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VALUATIONS OF STRUCTURES

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Abstract: This paper is a contribution to the development of the alternative set theory. A typical special result among those presented is the following: Let $\mathcal{a} = \langle a, f \rangle$ be a set-semigroup and let $\mathcal{a}/Q = \langle Q, f/Q^2 \rangle$ where $Q \subseteq a$ is a π-class be a substructure of $\mathcal{a}$. Then there exists a set-mapping $h: a \rightarrow RN(\geq 0)$ ($RN(\geq 0)$ is the class of non-negative rationals) such that $h(f(x,y)) \leq h(x) + h(y)$ and $h(x) \doteq 0 \equiv x \in Q$ holds for each $x, y \in a$. (As usual, we write $z \doteq 0$ if $|z| < n^{-1}$ for all finite natural numbers n.)

We present more general results; namely, they concern some richer structures than that of a semigroup, deal also with proper classes, and the universe Q of the substructure $\mathcal{a}/Q$ is a σ- or π-class.

As a consequence of our results we obtain a metrization theorem.

Key words: Structure, valuation, σ-class, π-class, metrization.

Classification: 02K10, 02K99, 08A05, 54J05

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§ 0. Introduction. Great numbers of important structures are constructed in the alternative set theory by using π-classes. For example, real numbers are constructed as factor-classes of the π-equivalence $\stackrel{\circ}{=}$ on the class RN of rational numbers. (See [V].) The topological structure is comprehended as a π-equivalence on a set-theoretically definable class. In this paper we study structures which are described by using σ-classes and π-classes only. Let us explain our problems in more detail on the structure $\langle a^2, \sim \rangle$, where a is a set and $\sim$ is a π-equivalence on a. Using some ideas of the proof of the classic metrization lemma, we can prove that there is a set-mapping $h: a^2 \longrightarrow RN(\geq 0)$ ($RN(\geq 0)$ denotes the class of non-negative rationals) such that $h(x,z) \leq h(x,y) + h(y,z)$, $h(x,y) = h(y,x)$, $h(x,y) \doteq 0 \equiv x \sim y$, $h(x,y) = 0 \equiv x = y$ hold. (h is called metric of $\sim$ on a.) We can say that h is a valuation of a^2 in $RN(\geq 0)$ such that h respect (in the sense mentioned above) the following couples of operations: the operation $\circ$ (the composition of pairs) and + ; the operation Cn of converse and the identity mapping Id. Moreover, the values of all elements of $\sim$ are exactly in $[\geq 0] = \{x \in RN(\geq 0); x \doteq 0\}$. We shall describe a class of structures of the type $\langle A,F,E \rangle$, where F is a binary function and E is a unary function, such that the following statement holds: if $\mathcal{A}$ is a set-structure of this class and $\mathcal{A}/Q$ is a substructure of $\mathcal{A}$ with the universe Q, which is a π-class, then the pair $\langle \mathcal{A}, \mathcal{A}/Q \rangle$ is valued in $\langle\langle RN(\geq 0),+,Id \rangle, \langle [\geq 0],+,Id \rangle\rangle$ by a set-mapping similarly as a set-metric of $\sim$ on a values $\langle\langle a^2, \circ, Cn \rangle, \langle \sim, \circ, Cn \rangle\rangle$ in $\langle\langle RN(\geq 0),+,Id \rangle, \langle [\geq 0],+,Id \rangle\rangle$.

Note that we do not work with set-structures only but the structure $\mathcal{A}$ mentioned can be generally a structure from a standard system $\mathfrak{M}$ and the universe Q of the substructure $\mathcal{A}/Q$ can be a $\pi^{\mathfrak{M}}$-or a $\sigma^{\mathfrak{M}}$-class. Then we construct a valuation of the pair $\langle \mathcal{A}, \mathcal{A}/Q \rangle$ as a class of $\mathfrak{M}$.
(For the notions of the standard systems and $\pi^{\mathfrak{M}}$-and $\sigma^{\mathfrak{M}}$-class see [M1].)

Let us mentione one consequence of our general results. Recall that $x \doteq y$ iff for each set-formula $\varphi(z)$ in FL we have

$\varphi(x) \doteq \varphi(y)$. The following statement holds: there is a metric of $\doteq$ on V which is an element of a revealment Sd_V^* of the codable class Sd_V of all set-theoretically definable classes (i.e., roughly speaking, there is a "formally set-theoretically definable" metric of $\doteq$ on V. (For the notion of the revealments see [S-V 1].)

Further results concerning the problems of valuations will be presented in another paper.

§ 1. Preliminaries

1.0.0. We use usual definitions and notions of the alternative set theory and definitions, notions and symbols introduced in [M1]. We shall use results obtained in [M1].

