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TENSOR PRODUCT AND QUASI-SPLITTING OF ABELIAN GROUPS

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Abstract: The purpose of this note is a study of some classes of torsion free groups characterized by quasi-splitting and tensor product.

Key words: Splitting, quasi-splitting and p-quasi-splitting of groups, tensor product, functor Ext.

Classification: 20K20, 20K21

If an abelian group G splits then each group H which is quasi-isomorphic to G, need not be splitting (see [1],[3]). In this note we shall deal with the class $\mathcal{E}$ of all torsion free groups A such that for each torsion group T a quasi-isomorphism $G \overset{\cdot}{\simeq} A \oplus T$ implies the splitting of the group G. It is shown that the class $\mathcal{E}$ contains a class $\mathcal{C}$ of torsion free groups whose definition is related with tensor product; in $\mathcal{C}$ is included the class $\mathcal{B}$ of all groups belonging to some Baer class Γ_α.

All groups in this note are supposed to be abelian and additively written. For the terminology and notation we refer to [2]. The symbol p represents always a prime. Furthermore, J_p (K_p resp., Q_p resp.) denotes the additive group of the ring Q_p^* of p-adic integers (of the field $\mathcal{K}_p$ of all p-adic numbers resp., of the ring Q_p of rational numbers with denominators prime to p resp.). All modules considered here are left and unitary. A Q_p^*-module G is said to be torsion free (divisible resp.) if its additive group (G;+) is torsion free (divisible resp.); the purity of a submodule H in G is defined analogously.

We begin with the following definition which will occur very useful in our investigations.

Definition 1. A torsion free Q_p^*-module G is completely decomposable if it is a direct sum of a divisible and a free modules.

At first we shall prove several elementary propositions concerning the just introduced notion.

Lemma 1. If a Q_p^*-module G is completely decomposable then each of its pure submodules H is completely decomposable as well.

Proof. Let H be a pure submodule in G and let U (V respectively) denote the maximal divisible submodule of H (of G resp.). Evidently, $U \subseteq V \cap H$; since $V \cap H$ is pure in V, it is divisible and therefore $V \cap H = U$. If we write $G = V \oplus G_1$ and $H = U \oplus H_1$ then

$$H_1 \cong H/U = H/(H \cap V) \cong (H+V)/V \subseteq G/V \cong G_1.$$

By assumption G_1 is free and each of its submodules is also free. Thus H_1 is free and hence H is completely decomposable.

Lemma 2. Let F be a free Q_p^*-module and let H be its pure submodule of finite rank. Then the module F/H is also free.

Proof. Consider F in the form $F = \sum_{i \in I} \oplus Q_p^* x_i$. Now we shall proceed by induction on the rank $r(H)$ of the free submodule H.

For $r(H) = 1$ we have $H = Q_p^* y$ where $\{y\}$ is a free basis in H. With respect to the relation $y \in F$ we can write $y = \alpha_1 x_{i_1} + \dots + \alpha_n x_{i_n}$ where $0 \neq \alpha_i \in Q_p^*$ $(i=1,\dots,n)$. Since the equation $px = y$ has no solution in H, there exists i_j $(1 \leq j \leq n)$ such that $p \nmid \alpha_{i_j}$. Hence the element α_{i_j} is invertible in Q_p^* and we have

$$F = Q_p^* y \oplus \sum_{k \neq i_j} \oplus\, Q_p^* x_k .$$

In this case F/H is free.

Suppose $r(H) = r > 1$ and write $H = \sum_{j=1}^{r} \oplus\, Q_p^* y_j$ where $\{y_1,\dots,y_r\}$ is a free basis of H. If we set $H_0 = \sum_{j=1}^{r-1} \oplus\, Q_p^* y_j$ then H_0 is pure in F and by induction F/H_0 is free. But H/H_0 is a rank one pure submodule in F/H_0 and by the preceding part $F/H \cong (F/H_0)/(H/H_0)$ is also free.

The following generalization is an immediate one and the proof will be omitted.

<u>Lemma 3.</u> If G is a completely decomposable Q_p^*-module and H its pure submodule of finite rank then G/H is also completely decomposable.

Now we shall continue by proving the following assertion.

<u>Lemma 4.</u> Let G be a reduced torsion free Q_p^*-module and let H be its pure submodule of finite rank. Then G/H is a reduced Q_p^*-module.

