Ladislav Bican

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Ladislav BICAN

Abstract: The purpose of this note is to characterize a class of mixed abelian groups G having the property that each pure subgroup of G splits. For the groups of countable (torsionfree) rank the problem is solved completely.

Key words: Splitting group, generalized p-height, increasing p-height ordering, generalized p-sequence, p-rank.

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By the word "group" we shall always mean an additively written abelian group. If M is a subset of a group G, then $\langle M \rangle$ denotes the subgroup of G generated by M. If g is an element of infinite order of a mixed group G then $h_p^G(g)$ ($\tau^G(g)$) denotes the p-height (the characteristic) of g in the group G. The rank of a mixed group G with the maximal torsion subgroup T is the rank of the factor-group G/T.

In what follows we shall deal with a mixed group G with the maximal torsion subgroup T and $\overline{G}$ will denote the factor-group G/T. The bar over the elements will denote the elements from $\overline{G}$. We say that a set $M = \{a_\lambda \mid \lambda \in \Lambda\}$ of elements of G is a basis of G if the set $\overline{M} = \{\overline{a}_\lambda \mid \lambda \in \Lambda\}$ is a basis of $\overline{G}$, i.e. a maximal linearly independent subset of $\overline{G}$.

A sequence $g_0, g_1, \dots$ of elements of a mixed group G is said to be a p-sequence of g_0 if $pg_{i+1} = g_i$, $i = 0,1,\dots$. Let U be any torsionfree subgroup of a mixed group G and let $g \in G \setminus U$ be an element of infinite order. If $h_p^{G/U}(g+U) = \infty$ then every sequence $g = g_0, g_1, \dots$ of elements of G such that $p(g_{i+1}+U) = g_i+U$, $i = 0,1,\dots$, is called a generalized p-sequence of g with respect to U.

Let $M = \{a_\alpha \mid \alpha < \mu\}$ (μ is an ordinal number) be a well-ordered basis of a mixed group G. We define the generalized p-height $H_p^G(a_\alpha)$ of the element a_α as the p-height of $a_\alpha + \sum_{\beta<\alpha} \langle a_\beta \rangle$ in $G/\sum_{\beta<\alpha} \langle a_\beta \rangle$. The well-ordering on M is said to be an increasing p-height ordering if $H_p^G(a_\alpha) \leq H_p^G(a_\beta)$ whenever $\alpha \leq \beta < \mu$.

It is well-known (see [6]) that if H is a torsionfree group of finite rank and K its free subgroup of the same rank then the number $r_p(H)$ of summands $C(p^\infty)$ in H/K does not depend on the particular choice of K and this number is called the p-rank of H.

<u>Lemma 1</u>: Let $M = \{a_\lambda \mid \lambda \in \Lambda\}$ be a basis of a mixed group G with the torsion part T. Then G splits if and only if there are non-zero integers m_λ, $\lambda \in \Lambda$, such that

(1) $\tau^G(a) = \tau^{\bar{G}}(\bar{a})$ for each element $a \in \sum_{\lambda\in\Lambda} \langle m_\lambda a_\lambda \rangle$,

(2) for every prime p there is an increasing p-height ordering $\{m_\alpha a_\alpha \mid \alpha < \mu\}$ on $\tilde{M} = \{m_\lambda a_\lambda \mid \lambda \in \Lambda\}$ such that $H_p^G(m_\alpha a_\alpha) = n_\alpha < \infty$ if and only if $\alpha < \nu$ and for every $\alpha < \nu$ there exists an element $x_\alpha \in G$ such that $p^{n_\alpha}(x_\alpha + \sum_{\beta<\alpha} \langle m_\beta a_\beta \rangle) = m_\alpha a_\alpha + \sum_{\beta<\alpha} \langle m_\beta a_\beta \rangle$ and every element $m_\gamma a_\gamma$, $\nu \leq \gamma < \mu$, has a generalized p-sequence with respect to $U = \langle x_\alpha \mid \alpha < \nu \rangle$.

Proof: See [1; Theorem].

The definition of p-rank of a torsionfree group H (of arbitrary rank) can be found in [7]. In this note we shall need only the following result.

Lemma 2: If H is a torsionfree group, then $r_p(H) = 0$ if and only if $r_p(K) = 0$ for each pure subgroup K of H of finite rank.

Proof: See [8; Corollary 2].