1.0.1. Throughout this paper let $\mathfrak{M}$ denote a standard system.

§ 2. e-structures. Valuations

2.0.0. By a structure we mean a m+n+1-tuple $\mathcal{A} = \langle A, F_i, R_j \rangle_{i \in m, j \in n}$, $m, n \in FN$, where, for each $i \in m$, F_i is a $a(i)$-ary function, $dom(F_i) = A^{a(i)}$, $F_i''A^{a(i)} \subseteq A$, $a(i) \in FN$ and, for each $j \in m$, $R_j \subseteq A^{b(j)}$, $b(j) \in FN$.

We say that a class B A is a universe in $\mathcal{A}$ iff, for each $i \in m$, $F_i''B^{a(i)} \subseteq B$ holds. A substructure of the structure $\mathcal{A}$ is a structure $\langle B, F_i \wedge B^{a(i)}, R_j \cap B^{b(j)} \rangle_{i \in m, j \in n}$ where B is a universe in $\mathcal{A}$. We denote the substructure presented by $\mathcal{A}/B$. If there is no danger of confusion, we write $\langle B, F_i, R_j \rangle$ instead of $\langle B, F_i / B^{a(i)}, R_j \cap B^{b(j)} \rangle_{i \in m, j \in n}$.

2.0.1. A covariant (contravariant resp.) e-structure is a structure $\langle A, F, E \rangle$ where F is a binary function, E is a unary function and the following holds: (1) F is associative on A,

(2) $E \circ E = Id$

(3) $F(E(x),E(y)) = E(F(x,y))$

$(F(E(x),E(y)) = E(F(y,x))$ resp.)

holds for each $x,y \in A$.

An e-structure is a covariant or a contravariant e-structure. An e-structure $\mathcal{A} = \langle A,F,E \rangle$ is a commutative e-structure iff F is commutative on A.
Then $\mathcal{A}$ is covariant and contravariant simultaneously. An e-structure $\langle A,F,Id \rangle$ is covariant. It is contravariant iff it is commutative. Let $\mathcal{A} = \langle A,F,E \rangle$ be an e-structure. We define the binary relation on A as follows:

$$x \lhd_{\mathcal{A}} y \equiv (\exists z \in A)(F(x,z) = y).$$

If there is no danger of confusion, we shall write simply $\lhd$ instead of $\lhd_{\mathcal{A}}$.

Proposition. The relation $\lhd_{\mathcal{A}}$ is transitive on A.

2.0.2. Examples. (1) A structure $\langle A,F \rangle$ is a semigroup iff $\langle A,F,Id \rangle$ is a covariant e-structure.

(2) $\langle N,+,Id \rangle$ is a commutative e-structure.

(3) Let $RN(\geq 0) = \{x \in RN; x \geq 0\}$, $RN(>0) = \{x \in RN; x > 0\}$. $\langle RN(\geq 0),+,Id \rangle$ and $\langle RN(>0),\cdot,^{-1} \rangle$ are commutative e-structures.

(4) We put, for $X \subseteq N$, $X_2 = \{2^{\alpha}; \alpha \in X\}$. $\langle N_2, \cdot ,Id \rangle$ is a commutative e-structure.

(5) Let a be a set, $a \neq 0$. Then $\langle P(a), \cup ,Id \rangle$, $\langle P(a), \cap ,Id \rangle$ are commutative e-structures.

(6) We define the mapping $F^o : (V^2 \cup \{0\})^2 \longrightarrow V^2 \cup \{0\}$ as follows: $F^o(\langle x,y \rangle, \langle u,v \rangle) = \langle x,v \rangle$ (0 resp.) iff $y = u$ $(y \neq u$

resp.) and $F^o(W,0) = F^o(0,W) = 0$ for each $W \in V^2 \cup \{0\}$.

F^o is an associative function on $V^2 \cup \{0\}$ and, consequently, $\langle V^2 \cup \{0\}, F^o, Id\rangle$ is an e-structure, which is not commutative. Let R be a transitive relation. Then $\langle R \cup \{0\}, F^o, Id\rangle$ is an e-structure and the following holds:

$(\forall u \in R \cup \{0\})(u \lhd 0) \& (\forall u \in R \cup \{0\})(0 \lhd u \equiv u = 0)$.

2.0.3. Lemma. Let $\langle A,F,E\rangle$ be an e-structure. Let A_0, A_1 be classes such that $A_0 \subseteq A_1 \subseteq A$ and $[\![F,E]\!](A_0,A_1)$ hold. Let $Q_i = E"A_i \cap A_i$ for $i = 0,1$.

Then $Q_0 \subseteq A_0 \subseteq Q_1 \subseteq A_1$ and, for $i = 0,1$, $F"Q_0^2 \subseteq Q_1$, $E"Q_i \subseteq$ $\subseteq Q_i$.