<u>Proof</u>. For an indirect proof suppose that the torsion free module G/H is not reduced. Then there exists a submodule G_1 in G such that $H \subseteq G_1$ and G_1/H is isomorphic to the Q_p^*-module K_p. If n is the rank of H then $n+1$ is the rank of

G_1; at the same time G_1 is a reduced torsion free Q_p^*-module. By the Prüfer-Kaplansky theorem [6, § 40] G_1 is a free Q_p^*-module and by Lemma 2 G_1/H is free as well. We get a contradiction with $G_1/H \cong K_p$ and hence G/H is reduced.

<u>Lemma 5.</u> Let G be a torsion free Q_p^*-module and H its pure submodule of finite rank. If the module G/H is completely decomposable then G is also completely decomposable.

<u>Proof</u> Denote by D(G) (D(H) resp.) the maximal divisible submodule in G (in H resp.) and write $H = D(H) \oplus H_1$; evidently, H_1 is reduced. Since H_1 is pure in G, $H_1 \cap D(G)$ is pure in D(G) and therefore $H_1 \cap D(G) = 0$. Thus there is in G a submodule G_1 such that $H_1 \subseteq G_1$ and $G = D(G) \oplus G_1$. Clearly $D(H) \subseteq D(G)$ and we have

$$G/H = (D(G) \oplus G_1)/(D(H) \oplus H_1) \cong D(G)/D(H) \oplus G_1/H_1.$$

The module G_1 is reduced and H_1 is its submodule which is pure and of finite rank. By Lemma 4, G_1/H_1 is reduced as well. From the complete reducibility of G/H it follows that G_1/H_1 is free. Hence, $G_1 = H_1 \oplus H_2$ where H_2 is free. Since H_1 is reduced and of finite rank, H_1 is also free [6, § 40]. Thus we have proved that G_1 is free and, therefore, $G = D(G) \oplus G_1$ is completely decomposable.

For any torsion free group A and any torsion free Q_p^*-module G the tensor product of abelian groups $G \otimes A$ may be considered as a torsion free Q_p^*-module. Thus we can formulate the following definition.

<u>Definition 2.</u> By the symbol $\mathcal{C}_p$ we shall denote the class of all torsion free groups A for which the Q_p^*-module $J_p \otimes A$ is completely decomposable (see [5]).

In the following proposition some elementary properties of the class $\mathcal{C}_p$ are concentrated.

Proposition 1. i) The class $\mathcal{C}_p$ is closed with respect to direct sums, tensor product and pure subgroups.
ii) If A is a torsion free group and S its pure subgroup of finite rank then $A \in \mathcal{C}_p$ if and only if $A/S \in \mathcal{C}_p$.

Proof. If $A = \sum_{i \in I} \oplus A_i$ with $A_i \in \mathcal{C}_p$ $(i \in I)$ then the relation $J_p \otimes A \cong \sum_{i \in I} \oplus (J_p \otimes A_i)$ (it represents a module isomorphism) implies $A \in \mathcal{C}_p$. Assume $A, B \in \mathcal{C}_p$. Then we have a module isomorphism $J_p \otimes (A \otimes B) \cong (J_p \otimes A) \otimes B$. By hypothesis the Q_p^*-module $J_p \otimes A$ is completely decomposable, therefore, $J_p \otimes A = \sum_{i \in I} \oplus G_i$, where each module G_i is isomorphic either to J_p or to K_p. Thus we have a relation

$$J_p \otimes (A \otimes B) \cong \sum_{i \in I} \oplus (G_i \otimes B)$$

where each module $G_i \otimes B$ is completely decomposable (if $G_i \cong K_p$ then $K_p \otimes B$ is divisible). This means that $A \otimes B \in \mathcal{C}_p$.

If $A \in \mathcal{C}_p$ and S is any pure subgroup of A then (see [2, 60.4])

$$(1) \qquad 0 \longrightarrow J_p \otimes S \longrightarrow J_p \otimes A \longrightarrow J_p \otimes (A/S) \longrightarrow 0$$

is a pure exact sequence of abelian groups. Thus the Q_p^*-module $J_p \otimes S$ is a pure submodule of the completely decomposable module $J_p \otimes A$ and hence, by Lemma 1, we get $S \in \mathcal{C}_p$. The proof of i) is complete.

