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REMARKS ON THE NON SELF-ADJOINT SCHRÖDINGER OPERATOR

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Abstract: Let h denote the closure in $L^2(\mathbb{R}^n)$ of the differential operator

$$Hu = -\Delta u + qu \qquad u \in C_0^\infty(\mathbb{R}^n)$$

where q is a complex valued function belonging to L^2_{loc}. If for a.e. $x \in \mathbb{R}^n$ $q(x)$ belongs to the sector $0 \le \arg \zeta \le \pi - \delta$ $(\delta \in]0,\pi])$, it is proved that h is the unique closed extension whose spectrum is contained in the above sector. Moreover conditions which are necessary and sufficient for the compactness of the resolvent of h, are obtained.

Key words: Compact resolvent, numerical range, m-sectorial operator.

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Introduction. It is well known that non-selfadjoint Schrödinger operators arise in quantum mechanical problems with energy dissipation.

Spectral properties of such operators have been studied by many authors (cf. e.g. [3,6,8,9]).

In this paper we study some properties of the operator h obtained by closure in $L^2(\mathbb{R}^n)$ of the differential operator H defined by

$$Hu = -\Delta u + q(x)u \qquad u \in C_0^\infty(\mathbb{R}^n)$$

where q is a complex valued function belonging to $L^2_{loc}(\mathbb{R}^n)$.

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If for a.e. $x \in R^n$ $q(x)$ belongs to a sector S (of the complex plane) defined by $0 \leq \arg \xi \leq \pi - \delta$ (with $\delta \in]0, \pi]$), it can be proved (cf. th. 2.2) that h is the unique closed extension of H whose spectrum $\sigma(h)$ is contained in S. Moreover conditions which are necessary and sufficient for the compactness of the resolvent of h, are obtained (cf. th. 2.3). Analogous results have been obtained in [3,6] for the one-dimensional Schrödinger operator.

Our study is mainly based on some well known results of the theory of the <u>sectorial</u> operators in Hilbert spaces (cf. e.g. [4,10,11]).

1. <u>Some preliminaries</u>. In this section we denote by E a separable Hilbert space with scalar product $(\cdot | \cdot)_E$ and norm $\| \cdot \|_E$; if T is a linear operator in E, D(T) denotes the domain and R(T) the range. $\sigma(T)$ denotes the spectrum of T. $\sigma(T)$ will be called discrete if it consists entirely of isolated eigenvalues of finite multiplicity. If T is closable, $\bar{T}$ denotes its closure.

If T is densely defined in E, T^* denotes the adjoint of T.

The numerical range N(T) of T is the set of all complex numbers $(Tu|u)_E$ where u changes over all $u \in D(T)$ with $\|u\|_E = 1$.

It is well known that the spectrum $\sigma(T)$ of T is not contained, in general, in $\overline{N(T)}$. However the following result holds (cf. e.g. th. 3.2 pg. 268 of [4] and ch. XIV of [11]).

<u>Theorem 1.1</u> - <u>Let</u> T <u>be a densely defined closable operator in</u> E <u>and suppose that</u> $\overline{N(T)}$ <u>is not a strip of a line</u>.

Then for each $\xi \in \mathbb{C} \setminus \overline{N(T)}$ $T-\xi I$ has closed range, dim Ker$(\overline{T}-$ $-\xi I)=0$ and codim $R(\overline{T}-\xi I)$ is constant for $\xi \in \mathbb{C} \setminus \overline{N(T)}$. Moreover there exists a closed extension $\hat{T}$ of T such that $\sigma(\hat{T}) \subset$ $\subset \overline{N(T)}$.

Now it is easily verified that if T satisfies the assumptions of theorem 1.1, a closed extension $\hat{T}$ of T satisfying the property

$$(1.2) \qquad \sigma(\hat{T}) \subset \overline{N(T)}$$

is maximal (in the sense that $\hat{T}$ has no proper extension satisfying (1.2)). Then such extension $\hat{T}$ is unique if the closure $\overline{T}$ of T satisfies (1.2). Therefore an interesting class of closable operator is the following one:

Definition 1.2 - A densely defined, closable operator T is called regular iff $\sigma(\overline{T}) \subset \overline{N(T)}$.