Proof. The relation $Q_i \subseteq A_i$, $i = 0,1$, is obvious. 1) We prove that $A_0 \subseteq Q_1$. Let $x \in A_0$. We have $E(x) \in A_1$, $x \in A_1$ and $x =$ $= E(E(x))$. Thus $x \in A_1 \cap E"A_1$. 2) We prove that $F"Q_0^2 \subseteq Q_1$, Let $x,y \in Q_0$. Thus $x,y \in A_0$ and $x = E(u)$, $y = E(v)$ hold with some $u,v \in A_0$. We have $F(x,y) \in A_1$, $F(u,v) \in A_1$ and $F(v,u) \in A_1$. Thus $F(x,y) = F(E(u),E(v)) \in E"A_1$ holds. We deduce from this that $F(x,y) \in A_1 \cap E"A_1$. 3) Let us prove that $E"Q_i \subseteq Q_i$ holds for $i = 0,1$. Let $x \in Q_i$. Then $x \in A_i$ and there is a $y \in A_i$ such that $x = E(y)$. Consequently, $E(x) \in A_i \cap E"A_i$ holds.

2.0.4. Let $\mathcal{A}$ be an e-structure. Let Q, B be universes in $\mathcal{A}$. The triple $\langle \mathcal{A}, \mathcal{A}/Q, \mathcal{A}/B\rangle$ is called a triad over $\mathcal{A}$. Let $\mathcal{A}(Q,B)$ denote this triad. A triad of the type $\sigma^{\mathfrak{M}}$ (or a $\sigma^{\mathfrak{M}}$-triad) is a triad $\mathcal{A}(Q,B)$ such that $\mathcal{A} \in \mathfrak{M}$, $B \in \mathfrak{M}$ and Q is a $\sigma^{\mathfrak{M}}$-class, We define a triad of the type $\pi^{\mathfrak{M}}$ (or a $\pi^{\mathfrak{M}}$-triad) analogously.

Examples. (1) $\langle N,+,Id\rangle (FN,\{0\})$, $\langle N_2, \cdot ,Id\rangle (FN_2,\{1\})$ are σ^o-triads.

(2) Let a be a set, $a \neq 0$ and let Q be an ideal on P(a).

Then $\langle P(a), \cup, Id\rangle (Q, \{0\})$ is a triad. Suppose, moreover, that Q is a σ (π resp.)-class. Then the triad presented is a σ-triad (π-triad resp.).

(3) The equivalence $\doteq$ on RN is defined as follows:

$$(\forall x,y \in RN)(x \doteq y \equiv (\forall n)(|x-y| < \frac{1}{n} \vee (x > n \,\&\, y > n) \vee (x < -n \,\&\, y < -n))).$$

We put $[\geq 0] = \{y \in RN(\geq 0); y \doteq 0\}$. Then $\langle RN(\geq 0), +, Id\rangle ([\geq 0], \{0\})$ is a π°-triad.

2.1.0. Let $\mathcal{A} = \langle A, F, E\rangle$, $\tilde{\mathcal{A}} = \langle \tilde{A}, \tilde{F}, \tilde{E}\rangle$ be e-structures. A mapping $H: A \to \tilde{A}$ is called <u>valuation of</u> $\mathcal{A}$ <u>in</u> $\tilde{\mathcal{A}}$ iff for each $x, y \in A$ holds:

$$H(F(x,y)) \lhd_{\tilde{\mathcal{A}}} F(H(x), H(y))$$

$$H(E(x)) = E(H(x)).$$

Let $\mathcal{A}(Q,B)$, $\tilde{\mathcal{A}}(\tilde{Q},\tilde{B})$ be triads. A mapping $H: A \to \tilde{A}$ is called <u>valuation of the triad</u> $\mathcal{A}(Q,B)$ <u>in the triad</u> $\tilde{\mathcal{A}}(\tilde{Q},\tilde{B})$ iff H is a valuation of $\mathcal{A}$ in $\tilde{\mathcal{A}}$ and we have for each $x \in A$:

$$x \in Q \equiv H(x) \in \tilde{Q}, \quad x \in B \equiv H(x) \in \tilde{B}.$$

<u>Example</u>. The mapping $H: N \to N_2$ sending α to 2^{α} is a valuation of $\langle N, +, Id\rangle (FN, \{0\})$ in $\langle N_2, \cdot, Id\rangle (FN_2, \{1\})$.

<u>Proposition</u>. Let $\mathcal{A}$ be an e-structure and let $\lhd_{\mathcal{A}}$ be reflexive on A. Let $\mathcal{A}(Q,B)$ be a triad over $\mathcal{A}$ and let $A' \subseteq A$ be an universe in $\mathcal{A}$.

(1) $\mathcal{A}/A'(Q \cap A', B \cap A')$ is a triad over $\mathcal{A}/A'$.

(2) Identity mapping Id is a valuation of $\mathcal{A}/A'(Q \cap A', B \cap A')$ in $\mathcal{A}(Q,B)$.