For the proof of ii) let us note that if S is of finite rank then $J_p \otimes S$ is of the same rank as Q_p^*-module (see [2, § 93]). Furthermore, with respect to (1) we have

$$(2) \qquad J_p \otimes (A/S) \cong (J_p \otimes A)/(J_p \otimes S).$$

If we assume $A \in \mathcal{C}_p$ then the relation $A/S \in \mathcal{C}_p$ follows from (2) by using Lemma 3. On the other hand, if $A/S \in \mathcal{C}_p$ then for the proof of $A \in \mathcal{C}_p$ we use (2) and Lemma 5.

Recall now the definition of Baer classes Γ_α . Firstly, Γ_1 is defined as the class of all countable torsion free groups. If $\alpha > 1$ then a torsion free group A belongs to Γ_α just if $A \notin \Gamma_\beta$ for each $\beta < \alpha$ and there exists a pure subgroup S in A of finite rank such that A/S is a direct sum of groups belonging to classes of indices less than α . By the symbol $\mathcal{B}$ we shall denote the class of all torsion free groups A such that there is an ordinal α with $A \in \Gamma_\alpha$.

<u>Lemma 6.</u> For every prime p we have the inclusion $\mathcal{B} \subseteq \mathcal{C}_p$.

<u>Proof</u>. We shall prove by induction that for every ordinal α it is $\Gamma_\alpha \subseteq \mathcal{C}_p$. The relation $\Gamma_1 \subseteq \mathcal{C}_p$ is a consequence of the Prüfer-Kaplansky theorem [2, 93.3]. Suppose now that $1 < \alpha$, $\Gamma_\beta \subseteq \mathcal{C}_p$ for each $\beta < \alpha$, and take $A \in \Gamma_\alpha$. By the definition there exists a pure subgroup S of finite rank in A such that $A/S = \sum_{i \in I} \oplus A_i$, $A_i \in \Gamma_{\beta_i}$ and $\beta_i < \alpha$ $(i \in I)$. Thus $A_i \in \mathcal{C}_p$ $(i \in I)$ and $A/S \in \mathcal{C}_p$ by Proposition 1. But using the same Proposition 1 we obtain $A \in \mathcal{C}_p$ and hence $\Gamma_\alpha \subseteq \mathcal{C}_p$. The proof by induction is finished.

For every prime p the class $\mathcal{B}$ may be extended in the following way:

<u>Definition 3.</u> By the symbol $\mathcal{B}_p$ we shall denote the class of all torsion free groups A such that $Q_p \otimes A \in \mathcal{B}$.

<u>Proposition 2</u>. For any prime p we have the inclusions

$\mathcal{B} \subseteq \mathcal{B}_p \subseteq \mathcal{C}_p$.

Proof. In order to prove the inclusion $\mathcal{B} \subseteq \mathcal{B}_p$ we shall prove the following sharper assertion: (*) If $A \in \Gamma_\alpha$ then there exists a $\beta \leqq \alpha$ such that $Q_p \otimes A \in \Gamma_\beta$. The proof will proceed by induction. If $A \in \Gamma_1$ then also $Q_p \otimes A \in \Gamma_1$ since both groups are countable. Assume $1 < \alpha$ and $Q_p \otimes A \notin \Gamma_\beta$ for each $\beta < \alpha$. Since $A \in \Gamma_\alpha$ $(1 < \alpha)$, there exists a pure subgroup S of finite rank in A with a direct decomposition $A/S = \sum_{i \in I} \oplus A_i$, $A_i \in \Gamma_{\beta_i}$, $\beta_i < \alpha$ $(i \in I)$. At the same time the sequence

$$0 \longrightarrow Q_p \otimes S \longrightarrow Q_p \otimes A \longrightarrow Q_p \otimes (A/S) \longrightarrow 0$$

is pure exact and we get an isomorphism

$$(Q_p \otimes A)/(Q_p \otimes S) \cong Q_p \otimes (A/S) \cong \sum_{i \in I} \otimes (Q_p \otimes A_i).$$

By inductive hypothesis $Q_p \otimes A_i \in \Gamma_{\gamma_i}$ where $\gamma_i \leqq \beta_i < \alpha$; this implies $Q_p \otimes A \in \Gamma_\alpha$ since $Q_p \otimes S$ is of finite rank. Thus the proof of (*) is finished and we have $\mathcal{B} \subseteq \mathcal{B}_p$.