Lemma 3: Let G be a mixed group with the torsion part T and p be a prime. Let $\{a_\alpha \mid \alpha < \mu\}$ be an increasingly p-height ordered basis of G such that $H_p^G(a_\alpha) = n_\alpha < \infty$ if and only if $\alpha < \nu$ and let $U = \langle x_\alpha \mid \alpha < \nu \rangle$ where $x_\alpha \in G$ are such that $p^{n_\alpha}(x_\alpha + \sum_{\beta<\alpha}\langle a_\beta \rangle) = a_\alpha + \sum_{\beta<\alpha}\langle a_\beta \rangle$. If the p-primary component T_p of T is a direct sum of a divisible and a bounded groups then every element a_γ, $\nu \leq \gamma < \mu$, has a generalized p-sequence with respect to U.

Proof: By hypothesis, $T_p = D \oplus V$ where D is divisible and $p^m V = 0$. Put $h_o = a_\gamma$ and assume that we have constructed the elements $h_o, h_1, \ldots, h_n$ in such a way that h_o+U, $h_1+U, \ldots$ $\ldots, h_n+U$ are of infinite p-height in G/U and $p(h_{i+1}+U) = h_i+U$, $i = 0,1,\ldots,n-1$.

Since h_n+U is of infinite p-height in G/U, there exist elements $h^{(s)} \in G$, $u^{(s)} \in U$, $s = 1,2,\ldots$, such that $p^{m+s}h^{(s)} = h_n+u^{(s)}$. Then $p^{m+1}(p^{s-1}h^{(s)}-h^{(1)}) = u^{(s)}-u^{(1)}$ and $p^{m+1}w^{(s)} = u^{(s)}-u^{(1)}$ for some $w^{(s)} \in U$, U being p-pure in G by [1, Lemma 4]. Consequently, $p^{s+1}h^{(s)}-h^{(1)}-w^{(s)} = d^{(s)}+v^{(s)}$, $d^{(s)} \in D$, $v^{(s)} \in V$. From the divisibility of D the existence of elements $d_s^{(s)} \in D$ follows, for which $p^{s-1}d_s^{(s)} = d^{(s)}$. Now, putting

$h_{n+1} = p^m h^{(1)}$ and $z_s = h^{(s)} - d_s^{(s)}$, we have $ph_{n+1} = p^{m+1}h^{(1)} = h_n + u^{(1)}$, $p^{m+s-1}z_s = p^m(h^{(1)} + w^{(s)} + v^{(s)}) = h_{n+1} + p^m w^{(s)}$ and the assertion follows easily.

Lemma 4: Let S be a pure subgroup of a mixed group G with the torsion part T. Let p be a prime and $a \in S$ be an element of infinite order, $\tilde{S} = S/S \cap T$, $\tilde{a} = a + S \cap T$. If $h_p^{\bar{G}}(\bar{a}) = h_p^G(a)$ then $h_p^S(a) = h_p^{\tilde{S}}(\tilde{a}) = h_p^G(a)$.

Proof: Obviously, $h_p^S(a) = h_p^G(a) = h_p^{\bar{G}}(\bar{a}) \geqq h_p^{\tilde{S}}(\tilde{a}) \geqq h_p^S(a)$, as desired.

Lemma 5: Let G be a mixed group of the form $G = \sum_{i=1}^{\infty}{}^{\oplus} \langle t_i \rangle \oplus A = T \oplus A$ where $\langle t_i \rangle$ is a cyclic group of order p^{ℓ_i}, $\ell_1 < \ell_2 < \dots$, and A is a torsionfree group of finite rank. If $r_p(A) > 0$ then G contains a non-splitting pure subgroup.

Proof: We shall divide the proof into several steps.

a) If A contains a rank one p-divisible pure subgroup B then $T \oplus B$ is pure in G and $T \oplus B$ contains a non-splitting pure subgroup by [2; Lemma 12].

b) If $\{a_1, a_2, \dots, a_n, a_{n+1}\}$ is an increasingly p-height ordered basis of A then there is $k \leqq n$ such that $H_p^A(a_i) < \infty$ for each $i = 1, 2, \dots, k$ and $H_p^A(a_i) = \infty$ for each $i = k+1, \dots, n+1$. Obviously, we can assume that $k = n$, since in the opposite case we can treat the pure closure B of $\langle a_1, a_2, \dots, a_k, a_{k+1} \rangle$ in A instead of A.