Proof. (1) follows from the fact that $Q \cap A'$ and $B \cap A'$ are universes in $\mathcal{A}/A'$. (2) Identity mapping is a valuation of $\mathcal{A}/A'$ in $\mathcal{A}$ (by using of the reflexivity of $\lhd_{\mathcal{A}}$).

<u>Proposition</u>. Let $\tilde{\mathcal{A}} = \langle \tilde{A}, \tilde{F}, \tilde{E}\rangle$ be a commutative e-structure and let $\tilde{\mathcal{A}}(\tilde{Q},\tilde{B})$ be a triad. Suppose that there exist

points a, q, $b\in\tilde{A}$ such that $b \lhd q \lhd a$ and $b\in\tilde{B}$, $q\in\tilde{Q}-\tilde{B}$, $a\in$ $\in\tilde{A}-\tilde{Q}$.

Then, for each triad $\mathcal{T}$, there is a valuation of $\mathcal{T}$ in $\tilde{\mathcal{a}}(\tilde{Q},\tilde{B})$.

Proof. Let H be a mapping, defined as follows: $H(x) = b \equiv x\in B$, $H(x) = q \equiv q\in Q-B$, $H(x) = a \equiv x\in A-Q$, where $\langle A,F,E\rangle (Q,B) = \mathcal{T}$. The H is the required valuation.

§ 3. Valuation lemmas

3.0.0. We shall prove two lemmas which have the important role for the construction of valuations of $\sigma^{\mathfrak{M}}$-triads and $\pi^{\mathfrak{M}}$-triads. At first, we introduce the following definition: let $\mathcal{a} = \langle A,F,E\rangle$ be an e-structure and let B be an universe in $\mathcal{a}$. A σ-string (π-string resp.) R is called σ (π resp.)-string in $\mathcal{a}$ over B iff $B = R(0)$, $A = R(dom(R)-1)$ and $[\![F,F_3]\!](R(\alpha),R(\alpha+1))$, $E"R(\alpha)\subseteq R(\alpha)$ holds for each $\alpha\in$ $\in dom(R)-1$ ($A = R(0)$, $B = R(dom(R)-1)$ and $[\![F,F_3]\!](R(\alpha+1), R(\alpha))$, $E"R(\alpha)\subseteq R(\alpha)$ holds for each $\alpha\in dom(R)-1$ resp.), where $F_3:A^3 \longrightarrow A$ is the function satisfying $F_3(x,y,z) =$ $= F(F(x,y),z)$.

3.0.1. σ-valuation lemma. The following holds in the sense of $\mathfrak{M}$: Let $\mathcal{a}$ be an e-structure and let B be an universe in $\mathcal{a}$. Let Q be a σ-string in $\mathcal{a}$ over B and let $\xi+1 =$ $= dom(Q)$.

Then there is a valuation H of the triad $\mathcal{a}(B,B)$ in $\langle N,+,Id\rangle(\{0\},\{0\})$ such that $Q(\alpha)\subseteq\{x\in A;H(x)\leq 2^{\alpha}\}\subseteq$ $\subseteq Q(\alpha+1)$ holds for each $\alpha\in\xi$.

π-valuation lemma. The following holds in the sense of $\mathfrak{M}$: Let $\mathcal{a}$ be an e-structure and let B be an universe in $\mathcal{a}$.

Let Q be a $\mathcal{T}$-string in $\mathcal{A}$ over B and let $\xi+1 = \mathrm{dom}(Q)$.

Then there is a valuation H of the triad $\mathcal{A}(B,B)$ in $\langle RN(\geq 0),+,Id\rangle(\{0\},\{0\})$ such that $Q(\alpha+1)\subseteq\{x\in A;\ H(x)\leq 2^{-(\alpha+1)}\}\subseteq Q(\alpha)$ holds for each $\alpha\in\xi$.

The $\mathcal{T}$-valuation lemma follows from the σ-valuation lemma. Really, let G be a valuation of $\mathcal{A}(B,B)$ in $\langle N,+,Id\rangle(\{0\},\{0\})$ such that $Q(\xi-\alpha)\subseteq\{x\in A;\ G(x)\leq 2^{\alpha}\}\subseteq Q_{(\xi-(\alpha+1))}$ holds for each $\alpha\in\xi$. We put $\beta=\xi-\alpha$. Thus, $Q(\beta)\subseteq\{x\in A;\ G(x)\leq 2^{\xi-\beta}\}\subseteq Q(\beta-1)$ holds for each $1\leq\beta\leq\xi$. The required valuation is the mapping $H = 2^{-\xi}.G$.

3.0.2. The proof of the σ-valuation lemma.

I. A path in A is a function t such that $\mathrm{dom}(t)\in N$ and $\mathrm{rng}(t)\subseteq A$. We construct the function $[F]$ with domain

$\bigcup\{\{t\}\times\{\langle\alpha,\beta\rangle;\ \alpha\leq\beta\ \&\ \beta\in\mathrm{dom}(t)\};\ t \text{ is a path in } A\}$

by induction over N:

$[F](t,\langle\alpha,\alpha\rangle) = t(\alpha)$

$[F](t,\langle\alpha,\beta+1\rangle) = F([F](t,\langle\alpha,\beta\rangle),t(\beta+1))$.

