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Abstract: A class of generalized modules is investigated.

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It is well known that the theory of medial quasigroups is equivalent to the theory of modules over a commutative noetherian ring (see e.g. [2]). This fact remains true for trimedial quasigroups (see [4]), however the underlying abelian groups of the modules are replaced by commutative Moufang loops (possibly non-associative). Such a module-like structure is called a quasimodule in the paper and some basic results concerning these quasimodules are proved. A special attention is paid to those properties which are important for the corresponding trimedial quasigroups.

1. Commutative Moufang loops. In this paragraph, we shall formulate some results concerning commutative Moufang loops. For complete proofs as well as further details, the

reader is referred to [1].

Let $Q(+)$ be a loop satisfying the identity $(x + x) + (y + z) = (x + y) + (x + z)$. Then, as is easy to see, the loop is commutative and it is called a commutative Moufang loop. For $a,b,c \in Q$, let $i_{a,b}(c) = ((c + a) + b) - (a + b)$ and $(a,b,c) = ((a + b) + c) - (a + (b + c))$.

1.1. Lemma. Let $a,b,c \in Q$. Then:
(i) $i_{a,b}$ is an automorphism of $Q(+)$.
(ii) $fi_{a,b} = i_{f(a),f(b)}f$ and $f((a,b,c)) = (f(a),f(b),f(c))$ for every endomorphism f of $Q(+)$.
(iii) If $a + (b + c) = (a + b) + c$ then the subloop generated by these elements is a group.

Proof. See [1, Lemma VII.3.3, Theorem VII.4.2].

1.2. Lemma. A subloop $P(+)$ of $Q(+)$ is normal iff $i_{a,b}(P) \subseteq P$ for all $a,b \in Q$.

Proof. See [1, Lemma IV.1.4].

Let $C(Q(+))$ denote the center of $Q(+)$. Further, put $C_0(Q(+)) = 0$ and $C_{n+1}(Q(+))/C_n(Q(+)) = C(Q(+)/C_n(Q(+))$ for $0 \leq n$.

1.3. Lemma. (i) Any subloop of $C(Q(+))$ is a normal subgroup of $Q(+)$.

(ii) For every $0 \leq n$, $C_n(Q(+))$ is a normal subloop of $Q(+)$.

Proof. Obvious (use 1.2).

1.4. Lemma. Let $P(+)$ be a subloop of $Q(+)$ and $G = P + C(Q(+)) = \{a + b \mid a \in P,\ b \in C(Q(+))\}$. Then $G(+)$ is a subloop of $Q(+)$, $P(+)$ is a normal subloop of $G(+)$ and $G(+)$ is a homomorphic image of the product $P(+) \times C(Q(+))$.

Proof. Easy.

1.5. Lemma. Let P(+) be a subloop of Q(+) such that Q(+) is generated by $P \cup C(Q(+))$. Then P(+) is a normal subloop of Q(+) and Q(+) is a homomorphic image of the product $P(+) \times C(Q(+))$.

Proof. Apply 1.4.

If J, K, L are non-empty subsets of Q then (J,K,L) denotes the subloop generated by all (x,y,z), $x \in J$, $y \in K$, $z \in L$.

Put $A_0(Q(+)) = Q$ and $A_{n+1}(Q(+)) = (A_n(Q),Q,Q)$ for every $0 \leq n$. Further, denote $A(Q(+)) = A_1(Q(+))$.

1.6. Lemma. (i) For every $0 \leq n$, $A_n(Q(+))$ is a normal subloop of Q(+).

(ii) $A_0(Q(+)) \supseteq A_1(Q(+)) \supseteq A_2(Q(+)) \supseteq \dots$.

Proof. Obvious.

The loop Q(+) is said to be nilpotent of class at most n if $A_n(Q(+)) = 0$. It is said to be nilpotent of class n if $A_n(Q(+)) = 0$ and either n = 0 or $A_{n-1}(\varphi(+)) \neq 0$

1.7. Lemma. Q(+) is nilpotent of class at most n iff $C_n(Q(+)) = Q$. In this case, $A_{n-1}(Q(+)) \subseteq C(Q(+))$.

Proof. By induction.