c) In view of a), b) and [1; Lemma 4] we can suppose that A contains no element of infinite p-height and that it has a basis $\{a_1, a_2, \dots, a_n, a\}$ such that $\langle N \rangle = \langle a_1, a_2, \dots, a_n \rangle$ is p-pure in A and $h_p^{A/\langle N \rangle}(a + \langle N \rangle) = \infty$, $h_p^A(a) = 0$. Thus, there are elements $b_i \in A$ with $p^{\ell_i} b_i = a + v_i$, $v_i \in \langle N \rangle$, $i = 1,2,\dots$. Put $s_i = b_i + t_i$, $i = 1,2,\dots$, $U = \langle N \cup \{s_1, s_2, \dots\} \rangle$ and $S = \{s \in G \mid ms \in U$ for some integer m, $(m,p) = 1\}$. Obviously, S is π'-pure in G where $\pi' = \pi \setminus \{p\}$, π being the set of all primes.

d) Now we are going to show that S is pure in G. Suppose, at first, that the equation $p^k x = u$, $u \in U$, has the solution x in G. Let $x = \sum_{i=1}^{r} \mu_i t_i + a'$, $a' \in A$, and $u = v + \sum_{i=1}^{r} \lambda_i s_i$, $v \in \langle N \rangle$. Then $\sum_{i=1}^{r} p^k \mu_i t_i + p^k a' = v + \sum_{i=1}^{r} \lambda_i b_i + \sum_{i=1}^{r} \lambda_i t_i$, and so ($G$ splits) $\sum_{i=1}^{r} p^k \mu_i t_i = \sum_{i=1}^{r} \lambda_i t_i$, $p^k a' = v + \sum_{i=1}^{r} \lambda_i b_i$. Hence $\lambda_i = p^k \mu_i + p^{\ell_i} \nu_i$ for some integer ν_i, $i = 1,2,\dots,r$. Let ℓ_j be such that $\ell_j \geq k$ and put $\nu = \sum_{i=1}^{r} \nu_i$, $u' = \sum_{i=1}^{r} \mu_i s_i + \nu p^{\ell_j - k} s_j$. Then $p^k u' = \sum_{i=1}^{r} \lambda_i s_i - \sum_{i=1}^{r} \nu_i (a + v_i) + \nu (a + v_j) = u - v - \sum_{i=1}^{r} \nu_i v_i + \nu v_j$. Further, $p^k(u' - x) = \nu v_j - v - \sum_{i=1}^{r} \nu_i v_i \in \langle N \rangle$ and $p^k v' = p^k(u' - x)$, $v' \in \langle N \rangle$, $\langle N \rangle$ being p-pure in A. So, $u = p^k x = p^k(u' - v')$ where $u' - v' \in U$.

Now the purity of S in G is easy to prove. If $p^k x = s$, $s \in S$, is solvable in G, then $ms = u \in U$ for some integer m, $(m,p) = 1$. So, there exist integers ϱ, σ with $m\varrho + p^k \sigma = 1$ and the preceding part yields the existence of $u' \in U$ such that $p^k u' = u$. Then $p^k(\varrho u' + \sigma s) = m\varrho s + p^k \sigma s = s$ and we are through.

e) Now we shall prove that $\langle t_j \rangle \cap S = 0$ for each $j = 1,2,\dots$. If $p^k t_j \in S$ for some $k < \ell_j$ then there exists a positive integer m relatively prime to p such that $mp^k t_j = v + \sum_{i=1}^{r} \lambda_i s_i = v + \sum_{i=1}^{r} \lambda_i b_i + \sum_{i=1}^{r} \lambda_i t_i$, $v \in \langle N \rangle$. We can clearly assume that $r \geqq j$. The above equality yields $\lambda_i = p^{\ell_i} \mu_i$, $i = 1, 2, \dots, r$, $i \neq j$, $mp^k = \lambda_j - p^{\ell_j} \mu_j$ and $0 = p^{\ell_j - k}(v + \sum_{i=1}^{r} \lambda_i b_i) = p^{\ell_j - k}(v + \sum_{i=1}^{r} \mu_i(a+v_i) + mp^k b_j) = (p^{\ell_j - k} \sum_{i=1}^{r} \mu_i + m)a+w$, $w \in \langle N \rangle$. Hence $p^{\ell_j - k} \sum_{i=1}^{r} \mu_i + m = 0$, $p^{\ell_j - k} \mid m$ - a contradiction showing that $\langle t_j \rangle \cap S = 0$.