Let us observe that if T is symmetric (i.e. T is densely defined and $T \subset T^*$), T is regular if and only if T is essentially self-adjoint.

If T is a positive definite (i.e. $(Tu|u)_E \geq 0 \ \forall u \in D(T)$), self-adjoint operator in E, $T^{1/2}$ denotes its square root. Let us recall the following well known results (cf. e.g. [1,10]).

Theorem 1.3 - Let T be a positive definite, self-adjoint operator in E, then the following statements are equivalent:

a) T has compact resolvent.

b) $T^{1/2}$ has compact resolvent.

c) The spectrum $\sigma(T)$ is discrete.

Theorem 1.4 - Let $a: D(a) \times D(a) \longrightarrow \mathbb{C}$ (D(a) dense in E) be a sesquilinear, symmetric form. Then if a is closed and bounded from below, there exists a self-adjoint operator A

(Friedrichs extension) with domain

$D(A) = \{x \in D(a) \mid \exists\, y \in E$ such that $a(x,z)=(y|z)_E \quad \forall z \in D(a)\}$

and defined by setting $Ax=y$ for $x \in D(A)$. A and a have the same lower bound; if a is positive (i.e. $a(u,u) \geq 0 \quad \forall u \in D(a)$), we have

$D(A^{1/2}) = D(a)$ and $a(u,v)=(A^{1/2}u|A^{1/2}v)_E \quad \forall u,v \in D(a)$

It is easy to prove the following

Theorem 1.5 - If T is a closed operator in E with $\sigma(T) \neq \mathbb{C}$ then the following statements are equivalent:

a) T has compact resolvent.

b) $D(T)$ equipped with the graph norm is compactly embedded in E.

Proof. a) $\Longrightarrow$ b) Let $\{(u_n)\} \subset D(T)$ $u_n \longrightarrow 0$ weakly in $D(T)$ equipped with graph norm. Then it is easily seen that if $\lambda \in \mathbb{C} \setminus \sigma(T)$, $(T-\lambda I)u_n \longrightarrow 0$ weakly in E; so, by virtue of a), we deduce that $u_n \longrightarrow 0$ in E.

b) $\Longrightarrow$ a) Let $\{v_n\} \subset E$ $v_n \longrightarrow 0$ weakly in E; if $\lambda \notin \sigma(T)$, we set $u_n = (T-\lambda I)^{-1}(v_n)$; then, by the continuity of $(T-\lambda I)^{-1}$, we deduce that $u_n \longrightarrow 0$ weakly in E and so $u_n \longrightarrow 0$ weakly in $D(T)$ equipped with the graph norm.

Therefore, by virtue of b), we deduce that $u_n \longrightarrow 0$ in E.

Let Ω be an open subset of $\mathbb{R}^n$. We shall use the following functional spaces:

- $L^p(\Omega)$ denotes the space of (equivalence classes of) functions on Ω which are (Lebesgue) measurable and satisfy

$$\|u\|_{0,p} = \begin{cases} \left(\int_\Omega |u(x)|^p dx\right)^{1/p} < +\infty & \text{for } p \in [1,+\infty[\\ \sup\limits_{x \in \Omega} \mathrm{ess}|u(x)| < +\infty & \text{for } p = +\infty \end{cases}$$

equipped with the norm $\|\cdot\|_{0,p}$. We shall set

$$(u|v)_0 = \int_\Omega u(x)\overline{v(x)}dx, \quad \|u\|_0 = \{(u|u)_0\}^{1/2} = \|u\|_{0,2},$$

moreover we set $L^p = L^p(\mathbb{R}^n)$.

- If m is a positive integer, $W^m(\Omega)$ is the Sobolev space of the functions $u \in L^2(\Omega)$ such that $D^\alpha u \in L^2(\Omega)$ for $|\alpha| \leq m$ and equipped with the norm

$$\|u\|_m = \{\sum_{|\alpha| \leq m} \|D^\alpha u\|_0^2\}^{1/2} .$$

  We shall set $W^m = W^m(\mathbb{R}^n)$.

