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ON RAMSEY GRAPHS WITHOUT CYCLES OF SHORT ODD LENGTHS

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Abstract: The Ramsey problem for classes of graphs without short odd cycles is solved. The proof uses category theoretical means.

Key words: Graph, Ramsey theorem, cycle.

Classification: 05C15, 05A17

Introduction. The existence of highly chromatic graphs without cycles of lengths $\leq p$ is a classic result of the graph theory. The problem started with Tutte-Zykov in the 40's and it ended in the 60's with a non-constructive proof of Erdős [2]and with a construction of Lovász [4] ; see [11] for the history of this subject. However, the structure of these graphs is far from being clear, in particular the Lovász construction based on the original idea of Tutte has so far been the only known one. No wonder that the connection of this topic with the recent progress in the Ramsey field leads to challenging questions (the first of them was formulated by F. Galvin and solved in [9] by the authors). It is the purpose of this paper to pursue the study of this problem and to prove that for every graph without cycles of short odd lengths there exists a Ramsey graph with the same property. In fact we characterize all the partition properties of classes of graphs without cycles of short odd lengths . Surprisingly, this depends on the lengths of the forbidden cycles . The method of proof generalizes the methods given in [7],[9], we prove [9] again. Let us remark that in the proof we do not use any "artificial" construction; it suffices to use the calculus outlined in [7]. We built our graphs by induction mainly by use of direct products (sometimes called also cardinal products, the Kronecker products, conjunctions). We may define the notion of an <u>ideal class</u> $\mathcal{K}$ of graphs. This is such a class of graphs which is closed on the formation of subgraphs and which satisfies the following: $G \in \mathcal{K}$ implies the direct product $G \times H \in \mathcal{K}$ for any graph H. The fact that graphs without short odd cycles form an ideal class is the main property used. We wish to thank the referee for many valuable comments, which improved the formal arrangement of this paper.

<u>Preliminaries and statement of results</u>. All graphs are supposed to be finite, undirected, without loops and multiple edges.

Let G, H be graphs, $G=(V,E)$, $H=(W,F)$ we write also $V=V(G)$, $E=E(G)$. A mapping $f:V \longrightarrow W$ is said to be an embedding (of G into H) if f is 1-1 and $[x,y] \in E \Longleftrightarrow [f(x),f(y)] \in F$. If $V \subseteq W$ and the inclusion is an embedding, then G is said to be a subgraph of H. Denote by $\binom{H}{G}$ the set of all subgraphs of H which are isomorphic to G (this notation is due to K. Leeb).

Thus $\binom{H}{G} \neq \emptyset$ if and only if there exists an embedding $G \longrightarrow H$. The Ramsey theory deals with the following notions and properties: We write $G \xrightarrow[k]{F} H$ (k is a positive integer, F, G, H are graphs) if for every mapping (called a colouring) c: $:\binom{H}{F} \rightarrow [1,k]$ there exists $G' \in \binom{H}{G}$ such that $\binom{G'}{F} \subseteq c^{-1}(i)$ for an $i \in [1,k]$ (here $[1,k] = \{1,2,\dots,k\}$). The negation of this statement will be denoted by $G \not\xrightarrow[k]{F} H$. In the particular case of $F=(\{1\},\emptyset)$ - the vertex ($F=(\{1,2\},\{[1,2]\})$ - the edge, respectively) we write $G \xrightarrow[k]{v} H$ ($G \xrightarrow[k]{e} H$, respectively). Denote by Gra the class of all finite graphs. Let $\mathcal{K}$ be a subclass of Gra, $F \in \mathcal{K}$. We say that $\mathcal{K}$ has the <u>F-partition property</u> if for every $G \in \mathcal{K}$ and every positive integer k there exists $H \in \mathcal{K}$ such that $G \xrightarrow[k]{F} H$.

For a fixed class $\mathcal{K}$ the basic question we are concerned with is the explicit description of the set of all F for which $\mathcal{K}$ has the F-partition property, see [7] - "the prototype theorem". For a positive integer p denote by Cyc(p) the class of all finite graphs without cycles of the lengths 3,5,...,2p+1. Put Cyc(0)=Gra.

We prove:

<u>Theorem A</u>. For every natural number p the class Cyc(p) has the edge-partition property.