f) Suppose now that the group S splits, $S = P \oplus B$, P torsion, B torsionfree. Obviously, there exists a positive integer k such that $p^k a_1, p^k a_2, \dots, p^k a_n, p^k a \in B$. Put $\tilde{N} = \{p^k a_1, p^k a_2, \dots, p^k a_n\}$ and take an index j such that $\ell_j > k$. For each $i > j$ the equality $p^{\ell_i} b_i = a+v_i$ yields $p^{\ell_j}(p^{\ell_i - \ell_j} b_i - b_j) = v_i - v_j = p^{\ell_j} w_i$, $w_i \in \langle N \rangle$, $\langle N \rangle$ being p-pure in A. Further, for each $i > j$ the equality $p^{k+\ell_i} b_i = p^k a + p^k v_i$, $v_i \in \langle N \rangle$, yields $p^{k+\ell_i} c_i = p^k a + p^k v_i$, $c_i \in B$, B being pure in G. Hence $p^{\ell_j}(p^{\ell_i - \ell_j} p^k c_i - p^k c_j) = p^k (v_i - v_j) = p^{\ell_j} p^k w_i$ and so $p^{\ell_i - \ell_j} p^k c_i = p^k c_j + p^k w_i$, B being torsionfree, $p^k w_i \in \langle \tilde{N} \rangle \subseteqq B$. We have shown that $p^k c_j + \langle \tilde{N} \rangle$ is of infinite p-height in $G/\langle \tilde{N} \rangle$. Similarly, the element $p^k b_j + \langle \tilde{N} \rangle$ is of infinite p-height in $G/\langle \tilde{N} \rangle$ and the same property has the element $p^k b_j - p^k c_j + \langle \tilde{N} \rangle$. On the other hand, $p^{\ell_j}(p^k b_j - p^k c_j) = 0$ shows that $p^k b_j - p^k c_j + \langle \tilde{N} \rangle$ lies in the torsion part $T + \langle \tilde{N} \rangle / \langle \tilde{N} \rangle \cong T$ of $G/\langle \tilde{N} \rangle$ and so $p^k b_j = p^k c_j \in B$. Consequently, $p^k t_j = p^k s_j - p^k b_j \in S$ - a contradiction (see e)) finishing the proof.

Definition: We say that a torsionfree group G belongs to the class $\mathcal{W}$ if for each prime p with $r_p(G) = 0$ each linearly independent subset N of G can be increasingly p-height ordered in such a way that $N = \{a_\alpha \mid \alpha < \mu\}$ and $H_p^G(a_\alpha) < \infty$

for each $\alpha < \mu$.

<u>Theorem 1</u>: Let G be a mixed group with the torsion part T such that $\bar{G} \in \mathcal{W}$. Then every pure subgroup of G splits if and only if

(i) G contains a basis M such that $\tau^{G}(a) = \tau^{\bar{G}}(\bar{a})$ for each element $a \in \langle M \rangle$ and

(ii) T_p is a direct sum of a divisible and a bounded groups for each prime p with $r_p(\bar{G}) > 0$.

<u>Proof</u>: Sufficiency. Let p be a prime such that $r_p(\bar{G}) = 0$. Since $\bar{G} \in \mathcal{W}$, there exists an increasing p-height ordering $\{\bar{a}_{\alpha}, \alpha < \mu\}$ on the basis $\bar{M}$ of $\bar{G}$ such that $H_p^{\bar{G}}(\bar{a}_{\alpha}) < \infty$ for each $\alpha < \mu$. In view of (i), $H_p^{G}(a_{\alpha}) < \infty$ for each $\alpha < \mu$.

Let p be a prime with $r_p(\bar{G}) > 0$ and let $\{a_{\alpha} \mid \alpha < \mu\}$ be an increasing p-height ordering on M such that $H_p^{G}(a_{\alpha}) = n_{\alpha} < \infty$ if and only if $\alpha < \nu$. By Lemma 3, each element a_{γ}, $\nu \leqq \gamma < \mu$, has a generalized p-sequence with respect to $U = \langle x_{\alpha} \mid \alpha < \nu \rangle$ where $x_{\alpha} \in G$ are such elements that $p^{n_{\alpha}}(x_{\alpha} + \sum_{\beta<\alpha} \langle a_{\beta} \rangle) = a_{\alpha} + \sum_{\beta<\alpha} \langle a_{\beta} \rangle$. Consequently, G splits by Lemma 1, $G = T \oplus A$.

Now let S be a pure subgroup of G and $N = \{a_{\lambda} \mid \lambda \in \Lambda\}$, be a basis of S. Then there exist non-zero integers m_{λ}, $\lambda \in \Lambda$, such that the basis $\tilde{N} = \{m_{\lambda} a_{\lambda} \mid \lambda \in \Lambda\}$ of S is contained in A. Hence $\tilde{N}$ satisfies condition (i) by Lemma 4.