1.8. Lemma. Suppose that Q(+) is generated by n elements. Then:

(i) Q(+) is a group, provided $n \leq 2$.

(ii) Q(+) is nilpotent of class at most n - 1, provided $2 \leq n$.

Proof. See 1.1(iii) and [1, Theorem VIII.10.1].

1.9. Lemma. Q(+) is a group, provided it is simple.

Proof. See [1, Theorem VIII.11.1].

Put $B(Q(+)) = \{x \mid x \in Q, 3x = 0\}$ and $D(Q(+)) = 3Q =$

$= \{3x \mid x \in Q\}$.

1.10. Lemma. (i) $B(Q(+))$ and $D(Q(+))$ are normal subloops of $Q(+)$.

(ii) $A(Q(+)) \subseteq B(Q(+))$ and $D(Q(+)) \subseteq C(Q(+))$.

(iii) If $P(+)$ is a subloop of $Q(+)$ such that $P \cap C(Q(+)) = 0$ then $P \subseteq B(Q(+))$.

Proof. See [1, Lemma VII.5.7].

Let n be an integer and f be a transformation of the set Q. We shall say that f is central of type (n) if $nx + f(x) \in C(Q(+))$ for every $x \in Q$.

1.11. Lemma. Let n be an integer, $n = 3m + k$, where $k \in \{0,1,2\}$.

(i) The mapping $x \longrightarrow nx$ is a central endomorphism of type (k).

(ii) If f is a central transformation of type (n) then f is central of type (k).

Proof. Apply 1.8(i) and 1.10(ii).

1.12. Lemma. Let f be a transformation of Q. Suppose that f is central of types (k), (ℓ), where $k, \ell \in \{0,1,2\}$. Then either $k = \ell$ or $Q(+)$ is a group.

Proof. Evident.

1.13. Lemma. Let f and g be central endomorphisms of types (n) and (m), resp. Then:

(i) $f + g$ is a central endomorphism of type (n+m).

(ii) $-f$ is a central endomorphism of type (-n).

(iii) fg is a central endomorphism of type (-nm).

(iv) If f is an automorphism then f^{-1} is central of type (n).

Proof. Easy.

1.14. *Lemma*. Let f, g, h be central endomorphisms of $Q(+)$. Then $f + (g + h) = (f + g) + h$.

Proof. Easy.

1.15. *Lemma*. The set of all central endomorphisms of $Q(+)$ is an associative ring with unit.

Proof. Apply 1.13 and 1.14.

1.16. *Lemma*. Let f be a central endomorphism of type (n). Then $f(x) = -nx$ for every $x \in A(Q(+))$.

Proof. Put $g(x) = nx + f(x)$ for each x Q. Then g is an endomorphism and $g(Q) \subseteq C(Q(+))$. Consequently, $g(Q)$ is a group and $A(Q(+))$ is contained in Ker g.

1.17. *Lemma*. Let $a,b,c \in Q$ be such that $(a + b) + c =$ $= -a + (b + c)$. Then $a = -a$.

Proof. We have $(a + b) + (3a + c) = ((a + b) + c) + 3a =$ $= (-a + (b + c)) + 3a = 2a + (b + c) = (a + b) + (a + c)$. From this, $2a = 0$ and $a = -a$.

1.18. *Lemma*. Let $P(+)$ be a non-zero cyclic subgroup of $Q(+)$.

(i) If $P \cap C(Q(+)) = 0$ then P contains just three elements.

(ii) If the order of P is not divisible by three then $P \subseteq C(Q(+))$.

(iii) If $P(+)$ is normal then $P \cap C(Q(+)) \neq 0$.

(iv) If $P(+)$ is a minimal normal subloop then $P \subseteq C(Q(+))$.

Proof. Apply 1.10 and 1.17.

1.19. *Lemma*. Let $P(+)$ be a normal subloop of $Q(+)$ such that P contains at most five elements. Then P $C(Q(+))$.

Proof. With respect to 1.18(ii), we can assume that P contains either one or three elements. In both cases, the result follows from 1.18(iv).