For the proof of the inclusion $\mathcal{B}_p \subseteq \mathcal{C}_p$ suppose $A \in \mathcal{B}_p$. This means $Q_p \otimes A \in \mathcal{B}$ and $Q_p \otimes A \in \mathcal{C}_p$ by Lemma 6. Hence, the Q_p^*-module $J_p \otimes (Q_p \otimes A)$ is completely decomposable. But $J_p \otimes (Q_p \otimes A) \cong (J_p \otimes Q_p) \otimes A \cong J_p \otimes A$, therefore, $A \in \mathcal{C}_p$.

Note that the proposition proved just now concerns the largeness of the class $\mathcal{C}_p$.

In the following we shall use the next notation: If G is any group then t(G) denotes the maximal torsion subgroup of G; $G_{(p)}$ represents the p-primary component of t(G). Recall that two groups G, H are said to be quasi-isomorphic [3] (in symbols $G \doteq H$) if there are subgroups $U \subseteq G$, $V \subseteq H$ and a positive integer n such that $nG \subseteq U$, $nH \subseteq V$ and $U \cong V$. Now we give a localization of the notion just defined.

Definition 4. Two groups G, H are said to be p-quasi-isomorphic if they are quasi-isomorphic and if the corresponding number n may be found in the form $n = p^k$. In this case we shall write $G \overset{\cdot}{\underset{p}{\approx}} H$.

From the Definition 4 it may be deduced that the relation $G \overset{\cdot}{\underset{p}{\approx}} H$ implies $G/t(G) \overset{\cdot}{\underset{p}{\approx}} H/t(H)$. The following lemma is a modification of [3, Theorem 5].

Lemma 7. A group G is p-quasi-isomorphic to a splitting group if and only if the exact sequence

$$(3) \qquad 0 \longrightarrow t(G) \longrightarrow G \longrightarrow G/t(G) \longrightarrow 0$$

represents an element of the group $[\mathrm{Ext}(G/t(G), t(G))]_{(p)}$.

Now we shall define a further class $\mathcal{C}_p$ (depending on p) of torsion free groups.

Definition 5. By the symbol $\mathcal{C}_p$ ($\mathcal{C}$ resp.) we shall denote the class of all torsion free groups A such that for each torsion group T the relation $G \overset{\cdot}{\underset{p}{\approx}} A \oplus T$ ($G \overset{\cdot}{\approx} A \oplus T$ resp.) implies that the group G splits. Evidently, $\mathcal{C} \subseteq \mathcal{C}_p$ for every prime p.

Proposition 3. A torsion free group A is contained in the class $\mathcal{C}_p$ if and only if $[\mathrm{Ext}(A,T)]_{(p)} = 0$ for every torsion group T.

Proof. Assume $A \in \mathcal{C}_p$, T a torsion group and let

$$(4) \qquad 0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

be an exact sequence representing an element of the group $[\mathrm{Ext}(A,T)]_{(p)}$. By [3, Theorem 3], for a suitable integer n the sequence

$$0 \longrightarrow T \longrightarrow p^nG + T \longrightarrow p^nA \longrightarrow 0$$

is splitting exact. Hence $p^nG + T = A_0 \oplus T$, where $A_0 \cong p^nA \cong$ $\cong A$ and therefore $A_0 \in \mathcal{E}_p$. Since $p^nG \subseteq p^nG + T = A_0 \oplus T \subseteq G$, we have $G \overset{\cdot}{\underset{p}{\cong}} A_0 \oplus T$, which implies that G splits. This means that (4) represents the zero element and we conclude $[\mathrm{Ext}(A,T)]_{(p)} = 0$.

On the other hand, let A be a torsion free group such that $[\mathrm{Ext}(A,T)]_{(p)} = 0$ for every torsion group T. Take any torsion group T_0 and consider a group G satisfying $G \overset{\cdot}{\underset{p}{\cong}} A \oplus T_0$; thus, as we have noted, $G/t(G) \overset{\cdot}{\underset{p}{\cong}} A$. By Lemma 7, the exact sequence (3) represents an element of $[\mathrm{Ext}(G/t(G),t(G))]_{(p)}$. Using [4, Lemma 2], from the relation $G/t(G) \overset{\cdot}{\underset{p}{\cong}} A$ we deduce $\mathrm{Ext}(G/t(G),t(G)) \cong \mathrm{Ext}(A,t(G))$ and hence $[\mathrm{Ext}(G/t(G),t(G))]_{(p)} \cong$ $\cong [\mathrm{Ext}(A,t(G))]_{(p)} = 0$. This means that (3) represents the zero element, G splits and, therefore, $A \in \mathcal{E}_p$.