- W^m_{loc} denotes the projective limit of the spaces $W^m(\Omega_0)$ (Ω_0 open and bounded) with respect to the restriction mappings $u \in W^m_{loc} \longmapsto u|_{\Omega_0} \in W^m(\Omega_0)$.

- If φ is a positive function on $\mathbb{R}^n$ belonging to L^1_{loc}, we denote by $\overset{\circ}{\Gamma}_\varphi$ the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\overset{\circ}{\Gamma}_\varphi} = \{\int_{\mathbb{R}^n} (|u(x)|^2\varphi(x) + |\text{grad } u(x)|^2)dx\}^{1/2} .$$

2. <u>The results</u>. Let q be a complex valued function on $\mathbb{R}^n$ belonging to L^2_{loc}. Let us now consider the operator H in L^2 with domain $D(H) = C_0^\infty(\mathbb{R}^n)$ and defined by

$$Hu = -\Delta u + qu \qquad \forall u \in D(H).$$

It is easily verified that the adjoint H^* of H is densely defined, then H is closable; let us denote by h its closure (i.e. $h = H^{**}$).

In what follows we shall find conditions (on q) which are sufficient to guarantee that H is regular (cf. def. 1.2). Moreover conditions which are necessary and sufficient for the compactness of the resolvent of h, will be obtained.

Let us set

$$W^2_{comp} = \{u \in W^2 \mid u \text{ has compact support}\}$$

and prove the following

Lemma 2.1 - Let us assume that

(2.1) $q \in L^r_{loc}$ with $r = \max\{2, n/2\}$ if $n \neq 4$ and $r > 2$ if $n = 4$. Then W^2_{comp} is contained in $D(h)$, and

$\forall u \in W^2_{comp}: hu = -\Delta u + qu$ (in the sense of distributions).

Proof. Let $u \in W^2_{comp}$. Then there exists a ball B, centered at the origin, such that supp $u \subset B$. It is easy to prove (if the radius of B is sufficiently large) that there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^n)$ with supp $u_n \subset B$ for each $n \in \mathbb{N}$, and such that

$$u_n \longrightarrow u \text{ (for } n \longrightarrow \infty \text{) in } W^2. \tag{2.2}$$

By applying Hölder inequality and Sobolev embedding theorems, it is not difficult to obtain

$$\int_{\mathbb{R}^n} |q(u_n-u)|^2 dx = \int_B |q|^2 \, |u_n-u|^2 dx \leq \|q\|^2_{L^r(B)} \|u_n - u\|^2_{L^s(B)} \leq \tag{2.3}$$

$$\leq c_1 \|q\|^2_{L^r(B)} \|u_n - u\|^2_{W^2(B)},$$

where $s = \begin{cases} +\infty \text{ if } n < 4 \\ 2r/(r-2) \text{ if } n=4 \\ 2n/(n-4) \text{ if } n > 4 \end{cases}$ and c_1 is a positive constant.

Then from (2.2) and (2.3) we deduce that

(2.4) $$qu_n \longrightarrow qu \text{ (for } n \rightarrow \infty \text{) in } L^2.$$

Moreover from (2.2),(2.4) we deduce that

$$-\Delta u_n + qu_n \longrightarrow -\Delta u + qu \text{ (for } n \rightarrow \infty \text{) in } L^2.$$

Then we conclude that $u \in D(h)$ and $hu = -\Delta u + qu$.

Q.E.D.

<u>Theorem 2.2</u> - <u>Let us assume that</u> q <u>satisfies</u> (2.1) <u>Moreover assume that</u> q <u>satisfies the following property</u> (q <u>sectorial)</u>:

P) for a.c. $x \in \Omega$ $0 \leq \arg q(x) \leq \pi - \delta$, $\delta \in]0, \pi]$

<u>then</u> H <u>is regular</u>.