2. Quasimodules. Throughout the paper, let R be an associative ring with unit. A (left R-) quasimodule Q is an algebra $Q(+,rx)$ with one binary operation + and a set of unary operations $x \longrightarrow rx$, $r \in R$, satisfying the following identities: $(x + x) + (y + z) = (x + y) + (x + z)$, $(-1)x + (x + y) = y$, $r(x + y) = rx + ry$, $(r + s)x = rx + sx$, $(rs)x = r(sx)$, $1x = x$, $0x = 0y$. Then $Q(+)$ is a commutative Moufang loop, the inverse operation - of $Q(+)$ is given by $x - y + x + (-1)y$, the element $0 = 0x$ is the neutral element of $Q(+)$ and the unary operations $x \longrightarrow rx$ are endomorphisms of $Q(+)$. We denote by $\mathcal{N}$ (resp. $\mathcal{M}$) the class of quasimodules (resp. modules). Obviously, $\mathcal{N}$ is a variety and $\mathcal{M}$ is a subvariety determined in $\mathcal{N}$ by the identity $x + (y + z) = (x + y) + z$.

2.1. Lemma. Let P be a subquasimodule of a quasimodule Q such that $P(+)$ is a normal subloop of $Q(+)$. Then P is a normal subquasimodule.

Proof. There is a congruence t of $Q(+)$ such that P is one of its blocks (in fact, $x\, t\, y$ iff $x - y \in P$). However, P is a subquasimodule and it is easy to verify that t is a congruence of the quasimodule Q.

2.2. Lemma. Let P be a subquasimodule of a quasimodule Q and $G(+)$ be the least normal subloop containing P. Then G is a normal subquasimodule.

Proof. Denote by $\mathcal{G}$ the permutation group generated by $i_{a,b}$, all $a,b \in Q$. By 1.2, $G(+)$ is just the subloop generated by all $f(x)$, $x \in P$, $f \in \mathcal{G}$. If $x \in P$, $f \in \mathcal{G}$ and $r \in R$ then

$rf(x) = g(rx)$ for some $g \in \mathcal{G}$ (use 1.1(ii)). The rest is now clear.

Let P_j, $j \in J$, be a family of subquasimodules of a quasimodule Q. We denote by $\sum P_j$ (resp. $\sum^* P_j$) the (normal) subquasimodule generated by $\bigcup P_j$. Clearly, this (normal) subquasimodule is equal to the (normal) subloop generated by $\bigcup P_j$.

2.3. <u>Lemma</u>. Let Q be a quasimodule. Then, for every $0 \neq n$, $A_n(Q) = A_n(Q(+))$ is a normal subquasimodule of Q. Moreover, Q/A(Q) is a module.

Proof. The subloop $A_n(Q(+))$ is invariant under endomorphisms of the loop Q(+).

2.4. <u>Lemma</u>. Let Q be a quasimodule. Then:

(i) Every subquasimodule contained in C(Q(+)) is normal.

(ii) Every subquasimodule containing A(Q) is normal.

Proof. Apply 1.3(i), 2.1 and 2.3.

2.5. <u>Lemma</u>. Let Q be a quasimodule. Then B(Q) = B(Q(+)) and D(Q) = D(Q(+)) are normal subquasimodules of Q. Moreover, D(B(Q)) = 0 and B(Q/D(Q)) = Q/D(Q).

Proof. Similar to that of 2.3.

2.6. <u>Lemma</u>. Let Q be a quasimodule and $a \in Q$. Denote by P the subquasimodule generated by a. Then $P = Ra = \{ra \mid r \in R\}$ and P is a module.

Proof. Obvious.

3. <u>Preradicals</u>. Let $\mathcal{A}$ be a class of quasimodules closed under subquasimodules, homomorphic images, finite cartesian products and containing R. By a semipreradical t for $\mathcal{A}$ we mean a mapping of $\mathcal{A}$ into $\mathcal{A}$ such that the following two conditions

are satisfied:

(i) For every $Q \in \mathcal{A}$, $t(Q)$ is a subquasimodule of Q.

(ii) If $f:Q \longrightarrow P$ is a surjective homomorphism and $Q \in \mathcal{A}$ then $f(t(Q)) \subseteq t(P)$.