As an immediate consequence we obtain

Corollary 1. The class $\mathcal{E}_p$ is closed with respect to direct sums and summands. Analogously for the class $\mathcal{E}$ (see [4]).

Now we shall describe some further properties of the class $\mathcal{E}_p$.

Lemma 8. Let A_1, A_2 be two torsion free groups satisfying $A_1 \overset{\cdot}{\underset{p}{\cong}} A_2$. Then $A_1 \in \mathcal{E}_p$ if and only if $A_2 \in \mathcal{E}_p$. Further, if A is a torsion free group such that for every torsion group T any extension G of $A \oplus T$ splits whenever $G/(A \oplus T)$ is a bounded p-group, then $A \in \mathcal{E}_p$.

Proof. If $A_1 \overset{\cdot}{\underset{p}{\cong}} A_2$ then $\mathrm{Ext}(A_1,T) \cong \mathrm{Ext}(A_2,T)$ for each torsion group T (see [4, Lemma 2]). Our first assertion follows now by Proposition 3. Further, let A satisfy the hypothesis of the lemma and let G be a group with $G \underset{p}{\simeq} A \oplus T$ for a torsion group T. In view of the Definition 4, there are subgroups U, V and an integer n such that $p^n G \subseteq U \subseteq G$, $p^n(A \oplus T) \subseteq V \subseteq A \oplus T$ and $U \cong V$. Since $p^n(A \oplus T) = p^n A \oplus p^n T$, $p^n A \cong A$ and $U \cong V$, there is a subgroup $U_1 \subseteq U$ satisfying $U_1 \cong$ $\cong p^n(A \oplus T) \cong A \oplus p^n T$ and $p^n U \subseteq U_1$. Hence $p^{n+n} G \subseteq U_1$ and, by hypothesis, the group G splits. But this means that $A \in \mathcal{E}_p$.

Lemma 9. If $A \in \mathcal{E}_p$ then $Q_p \otimes A \in \mathcal{E}_p$ as well.

Proof. Assume $A \in \mathcal{E}_p$ and denote by E(A) the divisible hull of A. If we set $A_p = Q_p A \subseteq E(A)$ then there exists a natural isomorphism $Q_p \otimes A \cong A_p = Q_p A$. We shall prove that $A_p \in$ $\in \mathcal{E}_p$. In order to verify this fact, take any torsion group T_o and consider an extension G of $A_p \oplus T_o$ such that $G/(A_p \oplus T_o)$ is a bounded p-group. If we denote t(G) = T then $T_o \subseteq T$ and $G/(A_p \oplus T)$ is a bounded p-group as well. We can write T = $= T_{(p)} \oplus T^*_{(p)}$ where $T^*_{(p)}$ represents the direct sum of all primary components different to $T_{(p)}$. Let us denote by G_o the following set

$$G_o = \{g; g \in G, p^n g \in A_p \oplus T_{(p)} \text{ for a suitable } n\}.$$

Evidently, G_o is a subgroup of G containing $A_p \oplus T_{(p)}$ and satisfying $G_o \cap T^*_{(p)} = 0$. We shall prove that $G_o \oplus T^*_{(p)} = G$. For an indirect proof consider $g \in G \setminus (G_o \oplus T^*_{(p)})$. Obviously there is an integer n such that $p^n g \in G_o \oplus T^*_{(p)}$, therefore, $p^n g =$ $= g_o + t$ with $g_o \in G_o$, $t \in T^*_{(p)}$. But t may be written in the form $t = p^n t_o$, $t_o \in T^*_{(p)}$, and hence $p^n(g - t_o) = g_o \in G_o$. This contradicts with $g - t_o \notin G_o$ and the definition of G_o. Thus we have shown $G = G_o \oplus T^*_{(p)}$ and $A_p \oplus T_{(p)} \subseteq G_o$. This means that $T_{(p)}$

is the maximal torsion subgroup of G_o and we get an isomorphism of bounded p-groups