We shall write more simply $[F](t,\alpha,\beta)$ instead of $[F](t,\langle\alpha,\beta\rangle)$.

<u>Lemma 1</u>. Let t be a path in A, $\alpha\leq\gamma+1\leq\beta\in\mathrm{dom}(t)$. Then

$[F](t,\alpha,\beta) = F([F](t,\alpha,\gamma),[F](t,\gamma+1,\beta))$

holds.

This follows by induction on $\beta-\alpha$.

Let t be a path in A, $\mathrm{dom}(t) = \vartheta+1$. We define the path $\bar{t}$ with $\mathrm{dom}(\bar{t}) = \vartheta+1$ as follows: $\bar{t}(\alpha) = t(\vartheta-\alpha)$. $\tilde{F}:A^2\to A$ is the function so that $\tilde{F}(x,y) = F(y,x)$ holds for each $x,y\in A$. $[\tilde{F}]$ is defined similarly as $[F]$.

The following lemma can be proved by induction on $\beta-\alpha$.

Lemma 2. Let t be a path in A, $dom(t) = \vartheta + 1$. Then

$$[F](t,\alpha,\beta) = [\breve{F}](t, \vartheta-\beta, \vartheta-\alpha)$$

holds for each $\alpha \leq \beta \leq \vartheta$.

II. We put for each $x \in A$: $G_Q(x) = \min\{\alpha \leq \xi; x \in Q(\alpha)\}$. Thus, G_Q is a function, $G_Q: A \to N$, and we have $G_Q(x) \leq \alpha \equiv x \in Q(\alpha)$, $\alpha < G_Q(x) \equiv x \notin Q(\alpha)$ for each $\alpha \leq \xi$. We shall write more simply G instead of G_Q. The index Q denotes only that G_Q is constructed from Q and this notion will be used in 3.0.3.

We define the function G^*, $G^*: A \to N$, as follows: $G^*(x) = 0$ iff $x \in B$, $G^*(x) = 2^{G(x)}$ iff $x \in A-B$.

Let t be a path in A. We put

$$\mathcal{V}_Q(t) = \sum \{G^*(x); x \in rng(t)\}.$$

We shall write more simply $\mathcal{V}$ instead of $\mathcal{V}_Q$. $\mathcal{V}$ is a function, $rng(\mathcal{V}) \subseteq N$.

We deduce from the definition of $\mathcal{V}$ that $\mathcal{V}(t) = 0 \equiv rng(t) \subseteq B$ and $\mathcal{V}(t) = 0 \to (\forall \alpha, \beta \in dom(t))(\alpha \leq \beta \to [F](t,\alpha,\beta) \in B)$.

Let t be a path in A, $dom(t) = \sigma + 1$. Writing $[F](t)$ ($[\tilde{F}](t)$ resp.) we mean $[F](t,0,\sigma)$ ($[\tilde{F}](t,0,\sigma)$ resp.). Note that whenever $[F](t,\alpha,\beta)$ appears, then we assume that $\langle t, \langle \alpha, \beta \rangle \rangle$ is an element of $dom([F])$. We use the similar convention for the terms $[F](t)$, $[\tilde{F}](t,\alpha,\beta)$, $[\tilde{F}](t)$.

Lemma 3. Let $z \in A$ and suppose that $[F](t) = z$. Then

$$(*) \qquad \mathcal{V}(t) \neq 0 \to 2^{G(z)} \leq 2 \cdot \mathcal{V}(t)$$

holds.

Proof. By induction on dom(t).

(i) Suppose that $dom(t) = 2$. Assume, for example that $G(t(0)) \leq G(t(1))$. Thus $G(z) \leq G(t(1)+1$ holds and we have $2^{G(z)} \leq 2 \cdot 2^{G(t(1))}$. If $t(1) \in B$ then $G(t(1)) = 0$ and, consequently, $G(t(0)) = 0$. We deduce from this that $t(0) \in B$, which is a contradiction. Thus, $t(1) \notin B$ holds and we have $2 \cdot 2^{G(t(1))} \leq 2 \cdot (G^*(t(0)) + 2^{G(t(1))}) = 2 \cdot \vartheta(t)$.

(ii) Suppose that the statement (*) holds whenever $\mathrm{dom}(t) \leq \beta + 1$ and $\beta + 1 \geq 3$ is fixed. Let t be a path in A and let $\mathrm{dom}(t) = \beta + 2$. Let $[F](t) = z$ and assume that $\vartheta(t) \neq 0$. We shall prove that $2^{G(z)} \leq 2 \cdot \vartheta(t)$ holds.