Proof. Let us initially observe that $\overline{N(H)}$ is contained in the sector S of the complex plane defined by

$$0 \leq \arg \xi \leq \pi - \delta .$$

Then by virtue of the first part of theorem 1.1, it will be sufficient to prove that there exists $\xi \in \mathbb{C} \setminus S$ such that $R(h - \xi I) = L^2$.

Let $\xi \in \mathbb{C} \setminus S$ with $\operatorname{Re} \xi$, $\operatorname{Im} \xi < 0$ and consider $\omega \in R(h - \xi)^{\perp}$ (the orthogonal complement of $R(h - \xi I)$). We shall prove that $\omega = 0$.

Obviously

$$\forall \varphi \in C_0^{\infty}(\mathbb{R}^n): \int_{\mathbb{R}^n} \omega \, \overline{\Delta \varphi} \, dx = \int_{\mathbb{R}^n} \omega \, \overline{(q - \xi)\varphi} \, dx$$

and thus

$$\Delta \omega = \overline{(q - \xi)} \omega \quad \text{(in the sense of distributions).}$$

Then, by (2.1) and by well known regularity theorems, it can be deduced that ω belongs to W^2_{loc}. In the following we shall adapt to our case some tricks used by F. Browder in [2].

Let us consider $\chi \in C_0^\infty(\mathbb{R}^n)$ with

$$0 \le \chi(x) \le 1 \quad \forall\, x \in \mathbb{R}^n \text{ and } \chi(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 2 \end{cases}$$

and set $\chi_\varepsilon(x) = \chi(\varepsilon x)$.

Then, by lemma 2.1, $\chi_\varepsilon^2 \omega \in D(h - \zeta I)$ and

$$(h - \zeta I)(\chi_\varepsilon^2 \omega) = -\Delta(\chi_\varepsilon^2 \omega) + (q - \zeta)(\chi_\varepsilon^2 \omega)$$

therefore, remembering that $\omega \in R(h - \zeta I)^\perp$, we obtain

$$(2.5) \quad |(-\Delta(\chi_\varepsilon \cdot \omega) + (q - \zeta)(\chi_\varepsilon \cdot \omega) \mid \chi_\varepsilon \cdot \omega)_0| =$$
$$= |(-\Delta(\chi_\varepsilon^2 \cdot \omega) + (q - \zeta)(\chi_\varepsilon^2 \cdot \omega) \mid \omega)_0 + (-\Delta(\chi_\varepsilon \omega) \mid \chi_\varepsilon \cdot \omega)_0 +$$
$$+ (\Delta(\chi_\varepsilon^2 \omega) \mid \omega)_0| = |((h - \zeta I)(\chi_\varepsilon^2 \omega) \mid \omega)_0 + (-\Delta(\chi_\varepsilon \omega) \mid \chi_\varepsilon \cdot \omega)_0 +$$
$$+ (\Delta(\chi_\varepsilon^2 \cdot \omega) \mid \omega)_0| = |(-\Delta(\chi_\varepsilon \cdot \omega) \mid \chi_\varepsilon \cdot \omega)_0 + (\Delta(\chi_\varepsilon^2 \cdot \omega) \mid \omega)_0| .$$

