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PSEUDOHEREDITARY AND PSEUDOCOHEREDITARY PRERADICALS

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Abstract: M.L. Teply in [12] calls a torsion theory $(\mathcal{T},\mathcal{F})$ pseudohereditary, if every submodule of $\mathcal{T}(R)$ is $\mathcal{T}$-torsion. In this paper, pseudohereditary preradicals together with the related dual problems are studied.

Key words: Preradical, pseudohereditary and pseudocohereditary preradicals, injective and projective modules.

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Throughout this paper, R stands for an associative ring with unit element and R-mod denotes the category of all unitary left R-modules. The injective hull of a module M will be denoted by E(M), the direct product (sum) by $\prod_{i\in I} M_i$ ($\sum_{i\in I}^{\oplus} M_i$). A submodule N of M is called essential (superfluous) in M, if $K\cap N = 0$ implies $K = 0$ ($K + N = M$ implies $K = M$) for every submodule K of M. If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a short exact sequence of R-modules, then we shall say that B is an envelope of A (B is a cover of C), if f(A) is essential in B (f(A) is superfluous in B). A ring is called left perfect, if every module has a projective cover.

We start with some basic definitions from the theory of preradicals (for details see [1],[2] and [3]).

A preradical r for R-mod is a subfunctor of the identity functor, i.e. r assigns to each module M its submodule r(M) in such a way that every homomorphism of M into N induces a homomorphism of r(M) into r(N) by restriction.

A preradical r is said to be

- idempotent if r(r(M)) = r(M) for every module M,
- a radical if r(M/r(M)) = 0 for every module M,
- hereditary if $r(N) = N \cap r(M)$ for every submodule N of a module M,
- cohereditary if r(M/N) = (r(M) + N)/N for every submodule N of a module M,
- faithful if r(M) = 0 for every projective module M,
- cofaithful if r(M) = M for every injective module M.

As it is easy to see a preradical r is faithful if and only if r(R) = 0 and r is cofaithful if and only if r(E(R)) = = E(R). A module M is r-torsion if r(M) = M and r-torsionfree if r(M) = 0. We shall denote by $\mathcal{T}_r$ ($\mathcal{F}_r$) the class of all r-torsion (r-torsionfree) modules. If r and s are preradicals then we write $r \leq s$ if $r(M) \subseteq s(M)$ for all $M \in$ R-mod. The idempotent core $\bar{r}$ of a preradical r is defined by $\bar{r}(M) = \sum K$, where K runs through all r-torsion submodules K of M, and the radical closure $\tilde{r}$ is defined by $\tilde{r}(M) = \bigcap L$, where L runs through all submodules L of M with M/L r-torsionfree. Further, the hereditary closure h(r) is defined by h(r)(M) = $= M \cap r(E(M))$ and the cohereditary core ch(r) by ch(r)(M) = = r(R) M. The intersection (sum) of a family of preradicals r_i, $i \in I$ is a preradical defined by $(\bigcap_{i \in I} r_i)(M) = \bigcap_{i \in I} r_i(M)$ $((\sum_{i \in I} r_i)(M) = \sum_{i \in I} r_i(M))$. For a preradical r and modules

$N \subseteq M$ let us define $C_r(N:M)$ by $C_r(N:M)/N = r(M/N)$. For an arbitrary class of R-modules $\mathcal{A}$ we define $p_{\mathcal{A}}(N) = \Sigma \operatorname{Im} f$, f ranging over all $f \in \operatorname{Hom}_R(M,N)$, $M \in \mathcal{A}$ and $p^{\mathcal{A}}(N) = \bigcap \operatorname{Ker} f$, f ranging over all $f \in \operatorname{Hom}_R(N,M)$, $M \in \mathcal{A}$. It is easy to see that $p_{\mathcal{A}}$ is an idempotent preradical ($p^{\mathcal{A}}$ is a radical). Moreover, if M is an injective (projective) module, then $p^{\{M\}}$ is hereditary ($p_{\{M\}}$ is cohereditary). Further, M is a faithful module if and only if $p^{\{M\}}$ is faithful. Dually, M is a cofaithful module if and only if $p_{\{M\}}$ is cofaithful.

§ 1. Pseudohereditary preradicals

Definition 1.1. A preradical r is said to be pseudohereditary if every submodule of $r(R)^{(I)}$ is r-torsion for every finite index set I.

Proposition 1.2. Let r be a preradical. Then the following are equivalent:

(i) r is pseudohereditary,

(ii) $N \subseteq ch(r)(M)$ implies $N \in \mathcal{T}_r$ for every submodule N of a module M,

(iii) $r(N) \subseteq ch(r)(M)$ implies $r(N) = N \cap ch(r)(M)$ for every submodule N of a module M.

