At first we are going to show that $\mathfrak{M} \subseteq Sd_V$. If F is an automorphism then the codable class $F''\mathfrak{M}$ is a revealment of $\mathfrak{N}$ by the last section and therefore we get $F''\mathfrak{M} = \mathfrak{M}$. If X is an element of $\mathfrak{M}$ then $\{F''X;$ "F is an automorphism"$\}$ is a subclass of the codable class $\mathfrak{M}$ and hence even this class is codable. Thus according to § 1 [Č-V], X is a real class. Further the class X is fully revealed because $\mathfrak{M}$ satisfies Rv_1 and thence both X and V-X are revealed. Using § 5 ch. II [V] and again § 1 [Č-V] we get that X is set-theoretically definable. Therefore we have proved the inclusion $\mathfrak{M} \subseteq Sd_V$.

Let $\mathfrak{A}$ be the codable class consisting of all classes of the form $\{z \in K; \neg(\exists y)(S''\{z\} = \{x; \chi(x,y)\})\}$ where $\chi(z_1,z_2)$ is a set-formula of the language FL. Then the elements of $\mathfrak{A}$ are revealed classes because the coding pair $\langle K,S\rangle$ is fully revealed. Further $\mathfrak{A}$ is dually directed (w.r.t. inclusion) since the formulas $(\exists y)(S''\{z\} = \{x; \chi_1(x,y)\}) \vee (\exists y)(S''\{z\} =$ $= \{x; \chi_2(x,y)\})$ and $(\exists y)(S''\{z\} = \{x; (\exists q)((y = \langle q,0\rangle \;\&$ $\&\; \chi_1(x,q)) \vee (y = \langle q,1\rangle \;\&\; \chi_2(x,q)))\})$ are equivalent. Moreover $\bigcap\{X; X \in \mathfrak{A}\} = 0$ according to the previous part of the proof. Therefore $0 \in \mathfrak{A}$ by § 5 ch. II [V] and hence we can fix a set-formula $\chi(z_1,z_2)$ of the language FL such that the formula $(\forall X \in \mathfrak{M})(\exists y)(X = \{x;\ \chi(x,y)\})$ holds.

Put $M = \{y; \{x;\ \chi(x,y)\} \in \mathfrak{M}\}$. Thus $M = \{y; ((\exists X)(\forall x)(x \in \in X \equiv \chi(x,y)))^{(\mathfrak{M})}\}$ and hence M is fully revealed according to Rv_1. Moreover the coding pair $\langle M, \{\langle x,y\rangle;\ \chi(x,y)\}\rangle$ codes $\mathfrak{M}$.

If F is an automorphism then $y \in M \equiv \{x;\ \chi(x,y)\} \in \mathfrak{M} \equiv \equiv F" \{x;\ \chi(x,y)\} \in \mathfrak{M} \equiv \{x;\ \chi(x,F(y))\} \in \mathfrak{M} \equiv F(y) \in M$. However, this means that M is a figure in the equivalence $\triangleq$. Further this figure and its complement are closed by § 2 ch. III [V] and therefore $M \in Sd_0$ according to § 1 ch. V [V].

We proved just now that $\mathfrak{M}$ is Sd_0-codable. Since $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same restrictions of formulas of the language FL one can conclude the whole proof of our theorem using the first theorem of this section.

<u>Theorem</u>. If a coding pair $\langle K,S\rangle$ with $K,S \in Sd_V$ code $\mathfrak{N}$ then $\mathfrak{N}$ is its revealment and coding pairs of the form $\langle F"K, F"S\rangle$ where F is an automorphism code all revealments of $\mathfrak{N}$ and hence every revealment of $\mathfrak{N}$ is coded by a coding pair which is an element of the codable class $\{\langle L,R\rangle; L,R \in Sd_V\}$.