If $r_p(\bar{G}) > 0$ then T_p is a direct sum of a divisible and a bounded groups by hypothesis. However, $(S \cap T)_p$ is pure in T_p by [2 ; Lemma 7] and $(S \cap T)_p$ is a direct sum of a divisible and a bounded groups by [2; Lemma 9].

Finally, suppose that $r_p(\bar{G}) = r_p(A) = 0$. Since $\bar{G} \in \mathcal{W}$ and $\tilde{N}$ is a linearly independent subset of A, $\tilde{N}$ can be increasingly p-height ordered in such a way that $\tilde{N} = \{ m_\alpha a_\alpha \mid \alpha < \mu \}$ and $H_p^A(m_\alpha a_\alpha) = H_p^G(m_\alpha a_\alpha) = H_p^S(m_\alpha a_\alpha) < \infty$ for each $\alpha < \mu$.

Similar arguments as in the first part of the proof show that S splits.

Necessity. Condition (i) is necessary by Lemma 1. Assume that G does not satisfy the condition (ii). Thus for a prime p with $r_p(\bar{G}) > 0$ the p-primary component T_p is not a direct sum of a divisible and a bounded groups. Without loss of generality we can suppose that T_p is reduced and that $G = T' \oplus B$ splits. Then $r_p(B) = r_p(\bar{G}) > 0$ and Lemma 2 yields the existence of a pure subgroup A of B of finite rank with $r_p(A) > 0$. Each basic subgroup of T_p is unbounded by [2; Lemma 11] and so T_p contains a subgroup T pure in T' having the form $T = \sum_{i=1}^{\infty} {}^{\oplus} \langle t_i \rangle$ where $\langle t_i \rangle$ is a cyclic group of order ℓ_i, $\ell_1 < \ell_2 < \dots$. An application of Lemma 5 finishes the proof.

<u>Corollary 1</u>: Let $G = T \oplus A$, T torsion, A torsionfree, be a splitting group such that $A \in \mathcal{W}$. Then every pure subgroup of G splits if and only if T_p is a direct sum of a divisible and a bounded groups for each prime p with $r_p(A) > 0$.

<u>Proof</u>: Clearly, G satisfies condition (i) of Theorem 1 by Lemma 1.

<u>Lemma 6</u>: Every countable torsionfree group G belongs to the class $\mathcal{W}$.

<u>Proof</u>: Let p be such a prime that $r_p(G) = 0$ and let M be an arbitrary linearly independent subset of G. Choose $a_1 \in M$ in such a way that $h_p^G(a_1) = \min\{h_p^G(a) \mid a \in M\}$. It is obvi-

ous that $h_p^G(a_1) < \infty$ (since $r_p(G) = 0$). Suppose that we have constructed the elements $a_1, a_2, \dots, a_n$ such that $H_p^G(a_1) \leqq \leqq H_p^G(a_2) \leqq \dots \leqq H_p^G(a_n) \leqq H_p^G(a)$ for each $a \in M \setminus \{a_1, a_2, \dots, a_n\}$ and $H_p^G(a_n) < \infty$. Choose $a_{n+1} \in M \setminus \{a_1, a_2, \dots, a_n\}$ such that $h_p^{G/V}(a_{n+1}+V) = \min\{ h_p^{G/V}(a+V) \mid a \in M \setminus \{a_1, a_2, \dots, a_n\}\}$ where $V = \langle a_1, a_2, \dots, a_n \rangle$. Using Lemma 2 we see that $H_p^G(a_{n+1}) = h_p^{G/V}(a_{n+1}+V) < \infty$. Obviously, this procedure yields an increasing p-height ordering $\{a_1, a_2, \dots\}$ on M (M is countable by hypothesis) such that $H_p^G(a_i) < \infty$ for each $i = 1, 2, \dots$.

Theorem 2: Every pure subgroup of a mixed group G of countable (finite) rank splits if and only if

(i) G contains a basis M such that $\tau^G(a) = \tau^{\bar{G}}(\bar{a})$ for each element $a \in \langle M \rangle$ and

(ii) T_p is a direct sum of a divisible and a bounded groups for each prime p with $r_p(\bar{G}) > 0$.

Proof: It suffices to use Lemma 6 and Theorem 1.

Corollary 2: Let T be a torsion group and A be a countable torsionfree group. Then every pure subgroup of $G = T \oplus A$ splits if and only if T_p is a direct sum of a divisible and a bounded groups for each prime p with $r_p(A) > 0$.

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Matematicko-fyzikální fakulta

Universita Karlova

Sokolovská 83, 18600 Praha 8

Československo

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