Let t be a semipreradical for $\mathcal{A}$. We shall say that t is

- idempotent if $t(t(Q)) = t(Q)$ for every $Q \in \mathcal{A}$,
- hereditary if $t(P) = P \cap t(Q)$ for every $Q \in \mathcal{A}$ and a subquasimodule P of Q,
- normal if $t(Q)$ is a normal subquasimodule of Q for every $Q \in \mathcal{A}$,
- cohereditary if $f(t(Q)) = t(P)$ for every $Q \in \mathcal{A}$ and every surjective homomorphism $f:Q \longrightarrow P$,
- a preradical if $f(t(Q)) \subseteq t(P)$ for all $Q,P \in \mathcal{A}$ and a homomorphism $f:Q \longrightarrow P$,
- a semiradical if it is normal and $t(Q/t(Q)) = 0$ for every $Q \in \mathcal{A}$,
- a radical if it is both a semiradical and a preradical.

A quasimodule Q is said to be t-torsion (t-torsionfree) if $t(Q) = Q$ $(t(Q) = 0)$.

3.1. <u>Lemma</u>. Let t be a semipreradical for $\mathcal{A}$. Then:

(i) t is an idempotent preradical, provided it is hereditary.

(ii) t is a semiradical, provided it is cohereditary and normal.

(iii) t is a preradical iff $t(P) \subseteq t(Q)$ for every $Q \in \mathcal{A}$ and a subquasimodule P of Q.

Proof. Obvious.

3.2. <u>Proposition</u>. Every hereditary semipreradical for $\mathcal{A}$ is a normal idempotent preradical.

Proof. Let t be a hereditary semipreradical. By 3.1, t is an idempotent preradical. It remains to show that t is normal. For, let $Q \in \mathcal{A}$, $a,b \in Q$ and $c \in t(Q)$. Put $d = i_{a,b}(c)$ and take $r \in (0:c) = \{s \mid s \in R,\ sc = 0\}$. Clearly, $rd = 0$, $(0:c) \subseteq (0:d)$ and the cyclic submodule Rd is a homomorphic image of Rc. But Rc is t-torsion. Consequently, Rd is t-torsion and $d \in t(Q)$.

A non-empty set $\mathcal{F}$ of left ideals of R is called a filter if the following conditions are satisfied:

(F1) If $I \subseteq K$ are left ideals and $I \in \mathcal{F}$ then $K \in \mathcal{F}$.

(F2) If $I \in \mathcal{F}$ and $r \in R$ then $(I:r) \in \mathcal{F}$.

(F3) $\mathcal{F}$ is closed under finite intersections.

Moreover, $\mathcal{F}$ is said to be a radical filter if:

(F4) If $I \subseteq K$ are left ideals, $K \in \mathcal{F}$ and $(I:r) \in \mathcal{F}$ for every $r \in K$ then $I \in \mathcal{F}$.

3.3. <u>Lemma</u>. Let t be a hereditary preradical for $\mathcal{A}$. Put $\mathcal{F} = \{I \mid t(R/I) = R/I\}$. Then:

(i) $\mathcal{F}$ is a filter.

(ii) For $Q \in \mathcal{A}$, $t(Q) = \{x \mid x \in Q,\ (0:x) \in \mathcal{F}\}$.

(iii) $\mathcal{F}$ is a radical filter, provided t is a radical.

Proof. Easy.

3.4. <u>Lemma</u>. Let $\mathcal{F}$ be a filter and $t(Q) = \{x \mid x \in Q,\ (0:x) \in \mathcal{F}\}$ for every quasimodule Q. Then t is a hereditary preradical for $\mathcal{N}$. Moreover, t is a radical, provided $\mathcal{F}$ is a radical filter.

Proof. Easy.

Let $\mathcal{B}$ be a non-empty subclass of $\mathcal{A}$. For every $Q \in \mathcal{A}$, put $p_{\mathcal{B}}(Q) = \bigcap \operatorname{Ker} f$, $f: Q \to G$, $G \in \mathcal{B}$.

3.5. <u>Lemma</u>. $p_{\mathcal{B}}$ is a radical for $\mathcal{A}$.

Proof. Obvious.

