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A NOTE ON RING EPIMORPHISMS AND POLYNOMIAL INDENTITIES

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Abstract: The following question is considered. If A is a ring (associative) satisfying a polynomial identity (or a family of identities) and $f:A \longrightarrow B$ is an epimorphism, must B also satisfy the given identity (or family of identities)? It is well-known that the question has an affirmative answer when the identity in question is $xy = yx$. In this paper, among others, the standard identities and the identities of the form $x^m = x^n$ are treated, both for rings with identity elements and for rings in general.

Key words: Epimorphism, polynomial identity.

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Introduction. An epimorphism in a category is a morphism f with the property that if $gf = hf$ for morphisms g, h, then $g = h$. (Here gf means "first f, then g".) In categories of rings, epimorphisms need not be surjective, but $f:A \rightarrow B$ is an epimorphism exactly when the inclusion $Im(f) \longrightarrow B$ is an epimorphism, so in treating non-surjective epimorphisms we often need only to consider those of the form $X \longrightarrow Y$, $x \mapsto X$. Under such circumstances, Y is called an epimorphic extension of X.

All rings discussed in this paper will be associative, and we shall work in two categories: the category of all rings and the category of rings with identity elements. The latter, to avoid any ambiguity attaching to multiple uses of the word "identity", will be called <u>unital</u> <u>rings</u>.

If (in either of our categories) A is a subring of B, the <u>dominion</u> dom(A,B) of A in B is the subring

$$\{b \in B \mid (\forall C)(\forall g,h:B \to C)(g(a) = h(a)\ \forall a \in A \Rightarrow g(b) = h(b)\} .$$

Thus B is an epimorphic extension of A if and only if dom(A,B) = B. In the category of all rings, dom(A,B) consists of all elements of B of the form $a + XPY$, where $a \in A$, X is a row vector (of suitable size) over B, Y is a column vector over B and P is a matrix over the standard unital extension of A, such that XP and PY are matrices over A. In the category of unital rings, dom(A,B) consists of all XPY where everything is as above except that P can be any matrix over A itself. (For details see, e.g. Isbell [3].) Hence if B is an epimorphic extension of A, then B = A + AB (or, in the unital case, B + AB).

It is well-known that every epimorphic extension of a commutative unital ring is commutative (see, e.g., 9 , Proposition 1.3). Because of the intimate connection between epimorphisms of rings and epimorphisms of unital rings (see, e.g., [2] § 1 or [3] § 1) we can easily deduce that epimorphic extensions (and thus arbitrary epimorphisms) preserve commutativity in the category of rings. (A nice direct proof of this result has been given by Bulaszewska and Krempa ([1], Theorem 1)).

It is therefore natural to ask for which polynomial identities f = 0 (families Φ of polynomial identities) it is true that if a ring A satisfies $f = 0$ (satisfies each identity in Φ) then the same is true of every epimorphic extension of A. This is equivalent to asking when the variety generated by $f = 0$ (by Φ) is closed under epimorphic extensions or, equivalently, under arbitrary epimorphisms.

In this note we shall look at some polynomial identities from this point of view. In §1 we work in the category of all rings. Here we show that a large family of "composite" polynomial identities fail to be preserved by epimorphisms. These include all proper powers of all standard identities. Some information is also obtained about the standard identities themselves, the identities $x^m = x^n$ and a couple of others. In §2, we turn our attention to the category of unital rings, and here give a complete account of the epimorphic behaviour of the standard identities and the identities $x^m = x^n$.

We shall use the following notation: Z denotes the ring of integers, Z_n the ring of integers mod n, $S_n(x_1,x_2,\dots,x_n)$ the standard polynomial $\sum (-1)^{\sigma} x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(n)}$; $[x,y] = xy - yx$; var $(f_1=0,\dots,f_n=0)$, var(R), var (C) denote, respectively, the variety generated by a set $\{f_1=0,\dots,f_n=0\}$ of identities, a ring R and a class C of rings.

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1. The general case. In this section we shall work in the category of all rings. We begin by proving a result which enables us to present many examples of polynomial identities

which are not preserved by epimorphisms. The construction used in the proof was suggested by an example of Isbell [3] (p. 268).