On the other hand, if $1 > \eta > 0$, we have

$$(2.6) \quad |(-\Delta(\chi_\varepsilon \cdot \omega) + (q - \zeta)(\chi_\varepsilon \cdot \omega) \mid \chi_\varepsilon \cdot \omega)_0| =$$
$$= \left| \int_{\mathbb{R}^n} |\operatorname{grad}(\chi_\varepsilon \cdot \omega)|^2 dx + \int_{\mathbb{R}^n} (q(x) - \zeta) |\chi_\varepsilon(x) \omega(x)|^2 dx \right| \ge$$
$$\ge \frac{1}{2} \Big\{ \eta \left| \int_{\mathbb{R}^n} |\operatorname{grad}(\chi_\varepsilon \cdot \omega)|^2 dx + \int_{\mathbb{R}^n} \operatorname{Re}(q(x) - \zeta) |\chi_\varepsilon \cdot \omega|^2 dx \right| +$$
$$+ \int_{\mathbb{R}^n} |\operatorname{Im}(q(x) - \zeta)| \cdot |\chi_\varepsilon \cdot \omega|^2 dx \Big\} \ge \frac{1}{2} \Big\{ \eta \cdot \int_{\mathbb{R}^n} |\operatorname{grad}(\chi_\varepsilon \cdot \omega)|^2 dx +$$
$$+ \eta \int_{\Omega_+} \operatorname{Re}(q(x) - \zeta) |\chi_\varepsilon \cdot \omega|^2 dx - \eta \int_{\Omega_-} |\operatorname{Re}(q(x) - \zeta)| \cdot |\chi_\varepsilon \cdot \omega|^2 dx +$$
$$+ \int_{\mathbb{R}^n} |\operatorname{Im}(q(x) - \zeta)| \cdot |\chi_\varepsilon \cdot \omega|^2 dx \Big\}$$

where $\Omega_+ = \{x \in \mathbb{R}^n \mid \operatorname{Re}(q(x) - \zeta) > 0\}$, $\Omega_- = \mathbb{R}^n \setminus \Omega_+$.

Now, remembering that $q(x) - \zeta$ lies in the sector S for a.e. $x \in \mathbb{R}^n$, we can choose η and $\gamma > 0$ so small that

$$(2.7) \quad \frac{1}{2} \int_{\Omega_-} |\operatorname{Im}(q(x) - \zeta)| \cdot |\chi_\varepsilon \omega|^2 dx \ge (\eta + \gamma) \int_{\Omega_-} |\operatorname{Re}(q(x) - \zeta)| \cdot |\chi_\varepsilon \omega|^2 dx.$$

Then from (2.6),(2.7) we deduce that

$$(2.8)\quad |(-\Delta(\chi_\varepsilon\cdot\omega)+(q-\xi)(\chi_\varepsilon\cdot\omega)\,|\,\chi_\varepsilon\cdot\omega)|_0 \geq$$

$$\geq \frac{1}{2}\{\eta\int_{\mathbb{R}^n}|\mathrm{grad}(\chi_\varepsilon\cdot\omega)|^2dx+\eta\int_{\Omega_+}\mathrm{Re}(q(x)-\xi)|\chi_\varepsilon\omega|^2dx+$$

$$+\gamma\int_{\Omega_-}|\mathrm{Re}(q(x)-\xi)|\cdot|\chi_\varepsilon\omega|^2dx+\frac{1}{2}\int_{\mathbb{R}^n}|\mathrm{Im}(q(x)-\xi)|\cdot|\chi_\varepsilon\cdot\omega|^2dx\}\geq$$

$$\geq c_1\{\int_{\mathbb{R}^n}|\mathrm{grad}(\chi_\varepsilon\cdot\omega)|^2+|q(x)-\xi|\cdot|\chi_\varepsilon\cdot\omega|^2dx\}\geq c_2\|\chi_\varepsilon\cdot\omega\|_1^2$$

where c_1, c_2 are positive constants.

On the other hand it can be easily verified that

$$(2.9)\quad |(\Delta(\chi_\varepsilon^2\cdot\omega)\,|\,\omega)_0-(\Delta(\chi_\varepsilon\cdot\omega)|\chi_\varepsilon\omega)_0|=$$

$$=|\sum_i\int_{\mathbb{R}^n}\{2\;|\frac{\partial\chi_\varepsilon}{\partial x_i}|^2\cdot|\omega|^2+\chi_\varepsilon\frac{\partial^2\chi_\varepsilon}{\partial x_i^2}|\omega|^2+2\frac{\partial\chi_\varepsilon}{\partial x_i}\frac{\partial\omega}{\partial x_i}\cdot\bar\omega\cdot\chi_\varepsilon\}\,dx|$$

then from (2.5),(2.8),(2.9) we deduce that

$$(2.10)\quad I\equiv|\sum_i\int_{\mathbb{R}^n}\{2\,|\frac{\partial\chi_\varepsilon}{\partial x_i}|^2\cdot|\omega|^2+\chi_\varepsilon\frac{\partial^2\chi_\varepsilon}{\partial x_i^2}\;|\omega|^2+$$

$$+2\frac{\partial\chi_\varepsilon}{\partial x_i}\frac{\partial\omega}{\partial x_i}\,\bar\omega\,\chi_\varepsilon\}\,dx\,|\geq c_2\cdot\|\chi_\varepsilon\cdot\omega\|_1^2\ .$$