<u>Theorem</u>. If $\mathfrak{N}$ cannot be coded by a coding pair $\langle K,S\rangle$ with $K,S \in Sd_V$, then there is no codable class $\mathfrak{A}$ such that every revealment of $\mathfrak{N}$ can be coded by a coding pair which is an element of $\mathfrak{A}$.

Proof. Let $\mathfrak{M}$ be a revealment of $\mathfrak{N}$. If there is $X \in \in \mathfrak{M} - Sd_V$, then by § 2 [Č-V] the system of classes of the form F"X where F is an automorphism is not codable. Therefore we can suppose that $\mathfrak{M} \subseteq Sd_V$. Thus a part of the proof of the last but one theorem shows that there are $S \in Sd_V$ and a fully revealed class K such that the coding pair $\langle K,S\rangle$ extensionally codes $\mathfrak{M}$. By the assumption of our theorem $K \notin Sd_V$. Thus K is not real and the class $\{F"K;$ "F is an automorphism with $F"S = S\}$ is not codable by § 2 [Č-V].

§ 4. Revealments of the codable class Sd_V. Set-theoretically definable classes behave in many cases analogically as sets. On the other hand, the following theorem shows that a very important property of sets, namely the prolongation axiom, has no analogue in the codable class Sd_V.

Theorem. No coding pair $\langle K,S\rangle$ with $S \in Sd_V$ codes Sd_0.

Proof. Let us suppose that a coding pair $\langle K,S\rangle$ codes Sd_0 and that there is a set-formula $\varphi(z_1,z_2,z_3)$ of the language FL and a convenient parameter a so that $S = \{\langle x,y\rangle; \varphi(x,y,a)\}$. Put $Y = \{\langle y,z\rangle; \neg\varphi(\langle y,z\rangle,y,z)\}$. Thus $Y \in Sd_0$ and thence there would be $y_0 \in K$ such that the equality $Y = S"\{y_0\}$ holds. In this case we would have $\langle y_0,a\rangle \in Y \equiv \neg\varphi(\langle y_0,a\rangle,y_0,a) \equiv \langle\langle y_0,a\rangle,y_0\rangle \notin S \equiv \langle y_0,a\rangle \notin S"\{y_0\} \equiv \langle y_0,a\rangle \notin Y$ which is a contradiction.

Further as a consequence of the last theorem we get that the codable class Sd_V does not fulfil the condition Rv_2 (as the required formulas can serve the formulas $Z_1 = X"\{1\} \& \ldots$ $\ldots \& Z_n = X"\{n\}$ where $\langle FN,X\rangle$ is a coding pair of Sd_0) and hence Sd_V is not fully revealed. By the last section there are many revealments of Sd_V. Up to the end of this section the symbol Sd_V^* denotes one of them.

Theorem. $Sd_V \subset Sd_V^*$.

Proof. Let a set-formula $\varphi(z,z_1,\ldots,z_k)$ of the language FL be given. We have $((\forall y_1)\ldots(\forall y_k)(\exists X)(\forall x)(x \in X \equiv$

$\equiv \varphi(x,y_1,\dots,y_k)))^{(Sd_V)}$ and hence we obtain even the statement $((\forall y_1)\dots(\forall y_k)(\exists X)(\forall x)(x \in X \equiv \varphi(x,y_1,\dots,y_k)))^{(Sd_V^*)}$. Therefore we have proved that for every $y_1,\dots,y_k$ the class $\{x; \varphi(x,y_1,\dots,y_k)\}$ is an element of Sd_V^*.

The following theorem (which is a trivial consequence of the property Rv_2) is an analogue of the prolongation axiom holding in Sd_V^*. The next theorem summarize some results which show that Sd_V^* keeps those properties of Sd_V in which set-theoretically definable classes look like sets.

Theorem. If $\{X_n; n \in FN\} \subseteq Sd_V^*$ then there is $R \in Sd_V^*$ with $(\forall n)R''\{n\} = X_n$.