Let t be a preradical for $\mathcal{A}$. Denote by $\mathcal{B}$ the class of all t-torsionfree quasimodules and put $\tilde{t} = p_{\mathcal{B}}$. One can easily check that the radical $\tilde{t}$ is just the least semiradical containing t.

3.6. <u>Lemma</u>. Let t be a normal preradical for $\mathcal{A}$ and Q be a $\tilde{t}$-torsion quasimodule. Then there are an ordinal number α and a chain Q_β , $0 \leq \beta \leq \alpha$, of normal subquasimodules of Q such that $Q_0 = 0$, $Q_\alpha = Q$ and $Q_{\beta+1}/Q_\beta = t(Q/Q_\beta)$ for every $1 \leq \beta + 1 \leq \alpha$.

Proof. Easy.

3.7. <u>Proposition</u>. Let t be a hereditary preradical for $\mathcal{A}$. Then $\tilde{t}$ is a hereditary radical.

Proof. Put $\mathcal{F} = \{I \mid t(R/I) = R/I\}$. Then $\mathcal{F}$ is a filter. Denote by $\mathcal{R}$ the radical filter generated by $\mathcal{F}$. Let p be the hereditary radical corresponding to $\mathcal{R}$. Then $t \subseteq p$, and so $\tilde{t} \subseteq p$. Further, let Q be a t-torsionfree module and $a \in p(Q)$. Suppose that $a \neq 0$ and put $I = (0:a)$. We have $1 \notin I \in \mathcal{R}$ and by [3, Corollary 2.7], there exists $r \in R \setminus I$ such that $(I:r) \in \mathcal{F}$. However, $(I:r) = (0:ra)$, and hence $ra \in t(Q)$, $ra = 0$. On the other hand, $r \notin I$ and $ra \neq 0$, a contradiction. We have proved that Q is p-torsionfree. Now, let Q be an arbitrary quasimodule from $\mathcal{A}$. The factor-quasimodule $P = Q/\tilde{t}(Q)$ is t-torsionfree. Therefore $p(P) = 0$ and $p(Q) \subseteq t(Q)$. Thus $p \subseteq \tilde{t}$ and $p = \tilde{t}$.

Let $\mathcal{B}$ be a non-empty subclass of $\mathcal{A}$. For every quasimodule $Q \in \mathcal{A}$, let $q_{\mathcal{B}}(Q) = \sum \mathrm{Im}\, f$, $f: G \longrightarrow Q$, $G \in \mathcal{B}$ and $\overset{*}{q}_{\mathcal{B}}(Q) = \sum^{*} \mathrm{Im}\, f$. Further, let $q_{*\mathcal{B}}(Q)$ be the subquasimodule generated by all Im f such that Im f is normal in Q.

3.8. Lemma. (i) $q_{\mathcal{B}}$ is an idempotent preradical for $\mathcal{A}$.

(ii) $\overset{*}{q}_{\mathcal{B}}$ is a normal preradical for $\mathcal{A}$.

(iii) $\underset{*}{q}_{\mathcal{B}}$ is a normal idempotent semipreradical for $\mathcal{A}$.

(iv) $\underset{*}{q}_{\mathcal{B}} \subseteq q_{\mathcal{B}} \subseteq \overset{*}{q}_{\mathcal{B}}$.

Proof. Obvious.

A quasimodule Q is said to be diassociative if every subquasimodule of Q generated by at most two elements is a module.

3.9. Lemma. Suppose that every quasimodule from $\mathcal{A}$ is diassociative. Let $\mathcal{B}$ be closed under cyclic submodules and finite cartesian products. Then $q_{\mathcal{B}}$ is a hereditary preradical for $\mathcal{A}$.

Proof. Easy.

3.10. Proposition. (i) For every $0 \leq n$, A_n is a normal cohereditary radical for $\mathcal{N}$.

(ii) B is a hereditary preradical for $\mathcal{N}$ and $A \subseteq B$.

(iii) D is a normal cohereditary radical for $\mathcal{N}$.

Proof. Easy.