We put $c = \vartheta(t)$. Let σ be the maximal natural number such that $2^{\sigma} \leq c$. If $\sigma \geq \xi - 1$ then $2^{G(z)} \leq 2^{\xi} \leq 2^{\sigma+1} \leq 2 \cdot 2^{\sigma} \leq$ $\leq 2 \cdot c$ and, consequently, the statement in question is proved. Assume $\sigma < \xi - 1$.

(α) Suppose that $G^*(t(0)) \leq \frac{c}{2}$. Let $\gamma \in N$ be a maximal number such that

$$\vartheta(t \wedge \gamma + 1) = \sum_{\alpha=0}^{\gamma} G^*(t(\alpha)) \leq \frac{c}{2} .$$

Obviously, $0 \leq \gamma \leq \beta$. Moreover, $0 \neq G^*(t(\gamma + 1)) \leq c$ and $\sum_{\alpha=\gamma+2}^{\beta+1} G^*(t(\alpha)) \leq \frac{c}{2}$. We put $z_1 = [F](t, 0, \gamma)$, $z_3 = [F](t, \gamma + 2, \beta + 1)$.

Suppose that $\sum_{\alpha=0}^{\gamma} G^*(t(\alpha)) \neq 0$. We deduce from the induction hypothesis that $2^{G(z)} \leq 2 \cdot \frac{c}{2} = c$. Thus, the following relation holds:

(*) $\qquad G(z_1) \leq \sigma$. It is easy that

(**) $\qquad G(t(\gamma+1)) \leq \sigma$. We deduce as above that

(***) $\qquad G(z_3) \leq \sigma$

follows from $\sum_{\alpha=\gamma+2}^{\beta+1} G^*(t(\alpha)) \neq 0$.

The relations (*), (**), (***) hold too in the case if $\sum_{\alpha=0}^{\gamma} G^*(t(\alpha)) = 0$ or $\sum_{\alpha=\gamma+2}^{\beta+1} G^*(t(\alpha)) = 0$. We have $z = [F](t) =$ $= F(F(z_1, t(\gamma+1)), z_3) = F_3(z_1, t(\gamma+1), z_3)$ and $F_3" Q^3(\sigma) \subseteq Q(\sigma+1)$. We deduce from this that $z \in Q(\sigma + 1)$. Consequently, $G(z) \leq \sigma + 1$

holds, and

$$2^{G(z)} \le 2^{\delta+1} = 2\cdot 2^{\delta} \le 2\ c = 2\cdot \mathcal{V}(t)$$

follows immediately.

(β) Suppose that $G^*(t(0)) > \frac{c}{2}$. Then $G^*(t(\beta+1)) \le \frac{c}{2}$. Thus, $G^*(\bar{t}(o)) = G^*(t(\beta+1)) \le \frac{c}{2}$ holds. We have $[\tilde{F}](t) = z = [F](t)$ (by using the lemma 2). We deduce similarly as in the case (α) that $2^{G(z)} \le 2\cdot c$ holds.

III. The following definition of the function $H: A \to N$ is justified:

$$H(x) = \min \{\mathcal{V}(t);\ [F](t) = x\}.$$

We shall prove that H is the valuation in question.

(a) $H(x) = 0 \equiv x \in B$. Suppose that $H(x) = 0$. Then there exists a path t in A such that $H(x) = \mathcal{V}(t)$ and $[F](t) = x$. Thus, $x \in B$ holds. Suppose that $x \in B$. We have $G^*(x) = 0$ and $H(x) = 0$ follows from the relation $H(x) \le \mathcal{V}(\{\langle x,0\rangle\}) = G^*(x) = 0$.

(b) $Q(\alpha) \subseteq \{x \in A;\ H(x) \le 2^{\alpha}\} \subseteq Q(\alpha+1)$ holds for each $\alpha \in \xi$. At first, we prove that

($\times\times$) $x \in A-B \to 2^{-1}\cdot 2^{G(x)} \le H(x) \le 2^{G(x)}$ holds.

Proof. Let t be a path in A such that $[F](t) = x$ and $\mathcal{V}(t) = H(x)$. We have $\mathcal{V}(t) \neq 0$ and, consequently, $2^{-1}\cdot 2^{G(x)} \le \mathcal{V}(t) \le H(x)$. The statement ($\times\times$) follows from this and from the relation $H(x) \le \mathcal{V}(\{\langle x,0\rangle\}) = G^*(x) = 2^{G(x)}$. We are proving (b). Let $x \in A$ be such that $H(x) \le 2^{\alpha}$ and $x \in B$. We have $2^{G(x)-1} \le H(x) \le 2^{\alpha}$ and, consequently $x \in Q(\alpha+1)$ holds. Conversely, let $x \in Q(\alpha)-B$. We have $G(x) \le \alpha$. We deduce from this that $H(x) \le 2^{G(x)} \le 2^{\alpha}$.