On the other hand we have

$$(2.11)\quad \|\chi_\varepsilon\cdot\omega\|_1^2\geq\|\chi_\varepsilon\cdot\omega\|_0^2+\sum_i(\|\frac{\partial\chi_\varepsilon}{\partial x_i}\cdot\omega\|_0^2+\|\chi_\varepsilon\frac{\partial\omega}{\partial x_i}\|_0^2-$$

$$-2\|\frac{\partial\chi_\varepsilon}{\partial x_i}\omega\|_0\cdot\|\chi_\varepsilon\frac{\partial\omega}{\partial x_i}\|_0)\geq\|\chi_\varepsilon\cdot\omega\|_0^2+\sum_i(\|\frac{\partial\chi_\varepsilon}{\partial x_i}\cdot\omega\|_0^2+$$

$$+\|\chi_\varepsilon\frac{\partial\omega}{\partial x_i}\|_0^2-\varepsilon c_3(\|\omega\|_0^2+\|\chi_\varepsilon\frac{\partial\omega}{\partial x_i}\|_0^2))$$

where $c_3=\sup|\mathrm{grad}\,\chi(x)|$.

Moreover it is easy to see that

$$(2.12)\quad I\leq c_4(\varepsilon^2\|\omega\|_0^2+\varepsilon(\|\omega\|_0^2+\sum_i\|\frac{\partial\omega}{\partial x_i}\chi_\varepsilon\|_0^2))$$

where c_4 is a positive constant.

Let us set

$$I_1 = c_4(\varepsilon^2 \|\omega\|_0^2 + \varepsilon\|\omega\|_0^2),\ I_2 = \varepsilon c_4 \cdot \sum_i \|\frac{\partial\omega}{\partial x_i}\chi_\varepsilon\|_0^2 .$$

Then from (2.10),(2.11),(2.12) we have

$$I_1+I_2 \geq I \geq \|\chi_\varepsilon\omega\|_0^2 + \sum_i (\|\frac{\partial\chi_\varepsilon}{\partial x_i}\omega\|_0^2 + \|\chi_\varepsilon \cdot \frac{\partial\omega}{\partial x_i}\|_0^2 -$$

$$- \varepsilon c_3(\|\omega\|_0^2 + \|\chi_\varepsilon \frac{\partial\omega}{\partial x_i}\|_0^2))$$

From which we obtain

$$I_1 + \varepsilon c_3 n \|\omega\|_0^2 \geq \|\chi_\varepsilon \cdot \omega\|_0^2 + \sum_i (\|\frac{\partial\chi_\varepsilon}{\partial x_i}\omega\|_0^2 +$$

$$+\|\chi_\varepsilon \frac{\partial\omega}{\partial x_i}\|_0^2 - \varepsilon c_3' \|\chi_\varepsilon \frac{\partial\omega}{\partial x_i}\|_0^2) - I_2 .$$

Then we deduce, if ε is sufficiently small, that

$$\varepsilon\|\omega\|_0^2 \geq c_5 \|\chi_\varepsilon \cdot \omega\|_0^2$$

where c_5 is a positive constant.

Then

$$\varepsilon \int_{\mathbb{R}^n} |\omega(x)|^2 dx \geq c_5 \int_{\{|x|<1/\varepsilon\}} |\omega(x)|^2 dx$$

Letting $\varepsilon \longrightarrow 0$, we deduce that $\omega = 0$.

Q.E.D.