Proof: (i) implies (ii). Let $M \in$ R-mod and $N \subseteq ch(r)(M)$. There is an epimorphism $f: F \longrightarrow M$ with F free. Consider the epimorphism $\bar{f}: r(F) \longrightarrow ch(r)(M)$ induced by f. By (i) $\bar{f}^{-1}(N) \in \mathcal{T}_r$ and hence $N = \bar{f}(\bar{f}^{-1}(N)) \in \mathcal{T}_r$.

(ii) implies (iii). If $M \in$ R-mod, $N \subseteq M$ such that $r(N) \subseteq ch(r)(M)$ then $r(N) \subseteq ch(r)(M) \cap N$. By (ii) $K = ch(r)(M) \cap N \in \mathcal{T}_r$ and hence $ch(r)(M) \cap N \subseteq r(N)$.

(iii) implies (i). If $K \subseteq r(R)^{(I)}$ then clearly $r(K) \subseteq ch(r)(R^{(I)})$

and (iii) yields $r(K) = ch(r)(R^{(I)}) \cap K = K$.

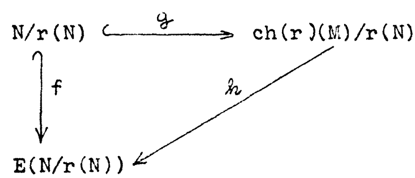
<u>Proposition 1.3</u>. Let r be a preradical. Then

(i) if r is pseudohereditary, then $F \in \mathcal{F}_r$ implies $E(F) \in \mathcal{F}_{ch(r)}$,

(ii) if r is a radical and $F \in \mathcal{F}_r$ implies $E(F) \in \mathcal{F}_{ch(r)}$ for every module F, then r is pseudohereditary.

<u>Proof</u>: (i). Let $F \in \mathcal{F}_r$. Since r is pseudohereditary, we have $K = F \cap ch(r)(E(F)) \in \mathcal{T}_r$. Thus $K = 0$ and $ch(r)(E(F)) = 0$.

(ii) Let $M \in R$-mod and $N \subseteq ch(r)(M)$. Consider the following commutative diagram



Now r is a radical and $N/r(N) \in \mathcal{F}_r$ implies $E(N/r(N)) \in \mathcal{F}_{ch(r)}$. On the other hand $N/r(N) = h(N/r(N)) \subseteq h(ch(r)(M)/r(N)) = h(ch(r)(M/r(N))) \subseteq ch(r)(E(N/r(N))) = 0$. Thus $N \in \mathcal{T}_r$.

<u>Proposition 1.4.</u>

(i) Every hereditary preradical is pseudohereditary.

(ii) Every faithful preradical is pseudohereditary.

(iii) If r is a cohereditary preradical, then r is pseudohereditary if and only if r is hereditary.

(iv) If ch(r) is hereditary, then r is pseudohereditary.

(v) If R is left hereditary, then r is pseudohereditary implies ch(r) is so.

(vi) If r_i, $i \in I$ is a family of preradicals, then $\bigcap_{i \in I} r_i$

is pseudohereditary provided each r_i is so.

(vii) If r is a preradical, then $\bigcap\{s, r \le s$, s pseudohereditary preradical$\}$ ($\bigcap\{s, r \le s$, s pseudohereditary radical$\}$) is the least pseudohereditary preradical (pseudohereditary radical) containing r.

(viii) If r is pseudohereditary, then $\bar{r}$ is so.

Proof follows immediately from Definition 1.1 and Proposition 1.2.

The next proposition is a modification of the well known result for hereditary radicals (see Jans [5]).

Proposition 1.5. Let r be a pseudohereditary radical. Then there is an injective ch(r)-torsionfree module Q such that $ch(r) = ch(p^{\{Q\}})$.

Proof: It is enough to put $Q = \prod_{A \in \mathcal{A}} E(A)$, where $\mathcal{A}$ is a representative set of cyclic r-torsionfree modules. As it is easy to see, Q is an injective ch(r)-torsionfree module, and therefore $ch(r) \le p^{\{Q\}}$. On the other hand it suffices to prove $\mathcal{T}_{p^{\{Q\}}} \subseteq \mathcal{T}_r$. For, let $T \in \mathcal{T}_{p^{\{Q\}}}$, $T \notin \mathcal{T}_r$. Without loss of generality we can assume that $T \in \mathcal{F}_r \cap \mathcal{T}_{p^{\{Q\}}}$ (take $T/r(T)$ instead T, if necessary). Therefore T contains a nonzero cyclic submodule C isomorphic to some $A \in \mathcal{A}$. Hence $Hom_R(C,Q) \neq 0$ and consequently $C \notin \mathcal{T}_{p^{\{Q\}}}$. On the other side $C \in \mathcal{T}_{p^{\{Q\}}}$ since p^{Q} is hereditary, a contradicition.