(c) $H(F(x,y)) \le H(x) + H(y)$ holds for each $x,y \in A$. This follows immediately from the construction of H.

(d) $H(E(x)) = H(x)$ holds for each $x \in A$.

We shall prove (d) by using the following lemma.

Lemma 5. Let t be a path in A, $dom(t) = \vartheta + 1$, and let $\alpha \le \beta \le \vartheta$. (1) $\mathcal{V}(E \circ t) \le \vartheta(t)$.

(2) If $\mathcal{A}$ is covariant then $[F](E \circ t, \alpha, \beta) = E([F](t, \alpha, \beta))$.

(3) If $\mathcal{A}$ is contravariant then $[F](E \circ \bar{t}, \alpha, \beta) =$
$= E([F](t, \vartheta - \beta, \vartheta - \alpha))$.

The proof of this lemma is straghtforward and we omit it.

We prove that

$$(\square) \qquad H(y) \le H(E(y))$$

holds for each $y \in A$. Suppose that $E(y) = x$. Let t be a path in A such that $[F](t) = x$ and $\mathcal{V}(t) = H(x)$. Assume covariant $\mathcal{A}$. Then $[F](E \circ t) = E([F](t)) = E(x) = y$. Assume contravariant $\mathcal{A}$. Then $[F](E \circ \bar{t}) = E([F](t)) = E(x) = y$. We have $\mathcal{V}(E \circ \bar{t}) \le \mathcal{V}(E \circ t) \le \mathcal{V}(t) = H(x)$ and, consequently, ($\square$) is proved. We deduce from ($\square$) that

$$H(y) \le H(E(y)) \le H(E(E(y))) = H(y).$$

Thus, the statement (d) is proved. The proof of the σ-valuation lemma is finished.

3.0.3. Remark. (1) The valuation H from the previous proof is defined as follows: $\langle x,y \rangle \in H \equiv y \in A \;\&\; x = \min\{\mathcal{V}_Q(t); [F](t) = x\}$. Thus, there is a normal formula $\Phi'(x,y,X,Y)$ of the language FL such that

$$\langle x,y \rangle \in H \equiv \Phi'(x, y, \mathcal{A}, \mathcal{V}_Q).$$

The function $\mathcal{V}_Q$ is constructed by a normal formula again. We deduce from this that there exists a normal formula $\Phi(x,y,X,Y)$ of the language FL, satisfying

$$\langle x,y \rangle \in H \equiv \Phi(x, y, \mathcal{A}, Q).$$

(2) Let Q, R be σ-artings in $\mathcal{A}$ over B, where B is an universe in an e-structure $\mathcal{A} = \langle A.F.E \rangle$. Let $dom(Q) = dom(R)$

and suppose that $Q(\alpha) \subseteq R(\alpha)$ holds for each $\alpha \in \mathrm{dom}(Q)$. We put

$H^Q = \{\langle x,y\rangle;\ \Phi(x,y,\mathcal{a},Q)\}$, $H^R = \{\langle x,y\rangle;\ \Phi(x,y,\mathcal{a},R)\}$.

Then $H^R(x) \leq H^Q(x)$ holds for each $x \in A$.

Proof. Let x be an element of A. Then $G_R(x) \leq G_Q(x)$. (For G_Q see the previous proof.) We deduce from this that $\mathcal{V}_R(t) \leq \mathcal{V}_Q(t)$ for each path t in A. The required proposition follows from this immediately.

§ 4. Scales for $\sigma^{\mathfrak{M}}$-triads and $\pi^{\mathfrak{M}}$-triads

4.0.0. A triad $\mathcal{T}$ is called scale for the type $\sigma^{\mathfrak{M}}$ ($\pi^{\mathfrak{M}}$ resp.) iff $\mathcal{T}$ is a σ^{o} (π^{o} resp.)-triad and, for each triad $\widetilde{\mathcal{T}}$ of the type $\sigma^{\mathfrak{M}}$ ($\pi^{\mathfrak{M}}$ resp.), there exists a valuation H of $\widetilde{\mathcal{T}}$ in $\mathcal{T}$ such that $H \in \mathfrak{M}$.

4.0.1. Theorem

(1) The triad $\langle N,+,Id\rangle$ $(FN,\{0\})$ is a scale for the type $\sigma^{\mathfrak{M}}$.

(2) The triad $\langle RN(\geq 0),+,Id\rangle$ $([\geq 0],\{0\})$ is a scale for the type $\pi^{\mathfrak{M}}$.