Moreover, the method of the proof will be sufficiently strong to prove:

<u>Theorem B</u>. For every non-negative integer p and every graph $F \in$ Cyc(p) the following two statements are equivalent:

1) Cyc(p) has the F-partition property;

2) F is either a single vertex or a single edge; in the case

of p=1 F may be also any discrete graph (i.e. $F=(V,\emptyset)$ and in the case of p=0 F may be any discrete or complete graph.

It was proved earlier [8] that Cyc(0)=Gra has the F-partition property if and only if F is a complete or a discrete graph. In [7] it was proved that Cyc(1) has the F-partition property if and only if F is an edge or a discrete graph. The difference between the partition properties of the classes Cyc(1) and Cyc(p) for $p > 1$ is somewhat surprising.

In the proofs of Theorems A and B it is more convenient to work with ordered graphs - graphs with linearly ordered sets of vertices. This modification is essential for the proofs below.

In § 1 we give the basic definitions related to ordered graphs and we state Theorem C, which is the analogy of ordered graphs to Theorem A. In § 1 we also give schemes of the proofs of Theorems B and C.

In § 2 we prove Theorem C, which implies Theorem A.

In § 3 we prove Theorem B.

Related statements and problems are given in the concluding remarks.

§ 1. Ordered graphs - Theorem C. The basis of the proofs of Theorem A and Theorem B is the following strengthening of Theorem A stated below as Theorem C. In order to state Theorem C we have to introduce a few notions first.

An ordered graph is a graph with a fixed linear ordering of its vertices. This ordering will be always denoted by $\leq$ and called the standard ordering. An ordered graph will be denoted by $((V,E),\leq)$ or $(G,\leq)$ or simply G.

Given two ordered graphs G and H an (ordered) <u>embedding</u> $f: G \longrightarrow H$ is an embedding of the corresponding graphs, which is also a monotone mapping with respect to the standard orderings. Explicitly: $f:((V,E),\leq) \longrightarrow ((W,F),\leq)$ is an embedding if

1. $f: V \longrightarrow W$ is 1-1;
2. $[x,y] \in E$ iff $[f(x),f(y)] \in F$;
3. $x \leq y$ iff $f(x) \leq f(y)$.

If G is a subgraph of H and if moreover the standard ordering of H restricted to the set V(G) coincides with the standard ordering of G, then G is called an (ordered) <u>subgraph</u> of H.

The set of all ordered subgraphs of $(H,\leq)$ which are ordered isomorphic to $(G,\leq)$ will be denoted by $\binom{(H,\leq)}{(G,\leq)}$, or simply by $\binom{H}{G}$.

For the ordered graphs $(G,\leq)$ and $(H,\leq)$ and for a positive integer k we write $(G,\leq) \xrightarrow[k]{e} (H,\leq)$ $((G,\leq) \xrightarrow[k]{v} (H,\leq)$, respectively, if for every colouring $c: E(H) \longrightarrow [1,k]$ $(c: V(H) \longrightarrow [1,k]$, respectively) there exists an ordered subgraph $(G',\leq)$ of $(H,\leq)$, $(G',\leq) \simeq (G,\leq)$ such that c restricted to the set $E(G')$ $(V(G')$, respectively) is a constant mapping.

Let $\overrightarrow{\mathcal{K}}$ be a class of ordered graphs. We say that $\overrightarrow{\mathcal{K}}$ has the <u>edge-partition property</u> (the <u>vertex-partition property</u>, respectively) if for every $(G,\leq) \in \overrightarrow{\mathcal{K}}$ and for every positive integer k there exists $(H,\leq) \in \overrightarrow{\mathcal{K}}$ such that $(G,\leq) \xrightarrow[k]{e} (H,\leq)$ $((G,\leq) \xrightarrow[k]{v} (H,\leq)$, respectively).

Let $\overrightarrow{\mathrm{Gra}}$ denote the class of all finite ordered graphs.

