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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
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TERMS AND SEMITERMS

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Abstract: A construction of the algebra of terms is found which allows a notation making the study of subterm instances more comfortable. It turns out that it is useful to consider (together with terms) new ideal objects, called semiterms. Using the theory of semiterms, a representation of universal algebras in commutative semigroups is obtained.

Key words: Term, semiterm, universal algebra, commutative semigroup, envelope.

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0. Preliminaries and introduction. By a type we mean a set T of operation symbols; every operation symbol F is associated with a non-negative integer which is denoted by n_F and called the arity of F. For every non-negative integer n we put $T_{(n)}=\{F\in T; n_F=n\}$. The symbols from $T_{(n)}$ are called n-ary (nullary, unary and binary in the cases n=0, n=1 and n=2).

By a T-algebra we mean a non-empty set U together with a mapping, assigning to any $F\in T$ an n_F-ary operation on U. If the T-algebra is denoted by A, then for every $F\in T$ the corresponding n_F-ary operation on U will be denoted by F_A (sometimes only by F); the set U is called the underlying set of

A and it is sometimes denoted by the same letter (or group of letters) as A.

For the definitions of standard notions from universal algebra see e.g. [1].

Given a type T and a non-empty set X, there exists an absolutely free T-algebra A over X, i.e. a T-algebra generated by X such that every mapping of X into any T-algebra can be extended to a homomorphism. Evidently, A is determined by T and X uniquely up to isomorphism over X. It is customary to take one concrete fixed absolutely free T-algebra over X and call its elements T-terms over X (or only terms). Usually terms are defined as formal inscriptions consisting of elements of X, operation symbols from T and parantheses, in the following inductive way: every element of X is a term (these terms are called variables); every symbol from $T_{(o)}$ is a term (these terms are called constants); if $F \in T \setminus T_{(o)}$ and if $t_1, \ldots, t_{n_f}$ are terms, then the inscription $F(t_1, \ldots$ $\ldots, t_{n_F})$ is a term (these terms are called composed terms). This definition has various advantages and disadvantages. The disadvantages are evident e.g. if in some connection we are forced to introduce a general study of subterm instances (occurences of subterms, positions of subterms). In this paper we develop a method for denoting terms in such a way that e.g. the investigation of subterm instances becomes more comfortable; this means a new construction of the algebra of terms.

For example, consider the type consisting of a single binary operation symbol F. Then every composed term has the left and the right parts. Every subterm instance in any term t can be obtained from t by a finite sequence of operations of taking the left and right parts; this finite sequence is uniquely determined by the subterm instance in t. Denote by F_1 the operation of taking the left part and by F_2 the operation of taking the right part. Thus subterm instances in t can be identified with certain elements of the free monoid M over the set $\{F_1,F_2\}$. For example, the subterm $F(x,x)$ has two distinct instances in the term $F(x,F(F(x,x),F(F(x,x),x)))$, namely F_2F_1 and $F_2F_2F_1$. Especially, the positions of variables in t are identified with certain elements of M. Since every term is uniquely determined by its variables and their positions, a term t is uniquely determined by the expression $u=e_1x_1+\ldots+e_rx_r$ where x_i are the variables occurring in t and $e_i \in M$ are their instances. (Here the operation + is supposed to be associative and commutative.) Thus the expression u can be considered to be a new notation for the term t; or we can identify t with u. For example, the new notation for the term $F(x,F(F(x,x),F(F(x,x),x)))$ is $F_1x+F_2F_1F_1x+F_2F_1F_2x+F_2F_2F_1F_1x+$ $+F_2F_2F_1F_2x+F_2F_2F_2x$.

Not all linear combinations $u=e_1x_1+\ldots+e_rx_r$ correspond to terms. We give in 1.1 a necessary and sufficient condition for u to correspond to a term. General linear combinations u will be called semiterms. In Section 2 the notion of semiterm is applied to finding short definitions of syntactic notions concerning terms. It turns out that the set of terms is a block of a congruence of the commutative semigroup of semiterms; this congruence is studied in Section 3. In Section 4 we apply the theory of semiterms to obtain a representation of arbitrary universal algebras in commutative semigroups. It is proved that for any T-algebra A there exists a commutative semigroup (S,+)

such that A is a subset of S and for every $F \in T \setminus T_{(o)}$ there exist endomorphisms $f_1, \dots, f_{n_F}$ of $(S,+)$ with $F_A(a_1, \dots, a_{n_F}) = f_1(a_1) + \dots + f_{n_F}(a_{n_F})$ for all $a_1, \dots, a_{n_F} \in A$.