Theorem. a) The universal class is the sole class X of Sd_V^* for which the formula $0 \in X \,\&\, (\forall x)(\forall y)(x \in X \longrightarrow \longrightarrow (x \cup \{y\}) \in X)$ holds.

b) The intersection of each element of Sd_V^* with a set is a set

c) If $\varphi(z,Z)$ is a normal formula of the language FL and if $X \in Sd_V^*$ then the class $\{x; \varphi(x,X)\}$ is an element of Sd_V^*, too. In particular, the formula $(\forall R \in Sd_V^*)(\forall x)(R''\{x\} \in \in Sd_V^*)$ holds.

Proof. The formula $(\forall X \in Sd_V)((0 \in X \,\&\, (\forall x)(\forall y)(x \in X \longrightarrow \longrightarrow (x \cup \{y\}) \in X)) \longrightarrow X = V)$ was accepted as the precise version of induction (cf. § 5ch. II [V]); moreover assuming that φ is a normal formula of the language FL we get formulas $(\forall X \in Sd_V)(\forall x)\mathrm{Set}(X \cap x)$ and $(\forall X \in Sd_V)(\exists Y \in Sd_V)Y = = \{x; \varphi(x,X)\}$. Therefore all statements of our theorem are implied by the assumption that Sd_V and Sd_V^* satisfy the same restrictions of formulas of the language FL.

The following theorem describes a holding in Sd^*_V analogue of an important consequence of the prolongation axiom.

Theorem. Let $\bigcup\{a_n; n \in FN\} = dom(R)$ and let $(\forall n)(R \restriction a_n \in Sd_V \,\&\, a_n \subseteq a_{n+1})$. Let Φ be a countable system of normal formulas of the language FL with one free variable so that we have $(\forall \varphi \in \Phi)(\forall n)\varphi(R \restriction a_n)$. Then there is $R^* \in Sd^*_V$ such that the formulas $(\forall n) R^* \restriction a_n = R \restriction a_n$ and $(\forall \varphi \in \Phi)\varphi(P^*)$ hold.

Proof. For every $n \in FN$ there is a set-formula φ_n of the language FL_V with $(\forall y \in a_n)(\varphi_n(x,y) \equiv \langle x,y\rangle \in R)$; moreover we have $R \restriction a_n \in Sd^*_V$. Hence it is sufficient to apply the condition Rv_2 to the countable class $\Phi \cup \{(\forall y \in (a_n \cap dom(Z))(\varphi_n(x,y) \equiv \langle x,y\rangle \in Z); n \in FN\}$ of normal formulas.

Especially if sets a_n in the previous theorem are finite we can replace the assumption $R \restriction a_n \in Sd_V$ by the condition $(\forall x \in a_n) R''\{x\} \in Sd_V$.

The last theorem substantiates that we are not able to define (e.g. adding some additional requirements) a uniquely determined convenient extension of the codable class Sd_V.

Theorem. Let $\mathfrak{M} = \{X; \varphi(X)\}$ be a codable class with $(\forall X \in \mathfrak{M})(\forall x)Set(X \cap x)$ where φ is a formula of the language FL. Then $\mathfrak{M} \subseteq Sd_V$.

Proof. Let us suppose that $Y \in \mathfrak{M} - Sd_V$. For every automorphism F we have $F''Y \in \mathfrak{M}$ since the formula $(\forall X)(\varphi(X) \equiv \varphi(F''X))$ holds. Therefore the system $\{F''Y;$ "F is an automorphism"$\}$ is a subclass of $\mathfrak{M}$ and hence it is codable. By § 1 [Č-V] the class Y must be real. Moreover we have $(\forall x)Set(Y \cap x)$ and hence Y is set-theoretically definable according to the same section. This contradicts our assumption.

R e f e r e n c e s

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Matematický ústav ČSAV
Žitná 25
Praha 1
Československo

Matematický ústav
Universita Karlova
Sokolovská 83
18600 Praha 8
Československo

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