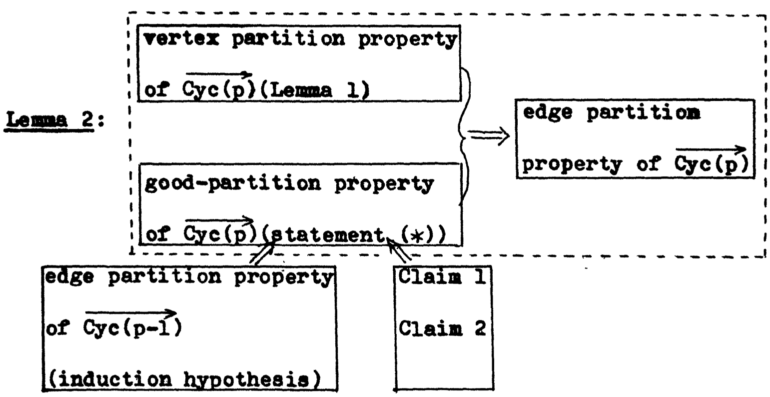
Let $\overrightarrow{\mathrm{Cyc}(p)}$ denote the class of all ordered graphs $(G,\leq)$,

$G \in Cyc(p)$ (p a non-negative integer).

Now we can state:

<u>Theorem C</u>. The class $\overrightarrow{Cyc(p)}$ has the edge-partition property.

Clearly Theorem C implies Theorem A, as every ordered embedding is an embedding. Consequently we shall prove Theorem C only. This will be done in § 2. The proof of Theorem C is complex, using several auxiliar results and constructions and proceeding in many steps. The following is a scheme of the proof of Theorem C:



There are at least three reasons for considering ordered graphs (rather than graphs without ordered sets of vertices). The first reason is the following property of the direct products of ordered graphs:

Let $G_i=(V_i,E_i)$, $(G_i,\leq_i)$ $i\in[1,n]$ be ordered graphs.

The graph $(\times_{i=1}^{m} G_i, \leq)=((V,E),\leq)$ is called an (ordered) product of graphs $(G_i,\leq_i)$ if $V=\times_{i=1}^{m} V_i$,

$[(x_i;i\in[1,n]),(y_i;i\in[1,n])]\in E$ iff for every $i\in[1,n]$

$$[x_i, y_i] \in E_i \text{ and } x_i \leq_i y_i$$

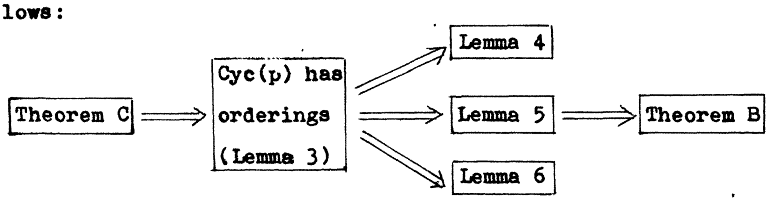
and the standard ordering $\leq$ is the lexicographic ordering induced by the ordering $\leq_i$, $i \in [1,n]$.
The crucial (and convenient) property of the ordered product is that every edge $e = [x,y] \in E$ is uniquely determined by its projections $\pi_i(e) = [\pi_i(x), \pi_i(y)] \in E_i$ (here $\pi_i : \times_{i=1}^{n} V_i \longrightarrow V_i$ is the projection).
As every edge $e \in E$ is uniquely determined by its projections, we shall write sometimes $e = (\pi_1(e), \ldots, \pi_n(e))$.

Remark: The above statements are not valid for the direct product of graphs without orderings. We have $K_2 \times K_2 = ([1,2] \times [1,2], \{[(1,1), (2,2)], [(2,1), (1,2)]\}$ and consequently two edges which have the same projections.

The second reason for considering ordered graphs is that for the classes $\overrightarrow{Cyc(p)}$ the products may be conveniently used. The following is true: $G \in \overrightarrow{Cyc(p)}$, $H \in \overrightarrow{Gra} \Longrightarrow G \times H \in \overrightarrow{Cyc(p)}$. Consequently, classes Cyc(p) are ideal classes of graphs in the sense of [7]. The last reason for considering ordered graphs is the fact that we may use Theorem C for the proof of Theorem B. The following notion is the key to Theorem B:

We say that a class $\mathcal{K}$ of graphs has orderings if for every graph $G=(X,E) \in \mathcal{K}$ there exists a graph $H=(Y,F) \in \mathcal{K}$ such that for every (linear) ordering $\leq$ of X and for every ordering $\preceq$ of Y there exists an embedding $f: G \longrightarrow H$, which is also a monotone mapping $f: (X, \leq) \longrightarrow (Y, \preceq)$. The validity of this statement for some particular G and H will be denoted by $G \xrightarrow{ord} H$. We prove that all classes Cyc(p) have orderings (§ 3, Lemma 3 below). This generalizes a result of O. Ore, who proved that there exists a graph H without triangles which contains for every ordering of its vertices a subgraph isomorphic to $C_4=([1,4],\{[1,2],[2,3],[3,4],[1,4]\})$ with naturally ordered vertices. (This is slightly weaker than $C_4 \xrightarrow{\text{ord}} H$.)

