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LEAST AND LARGEST INITIAL COMPLETIONS – II

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Abstract: Universal and largest initial completions of a concrete category are studied. This is a continuation of the first part of the paper, published in the same journal, the knowledge of which is assumed.

Key words: Initial completion of a concrete category, universal initial completion, largest initial completion, cartesian closed category.

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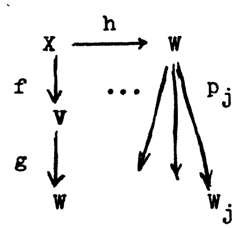
§ 3. Universal Completion. Analogously to the situation of the Mac Neille completion, a universal initial completion of a small category can be described by means of special sources (see [He_1]). We shall consider these sources for arbitrary categories and shall observe that if they form a legitimate conglomerate, then the category of all such sources is the universal initial completion. The converse is true as well, but its proof is more involved.

3.1 A source $S = (X \xrightarrow{f_i} |V_i|)$ is called semi-closed if it has the following properties:

(i) S is closed with respect to composition with morphisms from the left; i.e., given $X \xrightarrow{f} |V|$ in S, then $X \xrightarrow{\bar{g}\cdot f} |W|$ is in

S for each morphism $V \xrightarrow{g} W$.

(ii) S is closed with respect to initial lifts in the sense that given an initial source $(W \xrightarrow{p_j} W_j)$, then every structured map $X \xrightarrow{h} |W|$ belongs to S whenever each $X \xrightarrow{p_j \cdot h} W_j$ belongs to S.



The smallest semi-closed source containing a given source S is called the <u>semi-closed hull</u> of S. (In [HS_2] this is called the <u>standard enrichment</u> of S.)

3.2 <u>Theorem</u>. For a concrete category, $\mathcal{K}$, the following are equivalent:

(i) $\mathcal{K}$ has a universal initial completion;

(ii) the conglomerate of semi-closed sources in $\mathcal{K}$ is legitimate.

If these conditions hold, then the universal initial completion of $\mathcal{K}$ is the category of semi-closed sources.

<u>Proof</u>: (ii) $\Longrightarrow$ (i). This is a straightforward analogue of the proof when $\mathcal{K}$ is small; see [He_1].

(i) $\Longrightarrow$ (ii). Let $(\phi, \mathcal{L})$ be the universal completion (where ϕ is considered as the inclusion of $\mathcal{K}$ into $\mathcal{L}$). For each semi-closed source S from X we have its initial lift P_S in $\mathcal{L}$. It suffices to show that for distinct semi-closed sources S and S′, always $P_S \neq P_{S'}$. Then the conglomerate of all semi-closed sources will be codable by the class of objects of $\mathcal{L}$

and, hence, will be legitimate. To prove this we will define, for each semi-closed source $S = (X \xrightarrow{f_i} |V_i|)$, a special pair $(\phi, \mathcal{L}_S)$ where $\mathcal{L}_S$ is an initially complete category and ϕ is an initiality-preserving concrete functor from $\mathfrak{X}$ into $\mathcal{L}_S$.

<u>Objects of</u> $\mathcal{L}_S$ are triples (H,W,T) where W is an object of $\mathcal{L}$, $H \subseteq \hom_{\mathfrak{X}}(X,|W|)$ and T is an $\mathcal{L}$-source $T = (W \xrightarrow{r_j} \phi(U_j))_J$, subject to the following three conditions:

(a) for each $h \in H$ and each $j \in J$, $X \xrightarrow{r_j \cdot h} |U_j|$ is in S,

(b) T is maximal with respect to (a) (i.e., given an $\mathcal{L}$-morphism $W \xrightarrow{r} \phi(U)$ for which $h \in H$ implies $X \xrightarrow{r \cdot h} |U|$ is in S, then $W \xrightarrow{r} \phi(U)$ must be in T).

(c) H is maximal with respect to (a) (i.e., given $X \xrightarrow{h_0} |W|$ such that $j \in J$ implies $X \xrightarrow{r_j \cdot h_0} |U_j|$ is in S, then h_0 must be in H).

<u>Morphisms of</u> $\mathcal{L}_S$ $q:(H,W,T) \longrightarrow (H',W',T')$ are $\mathcal{L}$-morphisms $q: W \longrightarrow W'$ which are source maps $q:T \longrightarrow T'$. The <u>forgetful functor</u> sends $(H,W,T) \xrightarrow{q} (H',W',T')$ to $|W| \xrightarrow{q} |W'|$. Then $\mathcal{L}_S$ has the following properties:

(i) $\mathcal{L}_S$ <u>is a (legitimate) concrete category</u>.

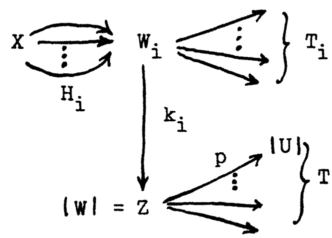
Proof: Legitimacy follows from the fact that each object (H,W,T) is determined by H and W, and W belongs to the class of objects of $\mathcal{L}$ while H belongs to the class of all subsets (!) of morphisms of $\mathcal{L}$. Amnesticity follows from the amnesticity of the forgetful functor for $\mathcal{L}$ and (c) above.