The results of this paper (although called theorems here) are not deep. This paper is only an attempt to develop a theory of notations that seems to be convenient in some respects. T. Kepka and the author are preparing a paper in which the methods of the present paper are specialized and applied to obtain some structure theorems on medial groupoids.

1. The algebra of semiterms and the algebra of terms. Let T be a type. We fix an injective mapping $(F,i) \longmapsto F_i$ of the set $\{(F,i);\ F \in T \setminus T_{(o)},\ i \in \{1,\dots,n_F\}\}$ into the class of unary operation symbols such that $F_1 = F$ for every $F \in T_{(1)}$. Further we fix a nullary symbol 0 not belonging to T. The type

$$T_{(o)} \cup \{0\} \cup \{F_i;\ F \in T \setminus T_{(o)},\ i \in \{1,\dots,n_F\}\} \cup \{+\}$$

will be denoted by T'; it consists of nullary and unary symbols and one binary symbol +.

We denote by M_T the free monoid over the set of unary symbols from T'. Every element $e \in M_T$ can be uniquely expressed in the form $e = \prod_{i=1}^{n} g_i$ where $n \ge 0$ and for every $i \in \{1,\dots,n\}$ g_i is a unary symbol from T'; the non-negative integer n is called the depth of e and it is denoted by $\partial(e)$. The unit element of M_T is denoted by 1; we have $\partial(1)=0$.

We denote by S_T the variety of T'-algebras satisfying the following identities:

$(x+y)+z = x+(y+z)$,

$x+y = y+x$,

$x+0=x$,

$F_i(x+y)=F_i x+F_i y$,

$F_i 0=0$

(where F is any symbol from $T \setminus T_{(o)}$ and $i \in \{1,\dots,n_F\}$). Thus algebras from S_T are essentially commutative semigroups with 0 in which several elements and 0-preserving endomorphisms are fixed.

Let T be a type and X be a non-empty set. We take one fixed free algebra in S_T over X (all such T'-algebras are isomorphic) and denote it by $SW'_{T,X}$. The elements of $SW'_{T,X}$ are called T-semiterms over X (or only semiterms over X or sometimes only semiterms). Evidently, every T-semiterm s over X can be expressed in the form $s=\sum_{i=1}^{r} e_i x_i$ where r is a non-negative integer and, for every $i \in \{1,\dots,r\}$, $e_i \in M_T$ and $x_i \in X \cup T_{(o)}$; this expression is unique up to the order of the summands. The non-negative integer r is called the length of s and is denoted by $\lambda(s)$. The non-negative integer $\sum_{i=1}^{r}(1+\partial(e_i))$ is called the total length of s and it is denoted by $\lambda^*(s)$. We have

$\lambda(s)=0 \iff \lambda^*(s)=0 \iff s=0$,

$\lambda^*(x)=1 \iff s \in X \cup T_{(o)}$.

Let us define a T-algebra $SW_{T,X}$ as follows: its underlying set is the set of all T-semiterms over X; if $F \in T_{(o)}$ then $F_{SW_{T,X}}=F$; if $F \in T \setminus T_{(o)}$ and $s_1,\dots,s_{n_F} \in SW_{T,X}$ then $F_{SW_{T,X}}(s_1,\dots$ $\dots,s_{n_F})=F_1 s_1+\dots+F_{n_F} s_{n_F}$. We shall usually omit the subscript $SW_{T,X}$ in $F_{SW_{T,X}}$. The T-algebra $SW_{T,X}$ is called the algebra of T-semiterms over X.

The subalgebra of $SW_{T,X}$ generated by X is denoted by $W_{T,X}$

and its elements are called T-terms over X (or only terms over X).

