Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 20 (1979), No. 4, 737--743

Persistent URL: http://dml.cz/dmlcz/105964

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IRREDUCIBILITY OF COMPOUND MATRICES

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Abstract: We investigate the influence of irreducibility or reducibility of a square matrix on irreducibility or reducibility of its compound matrices and generalized compound matrices. The case of additive compound matrices is solved by a graph-theoretical approach.

Key words: Compound matrix, directed graph, irreducible, reducible.

Classification: 15A75

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1. Introduction. As is well known [2], the k-th compound matrix $A^{(k)}$ to the $m \times n$ - matrix A ($1 \leq k \leq \min(m,n)$) is defined as follows: Let $M = \{1,\dots,m\}$, $N = \{1,\dots,n\}$; denote by $M^{(k)}$, $N^{(k)}$ respectively, the set of all k-tuples in M, N respectively, ordered lexicographically (e.g. for $m = 4$, $k = 2$, $M^{(k)} = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3.4)\}$). $A^{(k)}$ is then an $\binom{m}{k} \times \binom{n}{k}$ - matrix with row indices in $M^{(k)}$, column indices in $N^{(k)}$ whose entry with the row index $I \in M^{(k)}$ and column index $J \in N^{(k)}$ is det A(I,J), i.e. the determinant of the $k \times k$ submatrix of A with row indices in I and column indices in J.

For an indeterminate x, the k-th compound matrix $(A + xI)^{(k)}$, I being the identity matrix, is a matrix polynomial in x of degree k:

$$(A + xI)^{(k)} = A^{(k,0)}x^k + A^{(k,1)}x^{k-1} + \dots$$
$$\dots + A^{(k,k-1)}x + A^{(k,k)}.$$

The matrices $A^{(k,j)}$, $1 \leq j \leq k$, are sometimes called generalized compound matrices. In particular, the matrix $A^{(k,1)}$ is called k-th additive compound matrix and denoted by $A^{[k]}$.

We shall investigate what influence the irreducibility or reducibility of A has on irreducibility or reducibility of the generalized compound matrices. Let us recall that a square matrix A is reducible if it is of the form

$$\begin{pmatrix} A_{11}, & 0 \\ A_{21}, & A_{22} \end{pmatrix}$$

where A_{11}, A_{22} are square matrices of order at least one, or if A can be brought to such a form by a simultaneous permutation of rows and columns; otherwise, A is irreducible.

It is clear that if A has rank r, all compound matrices $A^{(k)}$ for $k > r$ will be zero matrices and, if $r < k < n$, $A^{(k)}$ will be reducible even if A is irreducible. Similar simple statements can be made for the generalized compound matrices $A^{(k,s)}$ if $r < s$.

We shall show, however, that the additive compound matrix (if of degree at least two) preserves the property of the original matrix to be irreducible or reducible. Moreover, each generalized compound matrix (if of degree at least two) of a reducible matrix is reducible as well. In the proof of the first theorem we shall use the notion of the directed graph

G(A) of a square matrix $A = (a_{ik})$: $G(A) = [N,E]$ where $N = \{1,2,\dots,n\}$ is the set of vertices and $(i,k) \in E$, the set of edges, iff $i \neq k$ (we omit diagonal entries) and $a_{ik} \neq 0$. It is well known [3] that a square matrix of order $n \geq 2$ is irreducible iff G(A) is strongly connected, i.e. if there is a path in G(A) from every vertex into any other vertex.

We shall also need the explicit formula for the additive compound matrix $A^{[k]}$ as shown in [1]:

If $A = (a_{ik})$ is square of order n, $N = \{1,\dots,n\}$, then the entry of $A^{[k]}$ with the row index $I \in N^{(k)}$ and column index $J \in N^{(k)}$ is ($|S|$ denotes the number of entries in the set S)

$$(1)\quad A^{[k]}(I,J) = \begin{cases} \sum_{i \in I} a_{ii} & \text{if } I = J, \text{ i.e. } |I \cap J| = k, \\ 0 & \text{if } |I \cap J| \leq k-2. \\ (-1)^{\sigma} a_{ij} & \text{if } |I \cap J| = k-1, \end{cases}$$

where $\{i\} = I \setminus (I \cap J)$, $\{j\} = J \setminus (I \cap J)$ and σ is the number of elements in $I \cap J$ between i and j (in the natural ordering of indices).