(5) $$G/(A_p \oplus T) \cong G_o/(A_p \oplus T_{(p)}).$$

At the same time we can write

(6) $$G_o/(A_p \oplus T_{(p)}) \cong [G_o/(A \oplus T_{(p)})]/[(A_p \oplus T_{(p)})/ /(A \oplus T_{(p)})]$$

and also

$$(A_p \oplus T_{(p)})/(A \oplus T_{(p)}) \cong A_p/A.$$

But $A_p/A = Q_pA/A$ is a divisible torsion group with p-component equal zero. Thus from (5) and (6) we deduce that the group $\overline{G} = G_o/(A \oplus T_{(p)})$ is a torsion group of the form $\overline{G} = \overline{G}_{(p)} \oplus \overline{G}^*_{(p)}$, where the p-component $\overline{G}_{(p)}$ is bounded and $\overline{G}^*_{(p)}$ is divisible. Let G_1 be the subgroup of G_o such that $A \oplus T_{(p)} \subseteq G_1$ and $G_1/(A \oplus T_{(p)}) = \overline{G}_{(p)}$; therefore, $G_1/(A \oplus T_{(p)})$ is a bounded p-group. Since $A \in \mathcal{C}_p$, the group G_1 splits: $G_1 = A_1 \oplus T_{(p)}$. From the construction of G_1 it follows that $G_o/G_1 \cong \overline{G}^*_{(p)}$, hence G_o/G_1 is a divisible torsion group with p-component equal zero. Now let us set

$A_o = \{g; g \in G_o,\ ng \in A_1$ for a suitable n and $p \nmid n\}$;

it is easily seen that A_o is a subgroup of G_o satisfying $A_1 \subseteq A_o$ and $A_o \cap T_{(p)} = 0$. The group $G_o/(A_o \oplus T_{(p)})$ is again torsion and divisible with p-component equal zero. By the same method as in the preceding part it may be shown that the assumption $G_o \neq A_o \oplus T_{(p)}$ makes a contradiction. Thus $G_o = A_o \oplus T_{(p)}$ and hence $G = G_o \oplus T^*_{(p)} = A_o \oplus T_{(p)} \oplus T^*_{(p)} = A_o \oplus T$. The group G splits, therefore, using Lemma 8 we conclude that $Q_p \otimes A \in \mathcal{C}_p$.

For the proof of the converse we shall use the following lemma.

Lemma 10. Let A, C be two torsion free groups with $pC \neq C$. If $C \otimes A \in \mathcal{C}_p$ then $A \in \mathcal{C}_p$.

Proof. Assume $C \otimes A \in \mathcal{C}_p$, take any torsion group T and set $E = \mathrm{Ext}(A,T)$; we shall prove that $E_{(p)} = 0$. Since the ring Z of rational integers is hereditary we have by [7, VI, Proposition 3.6 a)]

$$(7) \quad \mathrm{Ext}(C,\mathrm{Hom}(A,T)) \oplus \mathrm{Hom}(C,\mathrm{Ext}(A,T)) \cong$$
$$\cong \mathrm{Ext}(C \otimes A,T) \oplus \mathrm{Hom}(\mathrm{Tor}(C,A),T).$$

Both groups A, C are torsion free, therefore, $\mathrm{Tor}(C,A) = 0$. From (7) we obtain (up to an isomorphism) the inclusion

$$\mathrm{Hom}(C,\mathrm{Ext}(A,T)) \subseteq \mathrm{Ext}(C \otimes A,T).$$

Since A is torsion free, the group $E = \mathrm{Ext}(A,T)$ is divisible. If $E_{(p)} \neq 0$ then we should have a direct decomposition $E = E_0 \oplus Z(p^\infty)$ and hence

$$(8) \quad \mathrm{Hom}(C,E_0) \oplus \mathrm{Hom}(C,Z(p^\infty)) \subseteq \mathrm{Ext}(C \otimes A,T).$$

In view of Proposition 3, the relation $C \otimes A \in \mathcal{C}_p$ implies that $[\mathrm{Ext}(C \otimes A,T)]_{(p)} = 0$. But making use of the hypothesis $pC \neq C$ we conclude $\mathrm{Hom}(C,Z(p^\infty)) \neq 0$, which contradicts (8). Thus $E_{(p)} = 0$ and the Proposition 3 gives $A \in \mathcal{C}_p$.