Proposition 1.1. Let A be an idempotent ring, S a semigroup with identity. Let A[S] denote the corresponding semigroup ring. Then $\begin{bmatrix} A[S] & A[S] \\ A[S] & A[S] \end{bmatrix}$ is an epimorphic extension of $\begin{bmatrix} A & 0 \\ A[S] & A \end{bmatrix}$.

Proof. For brevity, let $R = \begin{bmatrix} A & 0 \\ A[S] & A \end{bmatrix}$ and $T = \begin{bmatrix} A[S] & A[S] \\ A[S] & A[S] \end{bmatrix}$.

We denote by $[a]_{rs}$ the matrix whose (r,s) entry is a and whose others are all 0.

Let a be any element of A. Then sice $A = A^2$, we can write $a = \sum u_i v_i w_i$, where $u_i, v_i, w_i \in A$. We then have

$$[a]_{12} = [\sum u_i v_i w_i]_{12} = \sum [u_i v_i w_i]_{12} = \sum [u_i]_{12} [v_i]_{21} [w_i]_{12},$$

where $[u_i]_{12}$ and $[w_i]_{12} \in T$, $[v_i]_{21} \in R$, $[u_i]_{12}[v_i]_{21} = [u_i v_i]_{11} \in R$ and $[v_i]_{21}[w_i]_{12} = [v_i w_i]_{22} \in R$. Hence $[a]_{12} \in \mathrm{dom}(R,T)$. Now let α be in S and let $a = \sum b_j c_j, b_j, c_j \in A$. Then

$$[a\alpha]_{11} = [\sum b_j c_j \alpha]_{11} = \sum [b_j c_j \alpha]_{11} = \sum [b_j]_{12} [c_j \alpha]_{21}$$

where $[b_j]_{12}$ (as shown above) is in dom(R,T) and $[c_j\alpha]_{21}$ is in R. Thus $[a\alpha]_{11} \in \mathrm{dom}(R,T)$. Similarly $[a\alpha]_{22} = \sum [b_j \alpha]_{21} [c_j]_{12} \in \mathrm{dom}(R,T)$. Finally,

$$[a\alpha]_{12} = \sum [u_i v_i w_i \alpha]_{12} = \sum [u_i]_{12} [v_i \alpha]_{21} [w_i]_{12} \in \mathrm{dom}(R,T).$$

It follows that $\mathrm{dom}(R,T) = T$. //

In the following theorem all variables are distinct.

Theorem 1.2. If Z satisfies the identities

$$(*) \quad f_1=f_1(x_{11},\dots,x_{1n_1}) = 0,\dots,f_m=f_m(x_{m1},\dots,x_{mn_m}) = 0 \dots.$$

and if all zerorings satisfy $g = g(x_1,\dots,x_m) = 0$, then $g(f_1,\dots,f_m) = 0$ is not preserved by epimorphisms. The same conclusion holds if some Z_n satisfies $(*)$ and there is at least one Z_n-algebra which does not satisfy $g(f_1,\dots,f_m) = 0$.

Proof. Let $R = Z$ or Z_n, as appropriate, and let $R\{X\} = R\{x_1,x_2,x_3,\dots\}$ be the polynomial ring over R in $\aleph_0$ non-commuting indeterminates $x_1,x_2,\dots$. Then $\begin{bmatrix} R & 0 \\ R\{X\} & R \end{bmatrix}$ satisfies $g(f_1,\dots,f_m) = 0$, since $\begin{bmatrix} R & 0 \\ R\{X\} & R \end{bmatrix} \Big/ \begin{bmatrix} 0 & 0 \\ R\{X\} & 0 \end{bmatrix} \cong R \oplus R$ satisfies $f_1 = 0,\dots,f_m = 0$ and $\begin{bmatrix} 0 & 0 \\ R\{X\} & 0 \end{bmatrix}$ satisfies $g = 0$.

But by Proposition 1.1, $\begin{bmatrix} R & 0 \\ R\{X\} & R \end{bmatrix}$ has $\begin{bmatrix} R\{X\} & R\{X\} \\ R\{X\} & R\{X\} \end{bmatrix}$ as an epimorphic extension, while the latter contains $\bar{x}_i = \begin{bmatrix} x_i & 0 \\ 0 & x_i \end{bmatrix}$ for each i. But then $g(f_1(\bar{x}_{11},\dots,\bar{x}_{1n_1}),\dots,f_m(\bar{x}_{m1},\dots,\bar{x}_{mn_m})) = \mathrm{diag}(g(f_1,\dots,f_m)) \neq 0$. //

Taking account of the commutativity of Z and the fact that Z_2 satisfies $x^m = x^n$ for every m, n, we get the following examples.