Using Theorem C we prove that Cyc(p) has orderings for every non-negative integer p and from it we derive Theorem B. A scheme of this proof (given in § 3) may be sketched as follows:



The whole paper is written in the finite set theory. For a set V let K_V be the complete graph with the vertex set V. K_n is the complete graph with the vertex set [1,n].

§ 2. Proof of Theorem C. We shall use the following weakening of the partition property of ordered graphs:

We write $G \xrightarrow[k]{\text{good},e} H$ for the ordered graphs G, H and a positive integer k if for every colouring $c: E(H) \longrightarrow [1,k]$ there exists $G'=(V',E') \in \binom{H}{G}$ (i.e. an ordered subgraph of H isomorphic to G) such that $c([y,x])=c([y',x])$ whenever $[y,x] \in E'$, $[y',x] \in E'$, $y<x$, $y'<x$ (i.e. the colour of an edge of G' depends on the "greater" vertex only).

The usefulness of this notion follows from the following two lemmas:

Lemma 1. $\overrightarrow{\text{Cyc}(p)}$ has the vertex partition property for every non-negative integer p.

Lemma 2. $\overrightarrow{\mathrm{Cyc}(p)}$ has the edge-partition property if and only if for every $G \in \overrightarrow{\mathrm{Cyc}(p)}$ and for every positive integer k there exists $H \in \overrightarrow{\mathrm{Cyc}(p)}$ such that $G \xrightarrow[k]{\mathrm{good},e} H$ (we abbreviate the latter statement by saying that $\overrightarrow{\mathrm{Cyc}(p)}$ has the good-partition property).

Proof of Lemma 1 follows easily from the existence of highly chromatic graphs without short cycles. This is done explicitly in [6] and therefore we omit the proof.

Proof of Lemma 2. Obviously $G \xrightarrow[k]{e} H$ implies $G \xrightarrow[k]{\mathrm{good},e} H$. To see the converse, let $G \in \overrightarrow{\mathrm{Cyc}(p)}$ be fixed. Choose $G' \in \overrightarrow{\mathrm{Cyc}(p)}$ with $G \xrightarrow[k]{v} G'$ (by Lemma 1) and $H \in \overrightarrow{\mathrm{Cyc}(p)}$ with $G' \xrightarrow[k]{\mathrm{good},e} H$. Then $G \xrightarrow[k]{e} H$ follows from the composition of the definitions of the arrows:

Let $c: E(H) \longrightarrow [1,k]$ be a fixed colouring. By $G' \xrightarrow[k]{\mathrm{good},e} H$ there exists a graph $\bar{G}' \in \binom{H}{G'}$ such that $c([y,x]) = c([y',x])$ whenever $[y,x] \in E(\bar{G}')$, $[y',x] \in E(\bar{G}')$, $y < x$, $y' < x$. Define the mapping $c': V(\bar{G}') \longrightarrow [1,k]$ by $c'(x) = c([y,x])$ for any $[y,x] \in E(\bar{G}')$, $y < x$ if such an edge $[y,x]$ exists, otherwise put $c'(z) = 1$. By the choice of $\bar{G}'$ this is a correct definition. By $G \xrightarrow[k]{v} \bar{G}' \simeq G'$ there exists $\bar{G} \in \binom{G'}{G}$ such that c' restricted to the set $V(\bar{G})$ is a constant. It is not difficult to see that from this it follows that the colouring c restricted to the set $E(G)$ is also a constant mapping.

Proof of Theorem C. We shall proceed by induction on p. Theorem C is valid for $p=0$, see [1],[8],[10]. Let $p > 0$ be a fixed integer and assume that Theorem C is valid for all p', $0 \leq p' < p$. Hence we prove by induction on $|V(G)|$ the following statement:

(∗) For every $G \in \overrightarrow{Cyc(p)}$ and for every positive integer k there exists $H \in \overrightarrow{Cyc(p)}$ such that $G \xrightarrow[k]{good,e} H$.