(ii) $\mathcal{L}_S$ <u>is initially complete</u>.

Proof: Each sink

$$(|(H_i,W_i,T_i)| \xrightarrow{k_i} Z)_I$$

has a final lift given by (H,W,T) where W is the final lift of $(|W_i| \xrightarrow{k_i} Z)_I$ in $\mathcal{L}$, T consists of all $W \xrightarrow{p} \phi(U)$



in $\mathcal{L}$ such that each $W_i \xrightarrow{p \cdot k_i} \phi(U)$ is in T_i (for some $i \in I$), and

$H = \{h : X \to Z \mid W \xrightarrow{p} \phi(U)$ in T implies $X \xrightarrow{p \cdot h} |U|$ is in S$\}$.

The <u>concrete functor</u> $\psi : \mathcal{X} \longrightarrow \mathcal{L}_S$ is defined by:

$\psi(V) = (H_V, V, T_V)$

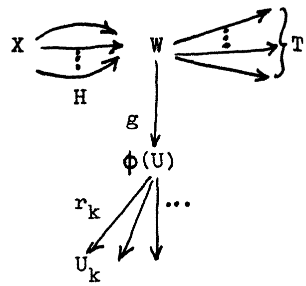
$\psi(f) = f$

where $H_V = S \cap \hom_{\mathfrak{X}}(X, |V|)$; $T_V = \bigcup (\hom_{\mathcal{X}}(V, \phi(U)))_{U \in \mathcal{X}}$

(iii) ψ <u>is a full embedding that preserves initiality</u>.

<u>Proof</u>: We must show that (H_V, V, T_V) is an object of $\mathcal{L}_S$. Clearly T_V is maximal for H_V. Also H_V is maximal for T_V since if $X \xrightarrow{h} |V|$ has the property that $X \xrightarrow{p \cdot h} |U|$ is in S for each $V \xrightarrow{p} \phi(U)$ in T_V, choose $p = 1_V$ to obtain $(h, V) \in S$. From this it is easily verified that ψ is a full embedding.

To show initiality preservation suppose that $(U \xrightarrow{r_k} U_k)$ is an initial source in $\mathcal{X}$. To establish the initiality of $(\psi(U) \xrightarrow{r_k} \psi(U_k))$ consider a map $|W| \xrightarrow{g} |U|$ and an object (H,W,T) such that for each k,



$r_k \cdot g:(H,W,T) \longrightarrow \psi(U_k)$ is a morphism in $\mathcal{L}_S$. By the definition of morphisms in $\mathcal{L}_S$ each $W \xrightarrow{r_k \cdot g} U_k$ is an $\mathcal{L}$-morphism. Since ϕ is initiality preserving, $W \xrightarrow{g} U$ must be an $\mathcal{L}$-morphism. To verify that $g:(H,W,T) \longrightarrow \psi(U)$ is an $\mathcal{L}_S$-morphism it remains to show that for each element of T_U, i.e. for each $\phi(U) \xrightarrow{r} \phi(U')$ in $\mathcal{X}$, we have $W \xrightarrow{r \cdot g} \phi(U')$ in T. Equivalently, for each $h \in H$, $X \xrightarrow{r \cdot g \cdot h} |U'|$ is in S. Since S is semiclosed it suffices to show that for each $h \in H$, $X \xrightarrow{g \cdot h} |U|$ is in S (see 3.1 (i)). But this follows from the fact that for each $k \in K$

$$X \xrightarrow{r_k \cdot g \cdot h} |U_k|$$

is in S (see 3.1 (ii)).

Thus we have an initiality preserving concrete functor ψ from $\mathcal{X}$ into the initially complete category $\mathcal{L}_S$. Thus (by 1.10 (ii)) ψ can be extended to an initiality preserving concrete functor $\psi^* : \mathcal{L} \longrightarrow \mathcal{L}_S$. But $(P_S \xrightarrow{f_i} V_i)$ is an initial source in $\mathcal{L}$, so that $(\psi^*(P_S) \xrightarrow{f_i} \psi(V_i))$ must be initial in $\mathcal{L}_S$. It is easily verified that the initial lift of $X \xrightarrow{f_i} \psi(V_i)$ in $\mathcal{L}_S$ is (H_0, P_S, S) where

$$H_0 = \{ X \xrightarrow{h} X \mid \text{each } X \xrightarrow{f_i \cdot h} |V_i| \text{ is in } S \}.$$

Therefore $\psi^*(P_S) = (H_0, P_S, S)$.