Examples 1.3. The following identities are not preserved by epimorphisms.

(i) $[x,y][z,w] = 0$;

(ii) $[x,y]^n = 0$, $n > 1$;

(iii) $[[x,y],[z,w]] = 0$;

(iv) $[x,y]^m - [x,y]^n = 0$, $m,n > 1$;

(v) $S_m(x_1,\dots,x_m)S_n(y_1,\dots,y_n) = 0$, $m,n > 1$;

(vi) $S_m(x_1,\dots,x_m)^n = 0$, $m,n > 1$;

(vii) $S_m(S_{n_1},\dots,S_{n_m}) = 0$, $m,n_1,\dots,n_m > 1$;

(viii) $(x^m - x^n)^k = 0$, $n \geq 1$, $k > 1$.

The identities described in (i) - (vii) can be regarded as generalizations of commutativity. In view of the "classical" result that commutativity is preserved by epimorphisms, they are of particular interest, and suggest the problem of finding epimorphically closed varieties properly containing the variety of commutative rings. There is a gap in the list above: we have said nothing of the standard identities themselves when the degree is > 2. We shall partially remedy this in our next result, but we need some notation first.

For any ring A, we denote by $M_n(A)$ the ring of n x n matrices over A, and by $\Lambda_n(A)$ the subring consisting of matrices whose non-zero entries occur only in the first column and on the main diagonal.

Theorem 1.4. The standard identity $S_m(x_1,\dots,x_m) = 0$ is not preserved by epimorphisms if $m > 3$.

Proof. Consider $\Lambda_k(F)$, where F is a field. In examining this ring with regard to standard identities, it is enough to check the matrix units e_{ij}. The relevant ones are $e_{11}, e_{21}, \dots$ $\dots e_{k1}, e_{22}, e_{33}, \dots, e_{kk}$. The longest non-zero products without repetitions are $e_{ii}e_{i1}e_{11}$, $i = 2,\dots,k$. It follows that

$\Lambda_k(F)$ satisfies $S_4(x_1,x_2,x_3,x_4) = 0$. But for any j, we have

$$e_{1j} = e_{1j}e_{j1}e_{1j} = \underset{\substack{\shortparallel \\ e_{11}\,e_{1j}}}{(e_{1j}e_{j1})e_{1j}} = \underset{\substack{\shortparallel \\ e_{1j}\,e_{jj}}}{e_{1j}(e_{j1}e_{1j})}$$

so $e_{1j} \in \mathrm{dom}(\Lambda_k(F), M_k(F))$ and then for any i, j, it follows that $e_{ij} = e_{i1}e_{1j} \in \mathrm{dom}(\Lambda_k(F), M_k(F))$, so $M_k(F)$ is an epimorphic extension of $\Lambda_k(F)$. (Compare this with Example 2.6 of Isbell [3]; it is asserted there that $M_k(F)$ is an epimorphic extension of the subring $\Lambda_k(F)$ consisting of matrices (a_{ij}) for which $a_{ii} = a_{jj}$ for every $i,j > 1$, but this is not clear.) Now as shown, $\Lambda_k(F)$ satisfies $S_4(x_1,x_2,x_3,x_4) = 0$, and therefore also $S_m(x_1,x_2,\ldots,x_m) = 0$ for any $m > 3$. But if $k \geq m$, then any polynomial identity satisfied by $M_k(F)$ must have degree $\geq 2k > k \geq m$. (See, e.g. Procesi [8], p. 22.) Hence for $m \geq 4$, the standard identity of degree m is not preserved by epimorphisms. //

The case of the standard identity of degree three remains open; to settle it, as we note in §2, non-unital rings must be used.

We next examine the identities $x^m = x^n$, $m > n$.

<u>Theorem 1.5</u>. (i) The identity $x^m = x$ is preserved by epimorphisms for every m.

(ii) If $n \geq 2$ and $m - n$ is even then the identity $x^m = x^n$ is not preserved by epimorphisms.

<u>Proof</u>. (i) Rings satisfying $x^m = x$ are regular, and therefore, as we showed in [2], have no proper epimorphic extensions.