1.1. Theorem. Let $t=\sum_{i=1}^{r} e_i x_i$ (where $x_i \in X \cup T_{(o)}$) be a T-semiterm over X. Then t is a term over X iff it satisfies the following four conditions:

(1) $r \geq 1$;

(2) if $i,j \in \{1,\dots,r\}$ and $e_i = e_j f$ for some $f \in M_T$ then $i=j$;

(3) if $i,j \in \{1,\dots,r\}$, $e_i = fF_k g$ and $e_j = fG_{\ell} h$ for some $f,g,h \in M_T$, $F,G \in T \setminus T_{(o)}$, $k \in \{1,\dots,n_F\}$ and $\ell \in \{1,\dots,n_G\}$, then $F=G$;

(4) if $i \in \{1,\dots,r\}$ and $e_i = fF_k g$ for some $f,g \in M_T$, $F \in T \setminus T_{(o)}$ and $k \in \{1,\dots,n_F\}$, then for every $\ell \in \{1,\dots,n_F\}$ there exists a $j \in \{1,\dots,r\}$ with $e_j = fF_{\ell} h$ for some $h \in M_T$.

Proof. It is easy to see that the set of all $t \in SW_{T,X}$ satisfying (1) - (4) is a subalgebra of $SW_{T,X}$ containing X. On the other hand, one can verify by induction on $\lambda^*(t)$ that if t satisfies (1) - (4) then $t \in W_{T,X}$.

1.2. Theorem. $W_{T,X}$ is an absolutely free T-algebra over X. For every T-term t over X exactly one of the following three cases takes place:

(1) $t \in X$;

(2) $t \in T_{(o)}$;

(3) there exists a unique symbol $F \in T \setminus T_{(o)}$ and a unique sequence $t_1,\dots,t_{n_F}$ of T-terms over X such that $t=F(t_1,\dots,t_{n_F})$.

Proof is easy.

2. Subterm instances. Let $t=\sum_{i=1}^{r} e_i x_i$ ($x_i \in X \cup T_{(o)}$) be a T-term over X. By a subterm instance in t we mean an element $e \in M_T$ such that $ef=e_i$ for some $f \in M_T$ and some $i \in \{1,\dots,r\}$.

The set of all subterm instances in t is denoted by $I(t)$.

By a T-pattern we mean a subset P of M_T satisfying the following four conditions:

(1) P is finite and $1 \in P$;

(2) if $ef \in P$ then $e \in P$;

(3) if $eF_k \in P$ and $eG_\ell \in P$ then $F=G$;

(4) if $eF_k \in P$ then $eF_\ell \in P$ for all $\ell \in \{1,\dots,n_F\}$.

2.1. Theorem. Let P be a subset of M_T. Then P is a T-pattern iff $P=I(t)$ for some T-term t over X. If P is a T-pattern and φ is a mapping of the set $N=\{e \in P;\ ef \in P$ implies $f=1\}$ into $X \cup T_{(o)}$ then $\sum_{e \in N} e\varphi(e)$ is a T-term over X and $P= =I(\sum_{e \in N} e\varphi(e))$.

Proof is easy.

2.2. Theorem. Let t be a T-term over X and e be a subterm instance in t. Then there exists a unique pair (w,u) such that w is a T-semiterm over X, u is a T-term over X and $t=w+eu$. Moreover, if v is an arbitrary T-term over X, then $w+ev$ is a T-term over X, too.

Proof. Put $t=\sum_{i=1}^{r} e_i x_i$ where $x_i \in X \cup T_{(o)}$. Denote by I the set of all $i \in \{1,\dots,r\}$ such that $e_i=ef$ for some $f \in M_T$; for every $i \in I$ define f_i by $e_i=ef_i$. Put $u=\sum_{i \in I} f_i x_i$ and $w= =\sum_{i \notin I} e_i x_i$. We have $t=w+eu$ and it follows from 1.1 that u is a term. Now let $t=w'+eu'$ where $w' \in SW_{T,X}$ and $u' \in W_{T,X}$. Suppose that for some $i \in I$, $e_i x_i$ is a summand in w'. Then we can take an $f \in M_T$ of maximal depth such that for some $g,h \in M_T$ and $x,y \in \in X \cup T_{(o)}$ $efgx$ is a summand in w' and $efhy$ is a summand in eu'; since $t \in W_{T,X}$, we would have $g=F_k\bar{g}$ and $h=F_\ell\bar{h}$ for some $\bar{g},\bar{h} \in M_T$, $F \in T \setminus T_{(o)}$ and $k,\ell \in \{1,\dots,n_F\}$; since $u' \in W_X$, u' would have a summand $efF_k\bar{\bar{h}}z$ for some $\bar{\bar{h}} \in M_T$ and $z \in X \cup T_{(o)}$, a contradiction with the maximality of $\partial(f)$. Hence $e_i x_i$ is a summand in eu' for all $i \in I$ and we get $u'=u$. Now $w'=w$ is evident. Using 1.1, it is obvious that $w+ev$ is a term.