Proof. Let $\mathcal{a} = \langle A,F,E\rangle$ be an e-structure and let $\mathcal{a}(Q,B)$ be a $\sigma^{\mathfrak{M}}$-triad over $\mathcal{a}$. We have $[\![F,E]\!](Q,Q)$. Thus, there is a σ-string S of Q, $S \in \mathfrak{M}$, and $B \subseteq S(0) \subseteq S(\alpha) \subseteq A$, $[\![F,E]\!](S(\alpha),S(\alpha+1))$ holds for each $\alpha+1 \in \mathrm{dom}(S)$. (This follows from [M1] 2.1.0). Put, for each $\alpha \in \mathrm{dom}(S)$,

$$\langle x,\alpha\rangle \in P \equiv x \in S(\alpha) \cap E"S(\alpha)$$

We deduce from 2.0.3 that P is a σ-string of Q and $B \subseteq P(0) \subseteq P(\alpha) \subseteq A$, $F"P^2(\alpha) \subseteq P(\alpha+1)$, $E"P(\alpha) \subseteq P(\alpha)$ hold for each $\alpha+1 \in \mathrm{dom}(P)$. Evidently, P is an element of $\mathfrak{M}$. Let $\delta \in N-FN$ be such that $2\delta < \mathrm{dom}(P)$. Let R be a relation, satis-

fying: $\mathrm{dom}(R) = \delta+1$, $R''\{0\} = B$, $R''\{\delta\} = A$, $1 \leq \alpha < \delta \rightarrow$
$\rightarrow R''\{\alpha\} = P(2\alpha)$. It is easy that $R \in \mathfrak{M}$ and R is a σ-string of Q. Moreover, R is a σ-string in $\mathcal{A}$ over B. We deduce from the σ-valuation lemma that there is a valuation $H \in \mathfrak{M}$ of $\mathcal{A}(B,B)$ in $\langle N,+,Id\rangle (\{0\},\{0\})$ and $x \in Q \equiv (\exists n)$ $(H(x) \leq 2^n)$ holds. Consequently, H is a valuation of $\mathcal{A}(Q,B)$ in $\langle N,+,Id\rangle (FN,\{0\})$ and the part (1) of the theorem is proved. The part (2) can be proved quite analogously as the part (1).

4.0.2. Remark. Let $\mathcal{A}(Q,B)$ be a triad and suppose that $\mathcal{A} \in Sd_V$, $B \in Sd_V$. Assume that Q is a σ-class which is not a σ^0-class. Then there exists a valuation H of $\mathcal{A}(Q,B)$ in $\langle N,+,Id\rangle (FN,\{0\})$ and $H \in Sd_V^*$. But no valuation of $\mathcal{A}(Q,B)$ in $\langle N,+,Id\rangle (FN,\{0\})$ is an element of Sd_V.

Proof. The existence of a valuation, which is a Sd_V^*-class, follows from the previous theorem (because $\mathcal{A}(Q,B)$ is a $\sigma^{Sd_V^*}$-triad).

Suppose that there is a valuation of $\mathcal{A}(Q,B)$ in $\langle N,+,Id\rangle (FN,\{0\})$ and let $H \in Sd_V$. Let $\xi \in N-FN$. Then $R =$ $= \{\langle x,\alpha\rangle ;\ H(x) < \alpha \ \&\ \alpha \in \xi\}$ is a σ-string of Q and $R \in Sd_V$. Thus Q is a σ^0-class, which is a contradiction.

4.1.0. Let Q be an equivalence on a class A. The mapping $H: A^2 \rightarrow RN(\geq 0)$ is called metric of Q on A iff the following holds for each $x,y,z \in A$:

$H(x,z) \leq H(x,y) + H(y,z)$, $H(x,y) = H(y,x)$, $H(x,y) \doteq 0 \equiv \langle x,y\rangle \in Q$, $H(x,y) = 0 \equiv x = y$.

Metrization theorem. Let Q be an equivalence on A, $A \in \mathfrak{M}$, and let Q be a $\sigma^{\mathfrak{M}}$-class. Then there exists a metric H of Q on A, $H \in \mathfrak{M}$.

Proof. Let $E^o: V^2 \cup \{0\} \to V^2 \cup \{0\}$ be the mapping defined as follows: $E^o(\langle x,y\rangle) = \langle y,x\rangle$, $E^o(0) = 0$. Then $\mathcal{A} = \langle A^2 \cup \{0\}, F^o, E^o\rangle$ is a contravariant e-structure and $\mathcal{T} = \mathcal{A}(Q \cup \{0\}, \{\langle x,x\rangle; x \in A\} \cup \{0\})$ is a $\pi^{\mathfrak{M}}$-triad. Let $G \in \mathfrak{M}$ be a valuation of $\mathcal{T}$ in $\langle RN(\geq 0),+,Id\rangle$ $([\geq 0],\{0\})$. A metric in question is the mapping $G \restriction A^2$.

Corollary. (1) There exists a metric H of $\stackrel{\circ}{=}$ on V so that $H \in Sd_V^*$.

(2) There is no metric of $\stackrel{\circ}{=}$ on V which is an element of Sd_V.

Proof. (1) follows from the metrization theorem. (2) follows from [M1], 1.0.7 and from 4.0.2. (For the equivalence $\stackrel{\circ}{=}$ see also § 0.)

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