4. Special quasimodules. Let Z designate the ring of integers. Further, denote by Z_3 a three-element field containing the elements 0, 1, 2 and let Φ be a mapping of R into Z_3. A quasimodule Q is said to be special of type (Φ) if the endomorphism $x \longrightarrow rx$ of Q(+) is central of type ($\Phi(r)$) for every $r \in R$. It is visible that the class $\mathcal{S}$ of special quasimodules of type (Φ) is a subvariety of $\mathcal{N}$. Moreover, $\mathcal{M} \subseteq \mathcal{S}$.

4.1. Lemma. Either $\mathcal{S} = \mathcal{M}$ or $-\Phi$ is a ring homomorphism of R onto Z_3 preserving the unit.

Proof. This is an immediate consequence of 1.12 and 1.13.

In the residual part of the paper, we shall assume that $-\Phi$ is a ring homomorphism preserving the unit. Moreover, the word quasimodule will mean always a special quasimodule of type (Φ).

A quasimodule Q is said to be primitive if $rx = -\Phi(r)x$ for all $r \in R$ and $x \in Q$. Clearly, the class $\mathcal{P}$ of primitive quasimodules is a subvariety of $\mathcal{S}$. Furthermore, $\mathcal{P}$ is equivalent to the variety of commutative Moufang loops satisfying $3x = 0$.

We can define a quasimodule structure on Z_3 induced by the homomorphism $-\Phi$. The corresponding quasimodule (which is isomorphic to the factorquasimodule $R/\mathrm{Ker}(-\Phi)$) is denoted by Z_3, too. Clearly, it is a primitive simple module.

4.2. Lemma. Let Q be a quasimodule. Then:

(i) Every subloop of A(Q) is a primitive subquasimodule of Q.

(ii) Every subloop containing C(Q) = C(Q(+)) is a subquasimodule of Q.

(iii) Q/C(Q) is a primitive quasimodule.

Proof. Apply 1.16.

4.3. Proposition. Let Q be a quasimodule which can be generated by n elements. Then:

(i) Q is a module, provided $n \leq 2$.

(ii) Q is nilpotent of class at most n - 1, provided $2 \leq n$.

Proof. There is a subset M of Q containing n elements such that M generates Q. Denote by G(+) and H(+) the subloop generated by M and $M \cup C(Q)$, resp. By 4.2, H is a subquasimodule, and so H = Q. On the other hand, by 1.5, Q(+) is a homomorphic image of the product $G(+) \times C(Q(+))$. The rest follows now from 1.8.

4.4. Lemma. Let Q be a quasimodule and $a,b \in Q$. Then:

(i) $i_{a,b}(rx) = (rx + \Phi(r)x) - \Phi(r)i_{a,b}(x)$ for all $r \in R$ and $x \in Q$.

(ii) The set $\{x \mid i_{a,b}(x) = x\}$ is a subquasimodule of Q.

Proof. Obvious.

4.5. Lemma. Let Q be a quasimodule generated by a set M. Let N be a subset of Q such that $(N,M,M) = 0$. Then $N \subseteq C(Q)$.

Proof. Use 4.4.

4.6. Proposition. Suppose that the ring R is left noetherian. Then every subquasimodule of a finitely generated quasimodule is finitely generated.

Proof. Let Q be a finitely generated quasimodule. By 4.3, Q is nilpotent of class n. We shall proceed by induction on n. For $n \leq 1$, Q is a module and the situation is clear. Assume $2 \leq n$ and put $G = A_{n-1}(Q)$, $H = A_{n-2}(Q)$, $C = C(Q)$, $K = Q/G$. Then G is contained in both H and C and K is nilpotent of class at most n - 1. Hence there exists a finite subset M of H such that H is generated by $M \cup G$. Further, Q is generated by a finite subset, say N. Denote by E the subquasimodule generated by $L = \{(x,y,z) \mid x \in M,\ y,z \in N\}$. Obviously, L is finite, $L \subseteq G$ and $E \subseteq G \subseteq C$. By 2.4, E is a normal submodule of Q. Let f be the natural homomorphism of Q onto $F = Q/E$. It follows from 4.5 that f(M) is contained in C(F). However, $f(G) \subseteq f(C) \subseteq C(F)$, and consequently $f(H) \subseteq C(F)$. But $f(H) = A_{n-2}(F)$. Thus F is nilpotent of class at most n - 1. Now, let P be a subquasimodule of Q and $T = P \cap E$. Then P/T is isomorphic to a subquasimodule of F and P/T is finitely generated. On the other hand, $T \subseteq E$ and E is a finitely generated module. Thus T is finitely generated and the rest is clear.

