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ATTRACTORS AND A FIXED POINT THEOREM IN LOCALLY CONVEX SPACE

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Abstract: Let X be a Hausdorff locally convex space and G be a non-empty complete convex subset of X, and let $f:G \rightarrow G$ be continuous. We prove that if (i) $\{f :n=1,2,...\}$ is equicontinuous and (ii) there exists $M \subseteq G$ which is an attractor for compact sets under f, then f has a fixed point.

Key words and phrases: Locally convex space, equi-continuous, attractor for compact sets under f, fixed point, compact open topology.

Classification: Primary 47H10
Secondary 54H25

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1. Introduction. Let X be a topological space and $f: X \rightarrow X$ be a map. A subset M of X is said to be an attractor for compact sets under f [8] if (i) M is non-empty compact and f-invariant and (ii) for any compact subset C of X and any open neighbourhood U of M, there exists an integer N such that $f^n(C) \subseteq U$ for all $n \geq N$. In [8] a conjecture suggested by F.E. Browder was stated as follows:

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Conjecture 1. Let G be a closed convex subset of a Banach space X and $f:G \to G$ be continuous. Assume that there exists a set $M \subseteq G$ which is an attractor for compact sets under f. Then f has a fixed point (?).

An affirmative answer to Conjecture 1 will give a generalization of the Schauder's fixed point theorem.

In [2] a partial solution to the above conjecture was obtained as follows:

Theorem (L. Janos and J.L. Solomon): Let G be a closed convex subset of a Banach space X and $f:G \to G$ be continuous. Assume that (i) there exists a subset $M \subseteq G$ which is an attractor for compact sets under f and (ii) the family $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Then f has a fixed point.

In this paper we generalize the above theorem to locally convex spaces and thus partially generalize the celebrated Schauder-Tychonoff fixed point theorem. Finally some remarks concerning attractors are also discussed.

2. Wallace's Theorem and its application. Let X be a Hausdorff topological semigroup. S is said to act on X if there is a continuous map $\pi : S \times X \to X$ such that $\pi(s_1, \pi(s_2,x)) = \pi(s_1 s_2, x)$ for any $s_1, s_2 \in S$ and $x \in X$. If $s \in S$, we denote by $\Gamma_n(s)$ the closure of the set $\{s^m : m \geq n\}$. Let $\Gamma(s) = \Gamma_1(s)$ and $K(s) = \cap\{\Gamma_n(s) : n \geq 1\}$. The following theorem can be found in [9]:

Theorem (Wallace): Suppose S acts on X. Let $s \in S$ be such that $\Gamma(s)$ is compact and let $A \subseteq X$ be nonempty compact such that $sA \supseteq A$. Then for each $s_1 \in \Gamma(s)$, $s_1A = A$ and s_1 acts as a homomorphism on A. In particular, the unit of K(s) acts as the identity map on A.

Note that if $\Gamma(s)$ is compact, then K(s) is a compact topological group (see [7]).

Lemma 1. Let X be a compact Hausdorff space and $f:X \longrightarrow X$ be continuous. If $A = \bigcap_{n=1}^{\infty} f^n(X)$, then $f(A) = A$.

Proof: That $f(A) \subseteq A$ is clear. To prove that $A \subseteq f(A)$, let $x \in A$. Then there are $x_n \in X$ such that $x = f^n(x_n)$ for all $n=1,2,\ldots$. Since X is compact, the sequence $(f^{n-1}(x_n))_{n=1}^{\infty}$ has a convergent subnet, say $f^{n_\alpha - 1}(x_{n_\alpha}) \longrightarrow x_0$. As f is continuous, we have $f^{n_\alpha}(x_{n_\alpha}) \longrightarrow f(x_0)$. But $f^{n_\alpha}(x_{n_\alpha}) = x$ for all α, it follows that $f(x_0) = x$. If x_0 were not in A, then there exist disjoint open sets U and V such that $x_0 \in U$ and $A \subseteq V$, since X is regular. As $\{f^n(X)\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty compact sets such that $\bigcap_{n=1}^{\infty} f^n(X) = A \subseteq V$, there exists a positive integer N such that $f^n(X) \subseteq V$ for all $n \geq N$. Since $(n_\alpha)_\alpha$ is a subnet of $(n)_{n=1}^{\infty}$, for this particular N, there exists α_0 such that $\alpha \geq \alpha_0 \Longrightarrow n_\alpha - 1 \geq N$. But then $\alpha \geq \alpha_0 \Longrightarrow f^{n_\alpha - 1}(X) \subseteq V$, so that the point x_0, being the limit point of $(f^{n_\alpha - 1}(X_{n_\alpha}))_\alpha$, belongs to $\overline{V}$. This contradicts the assumption that $x_0 \in U$ and $U \cap V = \Phi$. Hence $x_0 \in A$, and $x = f(x_0) \in f(A)$. Therefore $A \subseteq f(A)$.