(ii) Let $m > n \geq 2$ with $m - n$ even. Now for any $k, \Lambda_k(Z_2)$

has an ideal I (the set of zero-diagonal matrices) satisfying $xy = 0$ and $\Lambda_k(Z_2)/I \cong Z_2 \oplus \dots \oplus Z_2$ satisfies $x^2 = x$. Hence $\Lambda_k(Z_2)$ satisfies $(x^2 - x)^2 = 0$, and therefore (since it has characteristic 2) $x^4 = x^2$. Consequently the ring satisfies the identities

$$x^2 = x^4 = x^2x^2 = x^2x^4 = x^6 = x^2x^4 = x^2x^6 = x^8 = \dots = x^{2j} \forall j.$$

In particular, since $m - n$ is even, $\Lambda_k(Z_2)$ satisfies $x^2 = x^{m-n+2}$ and hence also

$$x^n = x^2x^{n-2} = x^{m-n+2}x^{n-2} = x^m.$$

But as we saw in the proof of Theorem 1.4, $M_k(Z_2)$ is an epimorphic extension of $\Lambda_k(Z_2)$ and moreover, if k is large enough, $M_k(Z_2)$ does not satisfy an identity of degree m; in particular it does not satisfy $x^m = x^n$. //

As we noted above, no examples appear to be known of non-trivial epimorphically closed varieties properly containing the commutative rings. There are non-commutative epimorphically closed varieties, however.

Example 1.6. For each n, the identity $x_1x_2 \dots x_n = 0$ is preserved by epimorphisms, since nulpotent rings have no proper epimorphic extensions ([3], p. 267).

Another example worth mentioning, consisting of nil, but not necessarily nilpotent rings, is the following

Example 1.7. $\mathrm{var}(x^2 = 0, 2x = 0)$ is epimorphically closed. Any ring A satisfying these two identities is commutative; so therefore is any epimorphic extension, B. The elements of $B = A + AB$ have the form $b = a + \sum a'b'$, $a, a' \in A$, $b' \in B$. Now $2b = 2a + \sum (2a')b' = 0$, and B is commutative, so $b^2 = a^2 + (\sum a'b')^2 = a^2 + \sum (a')^2(b')^2 = 0$.

A variety $\mathcal{V}$ is the <u>product</u> of varieties $\mathcal{V}_1,\dots,\mathcal{V}_n$ (notation: $\mathcal{V} = \mathcal{V}_1 \times \dots \times \mathcal{V}_n$) if every ring A in $\mathcal{V}$ is uniquely expressible as a direct sum $A = A_1 \oplus \dots \oplus A_n$, where $A_i \in \mathcal{V}_i$ for each i. An example of a product is the variety $\mathcal{V}$ generated by the class of all zerorings ($xy = 0$) and a finite set $\mathcal{F}$ of finite fields: $\mathcal{V} = \mathrm{var}(xy = 0) \times \mathrm{var}(\mathcal{F})$ (see Lee Sin-Min [5]).

Both the component subvarieties are epimorphically closed in this case.

<u>Proposition 1.8</u>. For any finite set $\mathcal{F}$ of finite fields, $\mathrm{var}(xy = 0) \times \mathrm{var}(\mathcal{F})$ is closed under epimorphisms.

<u>Proof</u>. Let $A = A_1 \oplus A_2$, where $A_1 \in \mathrm{var}(xy = 0)$ and $A_2 \in \mathrm{var}(\mathcal{F})$, and let B be an epimorphic extension of A. Then everything is commutative and we have

$$B = A + AB = A_1 + A_2 + (A_1+A_2)B = (A_1+A_1B) + (A_2+A_2B).$$

Now $A_1B = A_1(A_1+A_1B) + A_1(A_2+A_2B) = 0$, so A_1 is an ideal of B. Also, $A_2B = A_2(A_1+A_1B) + A_2(A_2+A_2B) = A_2+A_2B$, since $A_2A_1 = 0$ and $A_2^2 = A_2$. Thus $B = A_1+A_2B$. Now we have the following commutative diagram, where every map is an epimorphism:

$$\begin{array}{ccc}
A & \xrightarrow{\text{inc.}} & B \\
\downarrow{\scriptstyle\text{nat.}} & & \downarrow{\scriptstyle\text{nat.}} \\
A/A_1 & \xrightarrow{\text{inc.}} & B/A_1 = (A_1+A_2B)/A_1 \; . \\
\| & & \\
A_1 \oplus A_2/A_1 & & \\
\| & & \\
A_2 & &
\end{array}$$