The term u in 2.2 is called the e-th subterm of t and it is denoted by $t_{[e]}$. A T-term u over X is said to be a subterm of t if $u=t_{[e]}$ for some $e \in I(t)$. If $u=t_{[e]}$, then we say that e is an instance of u in t.

The term $w+ev$ in 2.2 is called the term resulting from t by substituting v for e.

More generally, let $e_1,\dots,e_n$ be subterm instances in t such that whenever $i,j \in \{1,\dots,n\}$ and $e_i=e_jf$ for some $f \in M_T$ then $i=j$. (We call such subterm instances independent.) Then it follows from 2.2 that there exists a unique $(n+1)$-tuple $s,u_1,\dots,u_n$ such that s is a T-semiterm over X, $u_1,\dots,u_n$ are T-terms over X and $t=s+e_1u_1+\dots+e_nu_n$. If $v_1,\dots,v_n$ are any T-terms over X, then $s+e_1v_1+\dots+e_nv_n$ is a T-term over X, too; it is evidently just the term resulting from t by substituting v_1 for $e_1,\dots,v_n$ for e_n (in arbitrary order).

3. <u>The congruence $\overline{\sigma}_X$ and irreducible semiterms</u>. Let T be a fixed type and X be a non-empty set.

Two semiterms u,v over X are called similar if we can write $u=\sum_{i=1}^{n} e_ix_i$ and $v=\sum_{i=1}^{n} e_iy_i$ for some $r \geq 0$, $e_i \in M_T$ and $x_i,y_i \in X \cup T_{(o)}$. Evidently, the set of all similar pairs of semiterms over X is a congruence of the T'-algebra $SW'_{T,X}$ (and so a congruence of the T-algebra $SW_{T,X}$, too). Evidently, if u,v are similar then u is a term iff v is a term.

If $u=\sum_{i=1}^{r} e_i x_i$ (where $x_i \in X \cup T_{(o)}$) is a semiterm over X, then the set $\{x_i;\ i=1,\dots,r\}$ is called the support of u and it is denoted by supp(u). The set $X \cap \mathrm{supp}(u)$ is called the variable support of u and it is denoted by var(u).

For every semiterm u and every $x \in X$ there exists a unique semiterm v similar to u such that $\mathrm{supp}(v)=\{x\}$; if $u=\sum_{i=1}^{r} e_i x_i$ then $v=\sum_{i=1}^{r} e_i x$.

We define a binary relation σ_X on $SW_{T,X}$ as follows: $(u,v) \in \sigma_X$ iff $u=w+ex$ and $v=w+eF_1x+\dots+eF_{n_F}x$ for some $w \in SW_{T,X}$, $e \in M_T$, $F \in T \setminus T_{(o)}$ and $x \in X$.

By a σ_X-proof we mean a finite sequence $u_o,\dots,u_n$ $(n \geq 0)$ of semiterms over X such that $(u_{i-1},u_i) \in \sigma_X \cup \sigma_X^{-1}$ for all $i \in \{1,\dots,n\}$. By a σ_X-proof from u to v we mean a σ_X-proof $u_o,\dots,u_n$ such that u is similar to u_o and v is similar to u_n.

We define a binary relation $\overline{\sigma}_X$ on $SW_{T,X}$ as follows: $(u,v) \in \overline{\sigma}_X$ iff there exists a σ_X-proof from u to v.

3.1. Theorem. $\overline{\sigma}_X$ is a congruence of the T'-algebra $SW'_{T,X}$ and $W_{T,X}$ is its block.

Proof. Evidently, $\overline{\sigma}_X$ is a congruence of $SW'_{T,X}$. It follows from 2.2 that if $(u,v) \in \sigma_X$ then u is a term iff v is a term; from this it follows that if $(u,v) \in \overline{\sigma}_X$ then u is a term iff v is a term. Let us fix an element $x \in X$. It is easy to prove by induction on $\lambda^*(t)$ that if t is a term then $(t,x) \in \overline{\sigma}_X$; hence $(t,t') \in \overline{\sigma}_X$ for all $t,t' \in W_{T,X}$.

By a minimal σ_X-proof from u to v we mean a σ_X-proof $u_o,\dots,u_n$ from u to v such that whenever $v_o,\dots,v_m$ is a σ_X-proof from u to v then $\lambda^*(u_o)+\dots+\lambda^*(u_n) \leq \lambda^*(v_o)+\dots+\lambda^*(v_m)$.