Let X be compact Hausdorff and $S = C(X,X)$ be the family of all continuous maps on X into itself equipped with compact open topology. For $f,g \in S$, define $f \cdot g = f \circ g$, the composition of g followed by f; then S is a Hausdorff topological semigroup. Define $\pi : S \times X \longrightarrow X$ by $\pi(f,x) = f(x)$, for all $f \in S$ and $x \in X$. Then π is (jointly) continuous (see

[3]), and thus S acts on X. If $f \in S$ is such that $\{f^n : n = 1,2,\ldots\}$ is equicontinuous (there is one and only one compatible uniform structure on X, see [3]), then, by Ascoli Theorem (see [3]), $\Gamma(f)$ is compact. Let $A = \bigcap_{n=1}^{\infty} f^n(X)$. Then, by Lemma 1, $f(A) = A$. Hence, by Wallace's Theorem, the unit, say r, of $K(f)$ acts as an identity map on A. We claim that r maps X onto A and in fact, each $g \in K(f)$ maps X onto A. Indeed, let $g \in K(f)$ and $x_0 \in X$, we shall show that $g(x_0) \in A$. Suppose $g(x_0) \notin A$, then there are disjoint open sets U and V such that $g(x_0) \in U$ and $A \subseteq V$. Since $(f^n(X))_{n=1}^{\infty}$ is a decreasing sequence of compact sets such that $\bigcap_{n=1}^{\infty} f^n(X) \subseteq V$, there is a positive integer N such that $f^n(X) \subseteq V$ for all $n \geq N$. Now g belongs to the closure of $F_n = \{f^m : m \geq n\}$ for all $n = 1,2,\ldots$, then for any neighbourhood W of g, $W \cap F_n \neq \Phi$ for all $n=1,2,\ldots$; pick arbitrary $f_{(W,n)} \in W \cap F_n$. Define a partial order $\leq$ on the set $D = \{(W,n) : W$ is a neighbourhood of g, $n = N, N+1, \ldots\}$ by $(W_1,n_1) \leq (W_2,n_2)$ if either $W_1 \supsetneq W_2$ or $W_1 = W_2$ and $n_1 \leq n_2$. Then $(f_{(W,n)})_{(W,n)\in D}$ is a net which converges to g. In particular, $f_{(W,n)}(x_0) \longrightarrow g(x_0)$. On the other hand, for any $(W,n) \in D$, $f_{(W,n)} \in W \cap F_n$, and hence $f_{(W,n)}(x_0) \in f^n(X) \subseteq V$ as $n \geq N$. It follows that $g(x_0) \in \overline{V}$ which contradicts the assumption that $g(x_0) \in U$ and $U \cap V = \Phi$. Thus $g(x_0) \in A$ for all $x_0 \in X$ and hence g maps X onto A. This gives us the following:

<u>Theorem 2</u>. Let X be compact Hausdorff, and $f : X \longrightarrow X$ be continuous. If $\{f^n : n=1,2,\ldots\}$ is equicontinuous, then each $g \in K(f)$ maps X onto $A = \bigcap_{n=1}^{\infty} f^n(X)$. In particular, the unit r of $K(f)$ is a retraction of X onto A.

3. Main result. Let L be a Hausdorff locally convex space and $K \subseteq L$ be nonempty. Then a family F of mappings from K into itself is said to be equicontinuous (on K) if for each $x \in K$ and each neighbourhood U of the origin 0, there exists a neighbourhood V of 0 such that $y \in K$ and $y - x \in V$ imply $f(y) - f(x) \in U$ for all $f \in F$. The proof of the following theorem is similar to that of Theorem 3.1 in [2].

Theorem 3. Let G be a nonempty complete convex subset of a Hausdorff locally convex space and $f: G \longrightarrow G$ be a map. If (i) $\{f^n : n=1,2,\ldots\}$ is equicontinuous and (ii) there exists $M \subseteq G$ which is an attractor for compact sets under f, then f has a fixed point.

Proof: Let $Y = \overline{Co}(M)$ be the closed convex hull of M. Then Y is a compact ([5]) subset of G. Let $X = \overline{\bigcup_{n=0}^{\infty} f^n(Y)}$ be the closure of the set $\bigcup_{n=0}^{\infty} f^n(Y)$, where $f^0(Y) = Y$. Clearly X is f-invariant. We shall show that X is totally bounded. Let U be any neighbourhood of 0. Then there is an open symmetric neighbourhood V of 0 such that $V + V + V \subseteq U$. As $M + V$ is an open neighbourhood of M and M is an attractor for compact sets under f, there exists a positive integer N such that $f^n(Y) \subset M + V$ for all $n \geq N$. Now for the compact set $M \cup \bigcup_{n=0}^{N-1} f^n(Y)$, there is a finite subset E of G such that $M \cup \bigcup_{n=0}^{N-1} f^n(Y) \subseteq E + V$. Thus $\bigcup_{n=0}^{\infty} f^n(Y) = \bigcup_{n=0}^{N-1} f^n(Y) \cup \bigcup_{n=N}^{\infty} f^n(Y) \subseteq$ $\subseteq (E+V) \cup (M+V) \subseteq (E+V) \cup (E+V+V) = E + V + V$. It follows that $X = \overline{\bigcup_{n=0}^{\infty} f^n(Y)} \subseteq E + V + V + V \subseteq E + U$. Therefore X is totally bounded. Furthermore X, being a closed subset of a complete set G, is complete and hence is compact ([4]). Let $A =$ $= \bigcap_{n=1}^{\infty} f^n(X)$. Then A is nonempty compact and $f(A) = A$ by