Corollary 1.6. Let r be a radical. Consider the following conditions:

(i) r is pseudohereditary,

(ii) there is an injective module Q such that $(0:Q) = r(R)$.

Then (i) implies (ii). Moreover, if R is a left hereditary ring then (ii) implies (i).

Proof: (i) implies (ii). By Proposition 1.5.

(ii) implies (i). By Proposition 1.4 (iv),(v).

§ 2. Pseudocohereditary preradicals.

Definition 2.1. A preradical r is said to be pseudocohereditary if for every module M and every epimorphism $M/h(r)(M) \longrightarrow A \quad A \in \mathcal{F}_r$.

Proposition 2.2. Let r be a preradical and Q be a faithful injective module. Then the following are equivalent:

(i) r is pseudocohereditary,

(ii) $h(r)(M) \subseteq C_r(N:M)$ implies $r(M/N) = (h(r)(M) + N)/N$ for every submodule N of a module M,

(iii) If I is an arbitrary index set and $Q^I/r(Q^I) \longrightarrow A$ an epimorphism, then $A \in \mathcal{F}_r$.

Proof: (i) implies (ii). Suppose $N \subseteq M$ and $h(r)(M) \subseteq C_r(N:M)$. Consider the natural epimorphism $M/h(r)(M) \longrightarrow M/(h(r)(M) + N)$.

According to (i) $(M/N)/((h(r)(M) + N)/N) \cong M/(h(r)(M) + N) \in \mathcal{F}_r$, and hence $r(M/N) \subseteq (h(r)(M) + N)/N$. The converse inclusion is obvious.

(ii) implies (i). If $M \in$ R-mod, $h(r)(M) \subseteq K \subseteq M$ and $M/h(r)(M) \longrightarrow M/K$ is a natural epimorphism, then we have $r(M/K) = (h(r)(M) + K)/K = 0$ by (ii).

(i) implies (iii). Obvious.

(iii) implies (i). Let A, $M \in$ R-mod and $g: M/h(r)(M) \longrightarrow A$ be an epimorphism. There is an epimorphism $f: F \longrightarrow M$ with F free. Since Q is faithful $p^{\{Q\}}(F) = 0$, and hence $F \overset{i}{\hookrightarrow} Q^J$ for some index set J. Further, i induces the inclusion $\bar{i}$: $:F/h(r)(F) \longrightarrow Q^J/h(r)(Q^J)$. Now consider the push-out diagram

$$\begin{array}{ccc} Q^J/r(Q^J) & \xrightarrow{\quad h \quad} & C \\ \uparrow{\scriptstyle \bar{i}} & & \uparrow{\scriptstyle j} \\ F/h(r)(F) & \xrightarrow[\quad g\circ \bar{f} \quad]{} & A \end{array}$$

where $\bar{f}:F/h(r)(F) \longrightarrow M/h(r)(M)$ is an epimorphism induced by f. As it is easy to see j is a monomorphism and h an epimorphism. According to (iii) $C \in \mathcal{T}_r$, and hence $A \in \mathcal{T}_r$.

Proposition 2.3. Let r be a preradical. Then:

(i) if r is pseudocohereditary and $0 \longrightarrow K \longrightarrow A \longrightarrow B \longrightarrow 0$ is a cover of B, then $B \in \mathcal{T}_r$ implies $A \in \mathcal{T}_{h(r)}$

(ii) if r is pseudocohereditary and $0 \longrightarrow K \hookrightarrow P \longrightarrow B \longrightarrow 0$ is an arbitrary projective presentation, then $B \in \mathcal{T}_r$ implies $h(r)(P) + K = P$,

(iii) if R is left perfect, r pseudocohereditary and $C(P) \xrightarrow{g_P} P$ a projective cover of P, then $P \in \mathcal{T}_r$ implies $C(P) \in \mathcal{T}_{h(r)}$,

(iv) if R is left hereditary, r pseudocohereditary and $B \in \mathcal{T}_r$, then there is a projective presentation $0 \longrightarrow K \longrightarrow P \longrightarrow B \longrightarrow 0$ with $P \in \mathcal{T}_{h(r)}$,

(v) if r is an idempotent preradical such that for each $B \in \mathcal{T}_r$ there is a projective presentation $0 \longrightarrow K \hookrightarrow P \longrightarrow B \longrightarrow 0$ with $P = K + h(r)(P)$, then r is pseudocohereditary.