3.2. Lemma. Let $u_o,\dots,u_n$ be a minimal σ_X-proof from u to v. Then there exists a $k\in\{0,\dots,n\}$ such that $(u_{i-1},u_i)\in\in\sigma_X^{-1}$ for all $i\in\{1,\dots,k\}$ and $(u_{i-1},u_i)\in\sigma_X$ for all $i\in\in\{k+1,\dots,n\}$.

Proof. Evidently it is enough to assume that there is an $x\in X$ with $supp(u_i)=\{x\}$ for all $i\in\{0,\dots,n\}$. Suppose that there is no such k, so that there exists an $m\in\{1,\dots,n-1\}$ with $(u_{m-1},u_m)\in\sigma_X$ and $(u_m,u_{m+1})\in\sigma_X^{-1}$. There are $w,w'\in\in SW_{T,X}$, $e,f\in M_T$ and $F,G\in T\setminus T_{(o)}$ with

$$u_{m-1}=w+ex,$$
$$u_m=w+eF_1x+\dots+eF_{n_F}x=w'+fG_1x+\dots+fG_{n_G}x,$$
$$u_{m+1}=w'+fx.$$

If e=f and F=G then evidently $u_{m-1}=u_{m+1}$ and $u_o,\dots,u_{m-1},u_{m+2},\dots,u_n$ is a σ_X-proof from u to v, a contradiction with the minimality of $u_o,\dots,u_n$. Thus we have either $e\neq f$ or $F\neq G$ and so there exists a $w''\in SW_{T,X}$ such that $w=w''+fG_1x+\dots+fG_{n_G}x$ and $w'=w''+eF_1x+\dots+eF_{n_F}x$. But then $u_o,\dots,u_{m-1}$, $w''+ex+fx$, $u_{m+1},\dots,u_n$ is a σ_X-proof from u to v, a contradiction with the minimality of $u_o,\dots,u_n$ again.

A semiterm t over X is called irreducible if there are no $w\in SW_{T,X}$, $e\in M_T$, $F\in T\setminus T_{(o)}$ and $x_1,\dots,x_{n_F}\in X\cup T_{(o)}$ with $t=w+eF_1x_1+\dots+eF_{n_F}x_{n_F}$.

3.3. Theorem. The following are true:

(1) For any semiterm t there exists an irreducible semiterm s with $(t,s)\in\overline{\sigma}_X$.

(2) If s,s' are two irreducible semiterms then $(s,s')\in\overline{\sigma}_X$ iff s,s' are similar.

(3) A term is irreducible iff it belongs to $X \cup T_{(o)}$.

(4) If $s=\sum_{i=1}^{k} e_i x_i$ is an irreducible semiterm (where $x_i \in X \cup \cup T_{(o)}$) and t is a semiterm then $(t,s) \in \bar{\sigma}_X$ iff $t=\sum_{i=1}^{k} e_i u_i$ for some terms $u_1,\dots,u_r$.

Proof. (1) can be proved by induction on $\lambda^*(t)$. (2) follows from 3.2. (3) is easy. (4) follows from 3.1 and 3.2.

It follows from 3.3 that for any semiterm t over X and any $x \in X$ there exists a unique irreducible semiterm s with $(t,s) \in \bar{\sigma}_X$ and $\mathrm{supp}(s)=\{x\}$; this s will be called the x-reduct of t.

3.4. Theorem. Let t,t' be two semiterms over X and let $x \in X$. Then $(t,t') \in \bar{\sigma}_X$ iff the x-reduct of t is the same as the x-reduct of t'.

Proof follows from 3.3.

4. Envelopes and universal envelopes. Let A be a T-algebra. By an envelope of A we mean an algebra $E \in S_T$ such that $A \subseteq E$ and $F_A(a_1,\dots,a_{n_F})=F_1(a_1)+\dots+F_{n_F}(a_{n_F})$ for all $F \in T \setminus T_{(o)}$ and all $a_1,\dots,a_{n_F} \in A$. (Here F_i means $(F_i)_E$ and $+$ means $+_E$.) Loosely speaking, an envelope of A is a commutative semigroup E with 0 such that A is a subset of E and for every $F \in T \setminus T_{(o)}$ there exist 0-preserving endomorphisms $f_1,\dots,f_{n_F}$ of E with $F_A(a_1,\dots,a_{n_F})=f_1(a_1)+\dots+f_{n_F}(a_{n_F})$ for all $a_1,\dots,a_{n_F} \in A$.