It follows from Lemma 2 that for a positive integer p the statement (∗) implies the validity of Theorem C.

We may start the induction procedure as if $|V(G)|=1$, then $G \xrightarrow[k]{good,e} G$.

Thus let $G=(V,E) \in \overrightarrow{Cyc(p)}$, $|V|>1$ be fixed. Fix a positive integer k.

Let $x \in V$ be the last vertex of G (in the standard ordering of G). Put $V(x)=\{y \in V;\ [x,y] \in E\}$. Derive two graphs, which will be denoted by G' and G^*:

$$G'=(V',E')=(V \setminus \{x\}, \{e \in E;\ x \notin e\});\ G^* =(V^*,E^*),$$

where $V^* = V' \setminus V(x) \cup \{x^*\}$, $x^* \notin V$, $y < x^*$ for every $y \in V'$, and $[x,y] \in E^*$ iff either $[x,y] \in E$, $\{x,y\} \subseteq V \setminus V(x)$ or $x=x^*$ and

$[x',y] \in E'$ for an $x' \in V(x)$, $y \in V \setminus V(x)$.

(In other words: G^* is the factor graph $G'/_{V(x)}$.)

Obviously $G' \in \overrightarrow{Cyc(p)}$. Moreover $G^* \in \overrightarrow{Cyc(p-1)}$, which may be demonstrated as follows: Assume $G^* \notin \overrightarrow{Cyc(p-1)}$. Then there exists a cycle C of the length $2p'+1$ for a $p' \in [1,p-1]$. Obviously, x^* has to be a vertex of C and consequently G would contain a cycle of the length $2p'+3$. This is a contradiction.

Let $|V'| = a$.

Let us write three rows of partition arrows (standard orderings are omitted):

I $\quad G' \xrightarrow[k]{good,e} H' \xrightarrow[m]{v} H'' \in \overrightarrow{Cyc(p)}$;

II $\quad G \xrightarrow[n]{e} H^* \in \overrightarrow{Cyc(p-1)}$;

III $\quad K_a \xrightarrow[k]{v} K_A \xrightarrow[q]{e} K_N$.

The meaning of the undefined symbols and the mechanism of this scheme is the following:

(i) H' exists by the induction hypothesis; $H' \in \overrightarrow{Cyc(p)}$;

(ii) put $n=k^r$, $r=|E(H')|$; H^* exists by the induction hypothesis;

(iii) the first arrow in III follows from the Dirichlet principle; it suffices to put $A=(a-1)\,k+1$. The second arrow in III is the classic Ramsey theorem; put $q=k^R$, $R=|E(H')|\cdot$ $\cdot|E(H^*)|$.

(iv) Put $m=k^B$, $B=|V(H^*)|\cdot N$, H'' exists by Lemma 1.

The choice of the numbers m,n,N,q and graphs H', H^*, H'', K_N is consistent. We may choose them subsequently in the above order (i.e. H', n, H^*, q, N, m, H'').

Define the ordered graph $H=(W,F)$ by $W=(V(H'')\times V(H^*)\times [1,N])$ $\cup V(H^*)$; the standard ordering $\preceq$ of W is given by the lexicographic ordering induced by the standard orderings of $V(H'')$, $V(H^*)$ and $[1,N]$ and by $y < z$ for every $z \in V(H^*)$ and every $y \in$ $\in V(H'')\times V(H^*)\times [1,N]$;

$[A, B] \in F$ iff either $A=(y,z,i)$, $B=(y',z',i')$, $y<y'$, $z<z'$, $i<i'$ and $[y,y'] \in E(H'')$, $[z,z'] \in E(H^*)$, $[i,i'] \in E(K_N)$, or $A=(y,z,i,)$, $B=z, y \in V(H'')$, $z \in V(H^*)$, $i \in [1, N]$.

Consequently, H is the ordered direct product $H''\times H^* \times K_N$ with added vertices of $V(H^*)$, which are joined with $H''\times H^*\times$ $\times K_N$ by "sheaves" of edges induced by the second coordinate H^*.

<u>Claim 1</u>. $H \in \overrightarrow{Cyc(p)}$.