Since A_2 is in $\mathrm{var}(\mathcal{F})$, it has no proper epimorphic extensions [2] and thus $A/A_1 = (A_1+A_2B)/A_1$. This means that $A_1 \oplus A_2 = A = A_1+A_2B = B$. Thus A can have no proper epimorphic extensions and therefore var$(xy = 0)$x var$(\mathcal{F})$ is closed under epimorphisms. //

As a special case (see [4]) we have

Corollary 1.9. For every n, var$((x+y)^n = x^n + y^n;\ (xy)^n = xy = x^ny^n)$ is closed under epimorphisms. //

It would be interesting to know whether, in general, the join of two epimorphiccally closed varieties is epimorphically closed.

2. Unital rings. In this section we consider the standard identities and the identities of the form $x^m = x^n$ for unital rings. It is possible, in this context, to say exactly which of these identities are preserved by epimorphisms.

Theorem 2.1. For unital rings, the standard identity $S_m(x_1,x_2,\ldots,x_m)$ is preserved by epimorphisms if and only if $m = 2$ or 3.

Proof. As Leron and Vapne ([6], p. 130) have noted, any unital ring which satisfies the standard identity of degree $2n + 1$ must also satisfy the standard identity of degree $2n$. Thus if A satisfies $S_3(x,y,z) = 0$, then A satisfies $S_2(x,y) = 0$, i.e. A is commutative. But then every epimorphic extension B of A is also commutative (see Introduction) and must therefore satisfy $S_3(x,y,z) = 0$. The rings used in the proof of Theorem 1.4 are unital, so the argument used there shows that the standard identities of degree ≥ 4 are not preserved by epimorphisms. //

For unital rings, odd-degree standard identities are largely irrelevant, and this simplifies the epimorphic preservation problem for standard identities in general. Something similar happens with the identities $x^m = x^n$. We need some lemmas to prepare for our theorem concerning these identities.

Lemma 2.2. (i) Let p be an odd prime. If Z_{p^k} satisfies $x^m = x^n$, then $m - n$ is even.

(ii) If Z_{2^k} satisfies $x^m = x^n$, then $m - n$ is even or $k = 1$.

Proof. (i) If Z_{p^k} satisfies $x^m = x^n$, then in particular $(p^k-1)^m \equiv (p^k-1)^n \bmod p^k$. But

$$(p^k-1)^m-(p^k-1)^n \equiv (-1)^m-(-1)^n = \begin{cases} 0 \text{ if } m \text{ and } n \text{ are even} \\ 2 \text{ if } m \text{ is even and } n \text{ is odd} \\ -2 \text{ if } m \text{ is odd and } n \text{ is even} \\ 0 \text{ if } m \text{ and } n \text{ are odd} \end{cases}.$$

Thus for odd p, we have $(p^k-1)^m \equiv (p^k-1)^n \bmod p$ if and only if m and n are both odd or both even, i.e. m-n is even. For p = = 2 and $k > 1$ we have the same conclusion while for $p^k = 2$ we always have equivalence. //

Lemma 2.3. If a unital ring R satisfies $x^m = x^n$ with $m > n$ then its additive group R^+ is bounded.

Proof. Let e be the identity element. Then $2^m e = (2e)^m = (2e)^n = 2^n e$, so $(2^m - 2^n)e = 0$ and hence $(2^m-2^n)a = (2^m-2^n)ea = 0$ for every $a \in R$. //

Lemma 2.4. For unital rings, $\mathrm{var}(x^m = x^n) = \mathcal{V}_{p_1} \times \dots \times \mathcal{V}_{p_n}$ where $p_1,\dots,p_n$ are primes, and $\mathcal{V}_{p_i} = \{A \in \mathrm{var}(x^m - x^n) \mid A^+ \text{ is a } p_i\text{-group}\}$, for each i.