4.7. Lemma. Let $0 \neq P$ be a normal subquasimodule of a nilpotent quasimodule Q. Then $0 \neq P \cap C(Q)$.

Proof. By induction on the nilpotence class of Q. We can assume that P is not contained in C(Q). Then $0 \neq f(P)$, f being the natural homomorphism of Q onto $G = Q/C(Q)$, and therefore $f(P) \cap C(G) \neq 0$. From this, we see that there exists $a \in P \cap C_2(Q)$ with $a \notin C(Q)$. Then $a + (b + c) = (a + b) + (c + d)$ for some $b,c,d \in Q$, $0 \neq d$. It is visible that $d \in P \cap C(Q)$.

4.8. Lemma. Every simple quasimodule is a module.

Proof. Let Q be a simple quasimodule. If $C(Q) \neq 0$ then $C(Q) = Q$ and Q is a module. Suppose $C(Q) = 0$. By 4.2, Q is primitive, and hence the loop Q(+) is simple and a group by 1.9. Consequently, Q is a module and $Q = 0$.

4.9. Corollary. Let P be a maximal normal subquasimodule of a quasimodule Q. Then P is a maximal subquasimodule and $A(Q) \subseteq P$.

4.10. Lemma. Let P be a proper subquasimodule of a nilpotent quasimodule Q. Then there is a subquasimodule G such that P is a normal subquasimodule of G and $P \neq G$.

Proof. By induction on the nilpotence class of Q. With respect to 1.4, we can assume that $C(Q) \subseteq P$. Then P/C(Q) is a proper subquasimodule of Q/C(Q).

4.11. Corollary. Every maximal subquasimodule of a nilpotent quasimodule is normal.

Let $\mathcal{C}$ be a class of all simple modules. Put $\mathcal{J} = p_{\mathcal{C}}$. Then $\mathcal{J}$ is a radical for $\mathcal{S}$ and $A \subseteq \mathcal{J}$. For a quasimodule Q, $\mathcal{J}(Q)$ is just the intersection of all maximal normal (resp. normal maximal) subquasimodules.

4.12. Proposition. Suppose that R is left noetherian. Let Q be a finitely generated quasimodule and f be the natural homomorphism of Q onto $Q/\mathcal{J}(Q)$. A subset N of Q generates Q iff f(N) generates f(Q).

Proof. Let f(N) generate f(Q). Denote by P the subquasimodule generated by N and assume $P \neq Q$. By 4.6, P is contained in a maximal subquasimodule, say G. Taking into account 4.11, we see that $\mathcal{J}(Q) \subseteq G$ and $f(P) \subseteq f(G) \neq f(Q)$, a contradiction.

4.13. Proposition. Let P be a minimal normal subquasimodule of a nilpotent quasimodule Q. Then P is a simple module and $P \subseteq C(Q)$.

Proof. This follows from 4.7.

Let S be a simple module. We put $Soc_S = q_{\{S\}}$. By 3.7 and 3.9, Soc_S is a hereditary preradical and $\widetilde{Soc_S}$ is a hereditary radical for $\mathcal{S}$. Further, put $\mathcal{K} = Soc_{Z_3}$.

4.14. Lemma. Let Q be a quasimodule. Then:

(i) Q is $\mathcal{K}$-torsion iff it is primitive.

(ii) $\mathcal{K}(Q)$ is the largest primitive subquasimodule of Q.

Proof. Obvious.

4.15. Lemma. Every $\mathcal{K}$-torsionfree quasimodule is a module.

Proof. Apply 4.14 and 4.2(i).

4.16. Lemma. Let S be a simple module not isomorphic to

Z_3. Then every $\widetilde{Soc}_S$-torsion quasimodule is a module.

Proof. Let Q be such a quasimodule and P = A(Q). Suppose $P \neq 0$. Then $Soc_S(P) \neq 0$ and P contains a submodule G isomorphic to S. On the other hand, G is primitive and it is isomorphic to Z_3, a contradiction.