<u>Proof</u>. Assume the contrary: let C be a cycle in H of the length $2p'+1$, $p' \leq p$. As $H''\times H^*\times K_N \in \overrightarrow{Cyc(p)}$, C has a com-

mon vertex with $V(H^*)$. Let C' be the part of C which belongs to $H''\times H^*\times K_N$, let π be the projection $H''\times H^*\times K_N \longrightarrow H^*$. π is a homomorphism and using the definition of H it follows that π maps C' onto a graph which contains an odd cycle of a length less than $2p'-1$ (we use $C'\neq C$). But this is a contradiction with $H^*\in \mathrm{Cyc}(p-1)$. In the case $p=1$ we claim that $\{y;[y,z]\in E(H)\}$ is an independent subset of H for each $z\in V(H^*)$.

<u>Claim 2:</u> $G \xrightarrow[k]{good,e} H$.

<u>Proof</u>. Let $c:E(H)\longrightarrow[1,k]$ be a fixed colouring. First, define the colouring $c'':V(H'')\longrightarrow[1,k]^{V(H^*)\times[1,N]}$ by $c''(y)=(c([(y,y^*,i),y^*]);\ y^*\in V(H^*),\ i\in[1,N])$. As $|[1,k]^{V(H^*)\times[1,N]}|=m$, it follows from the second part of line I that there exists a subgraph H''' of H'' isomorphic to H' such that c'' is a constant mapping on the set $V(H''')$. Secondly, define the colouring $d'':E(K_N)\longrightarrow[1,k]^{E(H'''\times H^*)}$ by $d''(e')=(c(e);\ e\in E(H'''\times H^*\times K_N),\ \pi_3(e)=e')$. Here $\pi_3:H''\times H^*\times K_N\longrightarrow K_N$ is the projection. Analogously denote the projections $\pi_1:H''\times H^*\times K_N\longrightarrow H''$ and $\pi_2:H''\times H^*\times K_N\longrightarrow H^*$. As every edge $e=[(y'',y^*,i),(z'',z^*,j)]\ E\ (H''\times H^*\times K_N)$ is uniquely determined by its projections $\pi_1(e)=e_1=[y'',z'']$, $\pi_2(e)=e_2=[y^*,z^*]$ and $\pi_3(e)=e_3=[i,j]$, we shall write also $e=(e_1,e_2,e_3)$. Using this convention we may restate the definition of the colouring d'' as $d''(e)=(c((e_1,e_2,e));\ e_1\in E(H'''),\ e_2\in E(H^*))$ so that really $d'':E(K_N)\longrightarrow[1,k]^{E(H''')\times E(H^*)}$. As $|[1,k]^{E(H'''\times H^*)}|=q$, it follows from the second part of line III that there exists a subgraph

$K^{\vee}$ of K_N isomorphic to K_A such that d'' restricted to the set $E(K^{\vee})$ is a constant mapping.

This means that we may define an auxiliary colouring $\bar{c}:E(H'^{\vee}\times H^*)\longrightarrow[1,k]$ by $\bar{c}(\bar{e})=c(e)$ for every $e\in E(H'^{\vee}\times H^*\times K^{\vee})$ which satisfies $(\pi_1(e),\pi_2(e))=\bar{e}$. (This is a correct definition as $c(e)=c(e')$, whenever $\pi_1(e)=\pi_1(e')$, $\pi_2(e)=\pi_2(e')$, $\{\pi_3(e),\pi_3(e')\}\subseteq E(K^{\vee})$.)

Thirdly, define the colouring $c^*:E(H^*)\longrightarrow[1,k]^{E(H'^{\vee})}$ by $c^*(e^*)=(\bar{c}(e);\ e\in E(H'^{\vee}\times H^*))$. As $|[1,k]^{E(H'^{\vee})}|=r$, it follows from line II that there exists a subgraph $G^{*\vee}$ of H^* isomorphic to G^* such that c^* is a constant on $E(G^{*\vee})$. Finally, define the colouring $c':E(H'^{\vee})\longrightarrow[1,k]$ by $c'(e')=c(e)$ for all $e\in E(H'^{\vee}\times G^{*\vee}\times K^{\vee})$ which satisfy $\pi_1(e)=e'$ (this is a correct definition by the above choice of $G^{*\vee}$ and $K^{\vee}$). By the first part of line I there exists a subgraph $G'^{\vee}$ of $H'^{\vee}$ isomorphic to G' such that the colour $c'(y,z)$, $y<z$ depends on z only.