Proof. By Lemma 2.3, every ring in $\mathrm{var}(x^m = x^n)$ has a bounded additive group. The set of prime divisors of additive orders of elements of rings in $\mathrm{var}(x^m = x^n)$ is finite, since otherwise $\mathrm{var}(x^m = x^n)$ would contain a ring of the form $\prod_{p \in S} R(p)$, where S is infinite and $R(p)^+$ is a non-zero p-group. But such a ring would not be bounded. //

<u>Lemma 2.5</u>. Let $\mathrm{var}(x^m = x^n) = \mathcal{V}_{p_1} \times \dots \times \mathcal{V}_{p_n}$ for primes $p_1, \dots, p_n$. Then $\mathrm{var}(x^m = x^n)$ is closed under epimorphisms if and only if each $\mathcal{V}_{p_i}$ is.

<u>Proof</u>. (i) Assume each $\mathcal{V}_{p_i}$ is epimorphically closed. If $A \in \mathrm{var}(x^m = x^n)$ and B is an epimorphic extension of A, let $A = A_{p_1} \oplus \dots \oplus A_{p_n}$. Then since $B = AB$ (see Introduction), the order of each element of B is a product of powers of $p_1, \dots, p_n$ and we can therefore write $B = B_{p_1} \oplus \dots \oplus B_{p_n}$. For each i, every map in the diagram

$$\begin{tikzcd} A \arrow[r, "\text{inc.}"] \arrow[d, "\text{proj.}"'] & B \arrow[d, "\text{proj.}"] \\ A_{p_i} \arrow[r, "\text{inc.}"'] & B_{p_i} \end{tikzcd}$$

is an epimorphism, whence it follows that $B_{p_i} \in \mathcal{V}_{p_i}$. Hence B is in $\mathrm{var}(x^m = x^n)$ and the latter is closed under epimorphisms.

(ii) Assume $\mathrm{var}(x^m = x^n)$ is closed under epimorphisms. If $A \in \mathcal{V}_{p_i}$ and B is an epimorphic extension of A, then since $B = AB$, B^+ is a p_i-group. Also B is in $\mathrm{var}(x^m = x^n)$ (since A is) so B is in $\mathcal{V}_{p_i}$. //

This completes the preliminaries. The following two propositions provide all the information needed for our theorem.

Proposition 2.6. If a unital ring R satisfies $x^m = x^n$, with $m > n$, then either R has characteristic 2 or $m - n$ is even.

Proof. Let e be the identity element of R. By Lemmas 2.3, 2.4 and 2.5, we may assume that the characteristic of R is p^k for some prime p. Then the subring $\langle e \rangle$ generated by e is isomorphic to Z_{p^k} and satisfies $x^m = x^n$. By Lemma 2.2, if p is odd or $k > 1$, $m - n$ must be even. //

Proposition 2.7. Let R be a unital ring satisfying $x^m = x^n$ where $m - n$ is odd. Then R has characteristic 2 and has no proper epimorphic extensions.

Proof. Proposition 2.6 takes care of the characteristic. For any $a \in R$, we have (denoting the identity element by e again),

$$(e+a)^m = e+ma + (\text{terms in } a^2,\dots,a^{m-1}) + a^m$$
$$= (e+a)^n = e+na + (\text{terms in } a^2,\dots,a^{n-1}) + a^n$$

and therefore, since $a^m = a^n$,

$$ma + (\text{terms in } a^2,\dots,a^{m-1}) = na + (\text{terms in } a^2,\dots,a^{n-1}).$$

Since $m - n$ is odd, just one of m, n is even, and we lose no generality by assuming that n is even and m is odd. But then (characteristic = 2) $ma = a$ and $na = 0$, so we have

$$a = (\text{terms in } a^2,\dots,a^{n-1},\dots,a^{m-1}).$$

Writing this as $a = \sum_{i=2}^{m-1} k_i a^i$ and writing $f(x) = \sum_{i=2}^{m-1} k_i x^i$, we have a polynomial identity $f(x) = x$ satisfied by R. If $f = 0$, then $R = 0$, while if $f \neq 0$, it follows from Theorem 13.2, p. 321 of Osborn [7], that R is periodic (with no nilpotent elements) and therefore regular. As shown in [2], R then has no proper epimorphic extensions. //

Theorem 2.8. For unital rings, the identity $x^m = x^n$ $(m>n)$ is preserved by epimorphisms if and only if either $n = 1$ or $m - n$ is odd.

Proof. Rings satisfying $x^m = x^n$ have no proper epimorphic extensions since they are regular [2]. If $m - n$ is odd, then by Proposition 2.7 we have a similar conclusion. The case $n>1$ and $m - n$ even can be settled as in the proof of Theorem 1.5, since all rings used there are unital. //

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