4.17. Proposition. Suppose that the ring R is left noetherian. Let S be a finite simple module. Then every finitely generated $\widetilde{Soc}_S$-torsion quasimodule is finite.

Proof. First, let Q be a finitely generated $\widetilde{Soc}_S$-torsion module. Then Q is a noetherian module, and hence it possesses a finite socle-sequence of submodules (see 3.6). The corresponding completely reducible factors are noetherian, hence finite, and consequently Q is finite. Now, let Q be a finitely generated $\widetilde{Soc}_S$-torsion quasimodule. We can assume that Q is not a module. By 4.16, S is isomorphic to Z_3. Further, Q is nilpotent and we shall proceed by the nilpotence class of Q. As we have proved, C(Q) is a finite module. However, Q/C(Q) is $\mathcal{K}$-torsion and the rest is clear.

4.18. Lemma. Let P be a normal cyclic submodule of a quasimodule Q. Then $P \subseteq C_2(Q)$ and either P = 0 or $P \cap C(Q) \neq 0$.

Proof. Let f denote the natural homomorphism of Q onto Q/C(Q). Then f(P) is normal in Q/C(Q). However, the last quasimodule is primitive and $f(P) \subseteq C(Q/C(Q))$ by 1.19. Thus $P \subseteq C_2(Q)$.

4.19. Corollary. Let Q be a quasimodule such that every subquasimodule of Q is normal. Then Q is nilpotent of class at most 2.

4.20. Lemma. $A \subseteq \mathcal{K} \subseteq B$, $A \subseteq \mathcal{J}$ and $D \subseteq C$. Moreover, C is a normal idempotent semipreradical for $\mathcal{S}$.

Proof. Obvious.

5. Subdirectly irreducible special quasimodules

5.1. Lemma. Let Q be a subdirectly irreducible quasimodule. Then C(Q) is a subdirectly irreducible module.

Proof. This follows immediately from 2.4(i).

5.2. Proposition. Let Q be a quasimodule with $C(Q) \neq 0$. The following conditions are equivalent:

(i) Q is subdirectly irreducible.

(ii) There exists a minimal submodule P of Q such that P is contained in every non-zero normal subquasimodule of Q.

Proof. Apply 5.1.

5.3. Proposition. The following conditions are equivalent for a nilpotent quasimodule Q:

(i) Q is subdirectly irreducible.

(ii) C(Q) is subdirectly irreducible.

Proof. Apply 5.1 and 4.7.

5.4. Proposition. Suppose that R is commutative and noetherian. Let Q be a subdirectly irreducible quasimodule such that Q is not a module. Then Q is $\tilde{\mathcal{K}}$ -torsion.

Proof. If $C(Q) = 0$ then Q is primitive and $\mathcal{K}$-torsion. Let $0 \neq C(Q)$. By 5.2, there is a minimal submodule P of C(Q). Since $A(Q) \neq 0$, $P \subseteq A(Q)$ and P is isomorphic to Z_3. However, Q/C(Q) is $\mathcal{K}$ -torsion and C(Q) is subdirectly irreducible. Hence P is an essential submodule of C(Q) and both C(Q) and Q are $\tilde{\mathcal{K}}$ -torsion.

5.5. Corollary. Suppose that R is commutative and noetherian. Let Q be a subdirectly irreducible quasimodule. Then there exists a simple module S such that Q is $\widetilde{Soc}_S$-torsion.

5.6. Proposition. The following conditions are equivalent for a non-zero nilpotent primitive quasimodule Q:

(i) Q is subdirectly irreducible.

(ii) C(Q) is isomorphic to Z_3.

Proof. This is clear from 5.3.

5.7. Corollary. Let Q be a nilpotent quasimodule of class $1 \le n$. Suppose that Q is primitive. Then Q is subdirectly irreducible iff every proper factor of Q is nilpotent of class at most n - 1.

5.8. Proposition. Suppose that the ring R is finitely generated and commutative. Then every finitely generated subdirectly irreducible quasimodule is finite.

Proof. By [2, Lemma 2.7], every simple module is finite. The rest follows from 5.5 and 4.17.

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