As a corollary we obtain:

Proposition 4. i) If A is a torsion free group then $A \in \mathcal{C}_p$ if and only if $Q_p \otimes A \in \mathcal{C}_p$. ii) The class of all torsion free groups which are not contained in $\mathcal{C}_p$, is closed with respect to the tensor product.

Proof. i) The implication $A \in \mathcal{C}_p \Rightarrow Q_p \otimes A \in \mathcal{C}_p$ is shown in Lemma 9, the converse follows from Lemma 10 setting

$C = Q_p$. ii) If C is torsion free and $pC = C$, then $Q_p \otimes C$ is divisible and, therefore, it is a direct sum of countable groups. Thus by [4, Theorems 2 and 3] (see also the following Lemma 11 and Corollary 1) we obtain $Q_p \otimes C \in \mathcal{E} \subseteq \mathcal{E}_p$, and in view of Lemma 10, $C \in \mathcal{E}_p$. Hence, assuming A, C torsion free and not contained in $\mathcal{E}_p$ we have $pC \neq C$ and using Lemma 10 we get $C \otimes A \notin \mathcal{E}_p$.

Before we prove the inclusion $\mathcal{C}_p \subseteq \mathcal{E}_p$ we recall the following known facts.

Lemma 11. The class $\mathcal{E}$ contains the class of all countable torsion free groups; for each prime p it is $J_p \in \mathcal{E}$.

Proof. See [4, Theorem 2].

Proposition 5. For each prime p we have the inclusion $\mathcal{C}_p \subseteq \mathcal{E}_p$.

Proof. If $A \in \mathcal{C}_p$ then the Q_p^*-module $J_p \otimes A$ is completely decomposable. Hence, the additive group $J_p \otimes A$ is a direct sum of the form $D \oplus A_1$, where D is divisible and A_1 is a direct sum of groups isomorphic to J_p. Using Corollary 1 and Lemma 11 we deduce that $A_1 \in \mathcal{E}$, $D \in \mathcal{E}$ and therefore $J_p \otimes A = D \oplus A_1 \in \mathcal{E} \subseteq \mathcal{E}_p$. From Lemma 10 we get $A \in \mathcal{E}_p$ and hence $\mathcal{C}_p \subseteq \mathcal{E}_p$.

This note we shall conclude by the following remarks.

Remark 1. If P denotes the product of $\aleph_0$ exemplars of infinite cyclic group Z then $P \notin \mathcal{E}_p$ and, therefore, $P \notin \mathcal{C}_p$. Thus none of the classes $\mathcal{B}_p$, $\mathcal{C}_p$, $\mathcal{E}_p$ is closed with respect to direct product. This shows also that the reduced Q_p^*-module $J_p \otimes P$ is not completely decomposable.

Proof. If T is a torsion group such that $T_{(p)}$ is not expressible as a direct sum of a bounded and a divisible groups then by [1, Satz 4.1]it is $[Ext(P,T)]_{(p)} \neq 0$. Now it suffices to use Propositions 3, 5 and 2.

Remark 2. Let $\mathcal{C}$ denote the class of all torsion free groups A such that $A \in \mathcal{C}_p$ for every prime p. Then $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{E}$.

Proof. The inclusion $\mathcal{B} \subseteq \mathcal{C}$ follows immediately from Proposition 2. If $A \in \mathcal{C}$ then $A \in \mathcal{C}_p \subseteq \mathcal{E}_p$ for any prime p, by Proposition 5. In view of Proposition 3, $[Ext(A,T)]_{(p)} = 0$ for every torsion group T and hence, Ext(A,T) is torsion free whenever T is torsion. This implies (see [4, Theorem 1]) that $A \in \mathcal{E}$ and, therefore, $\mathcal{C} \subseteq \mathcal{E}$.

The inclusions $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{E}$ show that the class $\mathcal{E}$ is sufficiently large. Further, the well known Kuroš - Mal´cev - Derry invariants theory (see [2, § 93]) may be extended to the class $\mathcal{C}$. Thus there is a possibility (by making use of the existence theorem) to construct (non trivial) groups of arbitrary cardinality lying in $\mathcal{C}$ and, therefore, in $\mathcal{E}$.

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