Let $\Phi':G'\longrightarrow G'^{\vee}$, $\Phi^*:G^*\longrightarrow G^{*\vee}$ be ordered isomorphisms.

Put $x^{\vee}=\Phi^*(x^*)$. Define the colouring $d':V(K^{\vee})\longrightarrow[1,k]$ by $d'(i)=c([(\Phi'(y),x^{\vee},i),x^{\vee}])$ for all $y\in V(G')$. This is a correct definition as $c([(\Phi'(y),x^{\vee},i),x^{\vee}])=c([(\Phi'(y'),x^{\vee},i),x^{\vee}])$ for all $y,y'\in V(G')$ (this follows from the first step of this proof - see the definition of the colouring c''). By the first part of line III there exists a subgraph K of $K^{\vee}$ isomorphic to $K_{V'}$ such that $d'(i)$ is a constant for all $i\in V(K)$. Let $\Phi'':V'\longrightarrow V[K]$ be the monotone bijection.

Define the mapping $\Phi:V(G)\longrightarrow V(H)$ by

$\Phi(y)=(\Phi'(y),\Phi^*(y),\Phi''(y))$ for all $y\in V'\setminus V(x)$;

$\Phi(y)=(\Phi'(y),x^{\vee},\Phi''(y))$ for all $y\in V(x)$;

$\Phi(x)=x^{\vee}$.

It follows from the above construction of the embeddings Φ', Φ^*, Φ'' that Φ is a monotone mapping and an embedding $G \longrightarrow H$. Moreover $c([\Phi(y), \Phi(z)])$, $z<y$ depends on z only as it follows from the properties of the above embeddings. This finishes the proof of Claim 2 and consequently of Theorem C.

The following diagram may be of some help in understanding the proof of Theorem C:

$$
\begin{array}{ll}
\Phi' : G' = G - x \longrightarrow H'^{\vee} \longrightarrow & H'' \\
 & \times \\
\Phi^* : G^* = G'/_{V(x)} \longrightarrow & H^* \\
\Phi'' : K_{V(G')} \longrightarrow K_{A} \longrightarrow & K_{N} \\
\quad x \longrightarrow \Phi^*(x^*) = x^{\vee} & \\
\Phi : G \longrightarrow & H
\end{array}
$$

§ 3. <u>Proof of Theorem B</u>. Using Theorem C, Theorem B will follow from the following statements:

<u>Lemma 3</u>. The class Cyc(p) has orderings for every natural p.

<u>Lemma 4</u>. If Cyc(p) has the F-partition property, then $F \xrightarrow{ord} F$.

<u>Lemma 5</u>. For a graph F the following two statements are equivalent:

1) $F \xrightarrow{ord} F$;

2) F is either a complete or a discrete graph.

<u>Lemma 6</u>. For every $p>1$ the class Cyc(p) has not the $(V,\emptyset)$-partition property for any set V, $|V|>1$.

Proof of Lemma 3 (by a direct application of Theorem C). Let $G=(V,E)\in Cyc(p)$ be fixed. Let $G'\in Cyc(p)$ and an ordering $\leq$ of $V(G')$ be given with the property that for every ordering $\leq$ of V there exists a monotone embedding $(G,\leq)\longrightarrow(G',\leq)$. $(G',\leq)$ exists as we may take the disjoint union of all $(G,\leq)$, $\leq$ an ordering of V. Let $(G',\leq)\xrightarrow[k]{e}(H,\leq)$, $H\in Cyc(p)$. Let $\precsim$ be an arbitrary ordering of V(H). Define the colouring c of E(H) by

$c([x,y])=1$ if and only if $x\leq y$, $x\precsim y$;

$c([x,y])=2$ otherwise.

By the definition of $(G',\leq)\xrightarrow[k]{e}(H,\leq)$ we get either a monotone embedding $(G',\leq)\longrightarrow(H,\precsim)$ or a monotone embedding $(G'\geq)\longrightarrow$ $\longrightarrow(H,\precsim)$. Using the definition of G' we get $G\xrightarrow{ord}H$.