Let us now prove the following theorem

<u>Theorem 2.3</u> - <u>Let</u> q <u>satisfy</u> (2.1) <u>and the property</u> P). <u>Let us assume also that</u> $\inf_{x\in\mathbb{R}^n}$ ess $|q(x)| > 0$. <u>Then the following statements are equivalent</u>.

a) $\mathring{\Gamma}_{|q|}$ <u>is compactly embedded in</u> L^2.

b) h <u>has compact resolvent</u>.

Proof. By theorem 2.2, H is regular then $\sigma(h)$ is contained in the sector (of the complex plane) S defined by $0 \leq \arg \zeta \leq \pi - \sigma$.

Let us now prove that

a) $\Longrightarrow$ b)

By the same arguments used in proving the inequality (2.8), we have:

$$(2.13) \qquad \forall\, u \in C_0^\infty(\mathbb{R}^n)\ \|hu\|_0^2 + \|u\|_0^2 \geq |(hu|u)_0| \geq c_1 \|u\|^2_{\overset{o}{\Gamma}_{|\varphi|}}$$

where $c_1 > 0$ independent of u.

Then, by a limit procedure, it is easily deduced that (2.13) holds for each $u \in D(h)$. So we conclude that D(h), with the graph norm, is continuously embedded into $\overset{o}{\Gamma}_{|\varphi|}$. Then, if $\overset{o}{\Gamma}_{|\varphi|}$ is compactly embedded in L^2, h has compact resolvent by theorem 1.5.

Let us now prove that

b) $\Longrightarrow$ a)

Obviously there exists $\vartheta \in\,]-\pi/2, \pi/2[$ such that the spectrum $\sigma(e^{i\vartheta} h)$ of the operator $e^{i\vartheta} h$ is contained in a sector S′ defined by $|\arg \zeta| \leq \gamma < \pi/2$. Then $e^{i\vartheta} h$ is <u>m-sectorial</u> [4, ch. V p.280].

Let us now construct the operator B=Re $e^{i\vartheta}$ h, <u>real part</u> of the <u>m-sectorial</u> operator $e^{i\vartheta}$ h [4,ch. VI p. 336]. To this end let us observe that the sesquilinear form

$$a(u,v) = (e^{i\vartheta} hu|v)_0 \qquad u,v \in D(h)$$

is closable [4, Th.1.27 p.318]; let us set

$$b(u,v) = \frac{1}{2}\{\hat{a}(u,v) + \overline{\hat{a}(v,u)}\} \quad u,v \in D(\hat{a}) = D(b)$$

where $\hat{a}$ denotes the <u>closure</u> of a [4,ch. VI].

Obviously b is symmetric (i.e. $b(u,v) = \overline{b(v,u)}$), closed

[4, Th.1.31 p. 319] and positive (i.e. $b(u,u)\geq 0$), then there exists a unique self-adjoint operator B (Friedrichs extension) such that

$$D(B)\subset D(b) \text{ and } b(u,v) = (Bu|v)_0 \qquad \forall\, u,v\in D(B).$$

B is called the <u>real part</u> of the operator $e^{i\vartheta}h$.

Now $e^{i\vartheta}h$ has compact resolvent, then, by a well known result [4, Th.3.3 p. 337], B has compact resolvent: therefore, by theorem 1.3 and theorem 1.5, we deduce that $D(B^{1/2})$, equipped with the scalar product

$$(B^{1/2}u\ B^{1/2}v)_0 + (u|v)_0 = b(u,v)+(u|v)_0, \quad u,v\in D(b).$$

is compactly embedded in L^2.

On the other hand it is easily seen that we have

$$\forall\, u\in C_0^\infty(\mathbb{R}^n) \quad b(u,u)\leq c\, \|u\|^2_{\overset{\circ}{\Gamma}_{|q|}}$$

where $c > 0$ independent of u.

Then $\overset{\circ}{\Gamma}_{|q|}$ is continuously embedded into D(b). Therefore we conclude that $\overset{\circ}{\Gamma}_{|q|}$ is compactly embedded in L^2.

Q.E.D.