Finally, we shall make use of the well known König's theorem [4]: Let $A = (a_{ik})$ be a square matrix of order n, $N = \{1,\dots,n\}$. If there exist subsets N_1 and N_2 of N such that $|N_1| + |N_2| > n$ and $a_{ik} = 0$ whenever $i \in N_1$ and $k \in N_2$ then $\det A = 0$.

2. <u>Results</u>. First of all, we shall introduce the notion of the k-th additive compound graph of a directed graph with n vertices, $1 \leq k \leq n$.

(2,1) <u>Definition</u>. Let $G = [V,E]$ be a finite directed graph without multiple edges with the set of vertices V and set of edges E. Let k be an integer, $1 \leq k \leq n$ where $n = |V|$. The k-th additive compound graph $G^{[k]}$ of G is the graph $[V^{(k)}, E^{[k]}]$ where $V^{(k)}$ is the set of all k-tuples in V and $(I,J) \in E^{[k]}$ for $I \in V^{(k)}$, $J \in V^{(k)}$ iff $|I \cap J| = k-1$ and, for $\{i\} = I \setminus (I \cap J)$, $\{j\} = J \setminus (I \cap J)$, $(i,j) \in E$.

The name is justified by the following theorem:

(2,2) <u>Theorem</u>. Let A be a square matrix of order n, $A^{[k]}$ its k-th additive compound of A, $1 \leq k \leq n$. Then the directed graph $G(A^{[k]})$ of $A^{[k]}$ is isomorph to the k-th additive compound graph $H^{[k]}$ of the directed graph $H = G(A)$ of A.

Proof. Follows immediately from (1); in fact, if the vertices of V are numbered by the numbers $1,2,\ldots,n$ and $V^{(k)}$ by k-tuples in lexicographical order the ordering of the vertices of $H^{[k]}$ corresponds to that of the rows in $G(A^{[k]})$.

(2,3) <u>Theorem</u>. Let G be a strongly connected finite directed graph with n vertices. Then for each $k = 1,2,\ldots,n$, the k-th additive compound graph $G^{[k]}$ of G is strongly connected as well.

Proof. Since $G^{[1]} = G$ and $G^{[n]}$ has a single vertex, we shall assume that $2 \leq k < n$. Let $G = [V,E]$ and let V_1, V_2 be different subsets of V, $|V_1| = |V_2| = k$. We shall show that there exists a path in $G^{[k]}$ from the vertex of $G^{[k]}$ corresponding to V_1 to the vertex corresponding to V_2. We shall use induction with respect to the number $\nu(V_1,V_2) = k - |V_1 \cap V_2|$. Let first $\nu(V_1,V_2) = 1$, let $V_1 = V_o \cup \{i\}$, $V_2 = V_o \cup \{j\}$ where $V_o = V_1 \cap V_2$. G being strongly connected, there exists a path $P = (i,u_1,\ldots,u_s,j)$ in G from i to j: If none of the vertices

$u_1,\dots,u_s$ is contained in V_o then $(V_o \cup \{i\},\ V_o \cup \{u_1\},\dots$
$\dots,\ V_o \cup \{u_s\},\ V_o \cup \{j\})$ is a path in $G^{[k]}$ from V_1 to V_2. Otherwise, let u_{p_1} be the first vertex from P in V_o, then u_{q_1} the first vertex from P succeeding p_1 which is not in V_o, then u_{p_2} the first vertex from P succeeding u_{q_1} in V_o etc., till u_{q_t} the first vertex for which all vertices u_{q_t}, $u_{q_t+1},\dots,u_s, j$ are outside V_o. Then,