Proof of Lemma 4. Let $F\not\xrightarrow{ord}F$. This means that there are orderings $\leq_1$ and $\leq_2$ such that there exists no monotone isomorphism $(F,\leq_1)\longrightarrow(F,\leq_2)$. Let $(G,\leq)$ be a disjoint union of $(F,\leq_1)$ and $(F,\leq_2)$ and let $G\xrightarrow{ord}G'$. We prove $G'\not\xrightarrow[2]{F}H$ for every $H\in Cyc(p)$. The reason for this is the following: given a $H\in Cyc(p)$, fix an ordering of V(H) and define a colouring $c:\binom{H}{F}\longrightarrow[1,2]$ by $c(F^\vee)=1$ if $F^\vee$ with the restricted ordering $\leq$ of V(H) is monotone isomorphic to $(F,\leq_1)$; $c(F^\vee)=2$ otherwise. It follows from the definition of G' that there exists no $G^\vee\in\binom{H}{G'}$ with $|c\binom{G^\vee}{F}|=1$.

Proof of Lemma 5. $F\xrightarrow{ord}F$ ~~if~~ and only if for any two orderings $\leq_1$ and $\leq_2$ of V(F) the ordered graphs $(F,\leq_1)$ and $(F,\leq_2)$ are monotone isomorphic. As every permutation may be coded by an ordering, we get that $F\xrightarrow{ord}F$ if and only if every permutation of V(F) is an automorphism of F. Consequently F is either a complete or a discrete graph.

Proof of Lemma 6. For brevity put $V=(V,\emptyset)$, $n=([1,n],\emptyset)$. Assume first $|V|=2$. Let $p>1$ be fixed. Denote by C_n the cycle of the length n. By way of a contradiction let us assume that $C_{2p+2}\xrightarrow[k]{V} H$ Cyc(p), $k\geq 2$. Define a colouring $c:\binom{H}{V}\longrightarrow[1,2]$ by $c(\{x,y\})=1$ if and only if $\rho(x,y)=2$;

$c(\{x,y\})=2$ otherwise.

(Observe that $\binom{H}{V}$ is the set of independent subsets of H of the size 2; $\rho(x,y)$ is the distance of x and y). As $H\in$ Cyc(p) and $p>1$, H does not contain a pentagon and consequently every cycle of the length 2p+2 in H contains two vertices of the distance >2. This is a contradiction with $C_{2p+2}\xrightarrow[2]{V} H$.

Let $|V|=n>2$, $p>1$ be fixed.

Let G be the cycle of the length 2p+2 with added n-2 isolated vertices.

Let $G\xrightarrow{ord} G'\in$ Cyc(p) (the existence of G' follows from Lemma 3). We claim that $G'\not\xrightarrow[k]{V} H$ for every graph $H\in$ Cyc(p). This will follow by showing that $G'\xrightarrow[2]{V} H$ implies $G\xrightarrow[2]{W} H$ for $|W|=2$; this is impossible by the first part of the proof. Assume $G'\xrightarrow[k]{V} H$. Let $\leq$ be an ordering of V(H) and let $c:\binom{H}{W}\longrightarrow[1,2]$ be a fixed colouring. Define the colouring $c':\binom{H}{V}\longrightarrow[1,2]$ by $c'(\{x_1,\dots,x_n\})=i$ if and only if $x_1<\dots<x_n$ and $c(\{x_1,x_2\})=i$. By the assumption there exists $i\in[1,2]$ and a subgraph $G^{\vee}\in\binom{H}{G'}$ such that $\binom{G^{\vee}}{V}\subseteq c'^{-1}(i)$. By the ordering property of G' there exists a subgraph $G^{-}\in\binom{G^{\vee}}{G}$ such that the graph G^{-} (with the relative ordering of H) and the graph G (with an ordering where every isolated vertex is greater than every non-isolated vertex) are order-isomorphic. Using the

definition of the colouring c' we get $C_{2p+2} \xrightarrow[2]{W} H$, $|W|=2$, which is a contradiction with the first part of the proof.

Concluding remarks. The proof of Theorem B is a procedure which may be generalized in various ways for other classes of graphs, examples of this are given in [7] and [8]. On the other hand, the fact that the classes Cyc(p) generally do not possess the $(V,\emptyset)$-partition property is the main reason for dealing with the classes Cyc(p) separately.

The main problem of the Ramsey theory in this field is perhaps the problem whether the class of all graphs without C_3 and C_4 has the edge-partition property. This class certainly has the vertex-partition property (see [6]). The products, of course, may not be used.
This problem was pointed out on several occasions by P. Erdős (see [3]).

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