Remark 2.3 - Let us consider $\rho\in L^1_{loc}$ with inf ess $\rho(x) > 0$; then the embedding $\overset{\circ}{\Gamma}_{|q|}\hookrightarrow L^2$ is compact if [1, th. 3.1]

$$\int_{S(x)}(1/\rho(y))dy \longrightarrow 0 \text{ for } |x|\longrightarrow +\infty \tag{2.14}$$

where S(x) is the unit sphere in $\mathbb{R}^n$ centered at x. Observe that (2.14) is obviously satisfied if $\rho(x)\longrightarrow +\infty$ for $|x|\longrightarrow +\infty$.

By virtue of a well known theorem due to Molchanow [3,7], it is not difficult to prove that a necessary and sufficient condition for the compactness of the embedding $\overset{\circ}{\Gamma}_\rho \hookrightarrow L^2$

is the following one (Molchanov condition):

(M) <u>there exists</u> $\varepsilon > 0$ <u>s.t.</u>, <u>if</u> F <u>is any closed subset of</u> C(x) (C(x) <u>denotes the unit edge cube centered at</u> $x \in \mathbb{R}^n$) <u>with capacity</u> $c(F) < \varepsilon$

$$\int_{C(x)\setminus F} \rho(y)dy \longrightarrow +\infty \quad \text{for } |x| \longrightarrow +\infty .$$

So if q is a complex potential verifying the assumptions of theorem 2.3, then h has compact resolvent if and only if $|q| = \rho$ satisfies the above condition (M). Let us recall that an analogous result has been obtained for one dimensional Schrödinger operators [6].

Remark 2.4 - Observe that if the potential q does not satisfy the property P), the compactness of the embedding of $\overset{o}{\Gamma}_{|q|}$ into L^2 is not sufficient, in general, to guearantee the discreteness of the spectrum $\sigma(h)$: in fact there are Schrödinger operators h, with real potentials q diverging to $-\infty$ for $|x| \longrightarrow +\infty$, whose spectrum covers the entire real axis [3, Th. 2.8 ch. II].

Remark 2.5 - If we assume Re $q \geq 0$, it can be proved that H is regular without the "local regularity" assumption (2.1) on q. In fact: Let $\xi \in \mathbb{C}$ with Re $\xi < 0$, then, by following analogous arguments as in proving the first part of th. 2.2, we have only to prove that $\omega = 0$ is the unique solution of the equation

(2.15) $\Delta \omega = \overline{(q-\xi)}$ (in the distributional sense).

Now, by virtue of a well known inequality [5], by (2.15) we deduce

$$\Delta |\omega| \geq \text{Re}\{(\text{sign}\,\overline{\omega})\Delta\omega\} = \text{Re}\{\overline{(q-\xi)}\,|\omega|\} \geq 0$$

in the distributional sense. Then, by following the same arguments used in [5], it can be deduced that $\omega = 0$.

Let us finally observe that in such situation (i.e. if $q \in L^2_{loc}$ and Re $q \geq 0$) it can be also proved, by following analogous arguments as in proving th. 2.3, that h has compact resolvent if $\mathring{\Gamma}_{Req} \hookrightarrow L^2$ compactly (cf. remark 2.3). An analogous result has been proved [2, th. 2.6] under more restrictive conditions on the "growth of Req" at infinity and on its local "regularity".

Remark 2.5 - Let Re $q \geq 0$. Then it can be proved [12] that the formal differential operator $L = -\Delta + q$ has an m-accretive realization A in L^2 if $q \in L^r_{loc}$ with

$$r = \begin{cases} 2n/(n-2) & \text{for } n \geq 3 \\ 1 + \varepsilon \ (\varepsilon > 0) & \text{for } n = 2 \\ 1 & \text{for } n = 1 . \end{cases}$$

Moreover, if for a.e. $x \in \mathbb{R}^n$ $q(x)$ belongs to a sector S (of the complex plane) defined by $|\arg \xi| \leq \gamma < \pi/2$, it can be proved [12] that L has an m-accretive extension under the weaker assumption $q \in L^1_{loc}$.

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