By a universal envelope of A we mean an envelope E of A such that the T'-algebra E is generated by A and whenever E' is another envelope of A then there exists a (unique) homomorphism of E into E' over A. We shall show that every T-algebra has a universal envelope.

Let A be a T-algebra. We define a binary relation ε_A on $SW_{T,A}$ as follows: $(u,v) \in \varepsilon_A$ iff there exist $w \in SW_{T,A}$, $e \in M_T$, $F \in T$ and $x, x_1, \dots, x_{n_F} \in A$ with $x = F_A(x_1, \dots, x_{n_F})$, $u = w + ex$, $v = w + eF$ if $n_F = 0$ and $v = w + eF_1 x_1 + \dots + eF_{n_F} x_{n_F}$ if $n_F \neq 0$. By an ε_A-proof from u to v we mean a finite sequence $u_0, \dots, u_n$ $(n \geq 0)$ of elements of $SW_{T,A}$ such that $u = u_0$, $v = u_n$ and $(u_{i-1}, u_i) \in \varepsilon_A \cup \varepsilon_A^{-1}$ for all $i \in \{1, \dots, n\}$. We define a binary relation $\sim_A$ on $SW_{T,A}$ as follows: $u \sim_A v$ iff there exists an ε_A-proof from u to v. Evidently, $\sim_A$ is a congruence of the T'-algebra $SW'_{T,A}$ (and so a congruence of the T-algebra $SW_{T,A}$, too); it is just the congruence of $SW'_{T,A}$ generated by the pairs $(F_A(x_1, \dots, x_{n_F}), F_{SW_{T,A}}(x_1, \dots, x_{n_F}))$ where $F \in T$ and $x_1, \dots, x_{n_F} \in A$.

4.1. <u>Theorem</u>. Let A be a T-algebra. Then:

(1) A has a universal envelope.

(2) Every two universal envelopes of A are isomorphic over A.

(3) If E is a universal envelope of A then there exists a (unique) isomorphism of E onto $SW'_{T,A}/\sim_A$ extending the canonical mapping of A into $SW'_{T,A}/\sim_A$.

(4) If E is a universal envelope of A and if E' is an arbitrary envelope of a T-algebra A', then every homomorphism of A into A' can be uniquely extended to a homomorphism of E into E'.

<u>Proof</u>. Denote by f the unique homomorphism of $W_{T,A}$ onto A extending the identity on A. If $a, b \in A$ and $a \sim_A b$, then there exists an ε_A-proof $u_0, \dots, u_n$ from a to b; since evidently $\varepsilon_A \subseteq \bar{\sigma}_A$, it follows from 3.1 that $u_i \in W_{T,A}$ for all i; now evidently $f(u_0) = f(u_1) = \dots = f(u_n)$, so that $f(a) = f(b)$, i.e. $a = b$.

This shows that the canonical mapping of A into $SW'_{T,A}/\sim_A$ is an injection and so there exists an algebra $E \in S_T$ and an isomorphism g of E onto $SW'_{T,A}/\sim_A$ such that A is a subset of E and g is an extension of the canonical mapping of A into $SW'_{T,A}/\sim_A$. It is easy to prove that E is a universal envelope of A and that it satisfies (4). (2) is evident.

For every T-algebra A we fix one universal envelope of A and denote it by E(A). If A,B are two T-algebras, then by 4.1(4) every homomorphism f of A into B can be uniquely extended to a homomorphism of E(A) into E(B); this extension will be denoted by E(f) (or, more exactly, by $E_{A,B}(f)$). Evidently, E is a functor from the category of T-algebras into the category S_T.

Let us remark that the appropriate modification of the definitions and results of previous sections enables us to prove the following:

For every T-algebra A there exists a commutative semigroup E such that A is a subset of E, for every $F \in T_{(1)}$ there exists an endomorphism f of E with $F_A(a)=f(a)$ for all $a \in A$ and for every $F \in T$ with $n_F \geq 2$ there exist automorphisms $f_1,\dots,f_{n_F}$ of E with $F_A(a_1,\dots,a_{n_F})=f_1(a_1)+\dots+f_{n_F}(a_{n_F})$ for all $a_1,\dots,a_{n_F} \in A$.

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