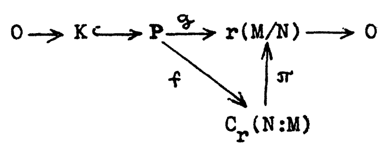
Proof: (i). If $0 \longrightarrow K \hookrightarrow A \longrightarrow B \longrightarrow 0$ is a cover of B and $B \in \mathcal{T}_r$, then $A/K = r(A/K) = (h(r)(A) + K)/K$ implies $A = h(r)(A) + K$, and hence $A \in \mathcal{T}_{h(r)}$, since K is superfluous in A.

(ii). This can be done in a similar fashion as in (i).

(iii). It follows immediately from (i).

(iv). Let $B \in \mathcal{T}_r$ and $0 \longrightarrow K \overset{f}{\hookrightarrow} P \overset{g}{\longrightarrow} B \longrightarrow 0$ be an arbitrary projective presentation. Since r is pseudocohereditary $h(r)(P) + K = P$ due to (ii). Now R is left hereditary and therefore $h(r)(P)$ is projective. Thus $h(r)(P) \in \mathcal{T}_{h(r)}$ and $g(h(r)(P)) = g(P) = B$.

(v). Suppose $N \subseteq M$ and $h(r)(M) \subseteq C_r(N:M)$ and consider the following commutative diagram



where the row is a projective presentation of $r(M/N)$ such that $K + h(r)(P) = P$ and π is a natural epimorphism. Now $r(M/N) =$ $= g(h(r)(P)) = \pi(f(h(r)(P))) \subseteq \pi(h(r)(M)) = (h(r)(M) + N)/N$ and consequently $r(M/N) = (h(r)(M) + N)/N$.

<u>Proposition 2.4</u>.

(i) Every cohereditary preradical is pseudocohereditary.

(ii) Every cofaithful preradical is pseudocohereditary.

(iii) If r is a hereditary preradical, then r is pseudocohereditary if and only if r is cohereditary.

(iv) If h(r) is cohereditary, then r is pseudocohereditary.

(v) If R is left hereditary, and r a pseudocohereditary preradical, then h(r) is cohereditary.

(vi) If r_i, $i \in I$ is a family of preradicals, then $\sum_{i \in I} r_i$ is pseudocohereditary provided each r_i is so.

(vii) If r is a preradical, then $\sum\{s, s \leq r$, s-pseudocohereditary preradical$\}$ ($\sum\{s, s \leq r$, s-pseudocohereditary idempotent preradical$\}$) is the largest pseudocohereditary (pseudocohereditary idempotent) preradical contained in r.

(viii) If r is pseudocohereditary, then $\tilde{r}$ is so.

Proof follows immediately from Definition 2.1 and Proposition 2.2.

Proposition 2.5. Let R be either left hereditary or left perfect and r be a pseudocohereditary idempotent preradical. Then $h(r) = h(p_{\{P\}})$ for some h(r)-torsion projective module P.

Proof: Let $\mathcal{A}$ be a representative set of cocyclic r-torsion modules and P be the direct sum of projective h(r)-torsion presentations of modules from $\mathcal{A}$ (the existence of P follows from Proposition 2.3(iii),(iv)). As it is easy to see P is a projective h(r)-torsion module, and therefore $p_{\{P\}} \leq h(r)$. On the other hand it suffices to show that $\mathcal{F}_{p_{\{P\}}} \subseteq \mathcal{F}_r$. For, let $F \in \mathcal{F}_{p_{\{P\}}}$ and $F \notin \mathcal{F}_r$. Without loss of generality we can assume that $F \in \mathcal{T}_r \cap \mathcal{F}_{p_{\{P\}}}$ (take r(F) instead F, if necessary). If C is a nonzero cocyclic factormodule of F, then $C \cong A$ for some $A \in \mathcal{A}$. Hence $Hom_R(P,C) \neq 0$ and $C \notin \mathcal{F}_{p_{\{P\}}}$. On the other hand $C \in \mathcal{F}_{p_{\{P\}}}$ since $p_{\{P\}}$ is cohereditary, a contradiction.

Corollary 2.6. Let r be an idempotent preradical for R-mod, where R is a left hereditary ring. Then the following are equivalent:

(i) r is pseudocohereditary,

(ii) there is a projective module P such that $r(M) = p_{\{P\}}(M)$ for every injective module M.

Proof: (i) implies (ii). By Proposition 2.5.

(ii) implies (i). By Proposition 2.4(iv),(v).

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