$(V_o \cup \{i\}, V_o \cup \{u_1\},\dots,V_o \cup \{u_{p_1-1}\}, V_o \cup \{u_{p_1-1}\} \cup \{u_{q_1}\} \setminus \{u_{q_1-1}\},$
$V_o \cup \{u_{p_1-1}\} \cup \{u_{q_1}\} \setminus \{u_{q_1-2}\},\dots,V_o \cup \{u_{p_1-1}\} \cup \{u_{q_1}\} \setminus \{u_{p_1}\},$
$V_o \cup \{u_{q_1}\}, V_o \cup \{u_{q_1+1}\},\dots,V_o \cup \{u_{p_2-1}\}, V_o \cup \{u_{p_2-1}\} \cup \{u_{q_2}\} \setminus$
$\setminus \{u_{q_2-1}\},\dots,V_o \cup \{u_{q_t}\}, V_o \cup \{u_{q_t+1}\},\dots,V_o \cup \{j\})$ is a path in $G^{[k]}$ from V_1 to V_2.

Now let $\nu(V_1,V_2) = \nu > 1$ and assume the assertion is true for all V_1', V_2' in $V^{(k)}$ satisfying $\nu(V_1',V_2') < \nu$. Then there exist vertices $u \in V_2 \setminus V_1$ and $v \in V_1 \setminus V_2$. Since $\nu(V_1, V_2 \setminus \{u\} \cup \cup \{v\}) < \nu$ and $\nu(V_2 \setminus \{u\} \cup \{v\}, V_2) = 1$, there exist paths from V_1 to $V_2 \setminus \{u\} \cup \{v\}$ as well as from $V_2 \setminus \{u\} \cup \{v\}$ to V_2 in $G^{[k]}$, hence also from V_1 to V_2.

(2,4) <u>Theorem</u>. Let A be an irreducible square matrix of order n. Then for each $k = 1,\dots,n$, the k-th additive compound matrix $A^{[k]}$ of A is irreducible as well.

Proof. As we mentioned in the introduction, a square matrix is irreducible iff its directed graph is strongly connected. Therefore, this theorem follows immediately from Theorem (2,3).

(2,5) <u>Theorem</u>. Let A be a square matrix of order n. If A is reducible then for each $k = 1,\dots,n-1$ and each $s = 1,\dots$

$\dots,k$, the generalized compound matrix $A^{(k,s)}$ is reducible as well.

Proof. Let $A = (a_{ik})$ be reducible, let $N = \{1,\dots,n\}$. Then there exists a non-void proper subset M of N such that $a_{ij} = 0$, whenever $i \in M$ and $j \notin M$. Let k be an integer, $1 \leqq k \leqq \leqq n-1$, let $N^{(k)}$ as above be the set of all k-tuples in N. We shall distinguish two cases:

Case A. $k \leqq |M|$. Define Z as the subset of $N^{(k)}$ consisting of all k-tuples contained in M. Observe that Z is a non-void and proper subset of $N^{(k)}$. Let us show that if $I \in Z$ and $J \notin Z$ then $\det A(I,J) = 0$, i.e. the entry of $A^{(k)}$ with the row index I and column index J is zero. Since $J \notin Z$, there exists an index $j \in J$ such that $j \notin M$. Therefore, the column in the submatrix $A(I,J)$ corresponding to the index j contains all zero entries so that $\det A(I,J) = 0$, indeed. However, the property that $\det A(I,J) = 0$ whenever $I \in Z$ and $J \notin Z$ means that $A^{(k)}$ is reducible.

Case B. $k > |M|$. Denote by $\hat{Z}$ the subset of $N^{(k)}$ consisting of those k-tuples which contain all indices in M. We shall show again that $\det A(I,J) = 0$ whenever $I \in \hat{Z}$ and $J \notin \hat{Z}$. If $J \notin \hat{Z}$, there exists an index $h \in M$, $h \notin J$. The matrix $A(I,J)$ of order k contains a zero block with k rows corresponding to indices in M and $k - (|M| - 1)$ columns corresponding to indices in $J \setminus (M \setminus \{h\})$. König's theorem mentioned in the introduction implies then $\det A(I,J) = 0$. Consequently, $A^{(k)}$ is reducible in this case as well.

Since $A + xI$ is reducible in the same manner as A,

$(A + xI)^{(k)}$ is reducible, which means that $A^{(k,s)}$ are reducible for all k, s, $1 \leq s \leq k \leq n-1$. The proof is complete.

R e f e r e n c e s

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(Oblatum 18.4. 1979)