Lemma 1 and hence $A \subseteq M$ since M is an attractor for compact sets under f. As $\{f^n : n=1,2,\dots\}$ is equicontinuous on X, then, by Theorem 2, there exists a retraction $r: X \longrightarrow A$. Note that $g = f \circ r$ maps Y continuously into itself. By Schauder-Tychonoff fixed point theorem, there exists $y_0 \in Y$ such that $g(y_0) = y_0$. As $r(y_0) \in A$ and A is f-invariant, the equalities $y_0 = g(y_0) = f(r(y_0))$ show that $y_0 \in A$. Thus $r(y_0) = y_0$ since r is an identity map on A. Therefore $y_0 = f(r(y_0)) = f(y_0)$. This completes the proof.

The above theorem generalizes Theorem 3.1 in [2] and is a partial generalization of Schauder-Tychonoff fixed point theorem.

4. <u>Some remarks on attractors</u>. Let X be a topological space and $f: X \longrightarrow X$ be a map. We call a subset M of X to be an attractor for neighbourhoods of points (or more appropriately "a local attractor") under f if (i) M is nonempty compact and f-invariant, and (ii) for any neighbourhood U of M and any $x \in X$, there is a neighbourhood V of x and a positive integer N such that $f^n(V) \subseteq U$ for all $n \geq N$. It is easy to see that if M is an attractor for neighbourhoods of points under f, then it is an attractor for compact sets under f. The converse clearly holds if the space X is locally compact. We note that Theorem 3 does not give a true generalization to the Schauder-Tychonoff fixed point theorem as it requires $\{f^n : n=1,2,\dots\}$ to be equicontinuous. However, the validity of the following conjecture would provide a true generalization:

Conjecture 2. Let G be a nonempty complete convex subset of a Hausdorff locally convex space, and $f:G \to G$ be continuous. If there exists a subset M of G which is an attractor for neighbourhoods of points under f, then f has a fixed point (?).

The above conjecture is not known even if G is a nonempty closed convex subset of a Banach space.

The following result can be found in [6]:

Theorem (Meyers): Let (X,d) be a metric space and $f: X \to X$ be continuous. If the following conditions hold, then for each $\lambda \in (0,1)$, there is a metric d_λ compatible with the topology on X such that $d(f(x),f(y)) \leq \lambda d(x,y)$, for all $x,y \in X$:

(i) there is $a \in X$ such that $f(a) = a$.

(ii) For each $x \in X$, $f^n(x) \to a$ as $n \to \infty$

(iii) there exists an open neighbourhood U of a such that for each open neighbourhood V of a there exists a positive integer N such that $f^n(U) \subseteq V$ for each $n \geq N$.

As an immediate consequence of Meyers' theorem, we have the following:

Theorem 4. Let X be a metrizable space, $f:X \to X$ be continuous and $a \in X$. Then the following are equivalent:

(1) For each λ, $0 < \lambda < 1$, there exists a metric d compatible with the topology on X such that $d(f(x),f(y)) \leq \lambda d(x,y)$ for all $x,y \in X$ and $f(a) = a$.

(2) The set $\{a\}$ is an attractor for neighbourhoods of points under f.

Proof: That (1) $\Rightarrow$ (2) is straightforward. It is easy to show that (2) implies conditions (i),(ii) and (iii) in Meyers' theorem, and thus (2) $\Longrightarrow$ (1).

The following result can be found in [1]:

Theorem (Janos, Ko and Tan): Let X be a metrizable space, $f:X \longrightarrow X$ be continuous and $a \in X$. Then the following are equivalent:

(3) There exists a metric d compatible with the topology on X, such that $d(f(x),f(y)) < d(x,y)$ for all $x,y \in X$ with $x \neq y$, and $f^n(x) \longrightarrow a$ for all $x \in X$.

(4) The set $\{a\}$ is an attractor for compact sets under f.

The statements (2) and (4) are equivalent when the space X is locally compact. Thus we have the following:

Theorem 5. Let X be a locally compact metrizable space, $f:X \longrightarrow X$ be continuous and $a \in X$. Then the statements (1),(3) and (4) are equivalent.

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