Now let $S' = X \xrightarrow{e_k} |V_k'|$ be another semi-closed source with $P_{S'} = P_S$. Then again $(\psi^*(P_S) \xrightarrow{e_k} \psi(V_k'))$ is an initial source in $\mathcal{L}_S$. Since $1_X : X \longrightarrow X$ is in H_0 and $e_k : S \longrightarrow T_{V_k}'$ is a source map, there follows that $(e_k, V_k') \in S$. Thus we have shown that $P_{S'} = P_S$ implies $S' \subseteq S$. By symmetry, $P_{S'} = P_S$ implies $S' = S$; which completes the proof.

3.3 Next, we consider fibre-small universal initial completions. Two structured maps from X

$$X \xrightarrow{f_1} |U_1| \text{ and } X \xrightarrow{f_2} |U_2|$$

are said to be $\approx$ -equivalent, $(f_1, U_1) \approx (f_2, U_2)$, iff as singleton sources they have the same semi-closed hull.

3.4 Definition. A concrete category is called very strongly fibre-small iff for each object X in $\mathcal{X}$ the conglomerate of all semi-closed sources from X is small; equivalently, iff the conglomerate of all $\approx$ -equivalence classes of structured morphisms $X \xrightarrow{f} |U|$ is small. Dual notion: very strongly co-fibre-small.

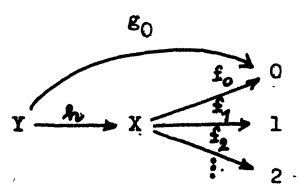
3.5 Theorem. A concrete category has a fibre-small universal initial completion iff it is very strongly fibre-small.

Proof: If the universal initial completion (= the category of semi-closed sources) is fibre-small, then for each X in $\mathcal{X}$ the conglomerate of semi-closed hulls of singleton sources from X is codable by a set.

Conversely, if $\mathcal{X}$ is a very strongly fibre-small category then for each object X in $\mathcal{X}$ there is a representative set A_X of structured maps from X, with respect to $\approx$. If a source is semi-closed then, with each $X \xrightarrow{f} |U|$, it contains all of its semi-closed hull . Hence the conglomerate of all semi-closed sources from X is codable by the set of all subsets of A_X. Thus the category of semi-closed sources is fibre-small.

3.6 <u>Example</u>: [HS_2] For k = 0,1,2,3,3 1/2, the category of topological T_k-spaces has a universal initial completion identical with its Mac Neille completion. E.g., for $T_{3\ 1/2}$ (= completely regular T_1) spaces, the completion is the category of all completely regular spaces. Compare with examples 1.9 and 1.11.

3.7 <u>Example</u>. Let $\mathfrak{X}$ be the category with objects $\{X,Y\} \cup$ $\cup$ <u>Ord</u> (<u>Ord</u> = the class of all ordinals) and morphisms given by: hom (Y,X) = $\{h\}$, hom (X,i) = $\{f_i\}$ for all $i \in$ <u>Ord</u>, hom (Y,i) = = $\{g_i\}$ for all $i \in$ Ord (where $g_i = f_i \cdot h$), hom (A,A) = 1_A for all objects A; and all other hom-sets empty.



Let $\mathfrak{X}$ be the subcategory obtained by deleting h. Then, considered as a concrete category via the embedding into $\mathfrak{X}$, $\mathcal{X}$ is fibre-small. However, $\mathcal{X}$ does not have a universal initial completion. Indeed, for each class $Q \subseteq$ Ord, we have the semi-closed source

$$S_Q = (X \xrightarrow{f_i} i)_{i \in Q}$$

Since $Q \neq Q'$ implies $S_Q \neq S_{Q'}$, the conglomerate of all semi-closed sources is not legitimate.

As the above example shows, one can easily construct categories with no universal initial completion; nevertheless most "everyday" categories do have a universal initial completion, that is even reflective. Specifically:

3.8 Theorem [HNST] If the faithful functor $||: \mathcal{X} \longrightarrow \mathcal{S}$ has a left adjoint, then $\mathcal{X}$ has a reflective universal initial completion if one of the following conditions holds:

(i) $\mathcal{X}$ is cocomplete and co-well-powered.

(ii) $\mathcal{X}$ is complete, well-powered, and co-well-powered.

(iii) $\mathcal{X}$ has (epi, $\underline{M}$)-factorizations and diagonalizations for some conglomerate of $\mathcal{X}$-sources $\underline{M}$.

§ 4. Largest Completion. There is a very natural initial completion of any (small) concrete category (over Set) already found by Antoine (+) [AN_1] and Day (+) [D] and treated generally by Herrlich [He_1]. This turns out to be the largest initial completion, defined again by means of special sources.

4.1 A source is called weakly-closed iff it is closed with respect to composition with morphisms from the left (cf. 3.1 (i)). The smallest weakly-closed source containing a given source S is called the weakly-closed hull of S. It consists of all $X \xrightarrow{h \cdot f} |U|$ for which $X \xrightarrow{f} |V|$ is in S and $V \xrightarrow{h} U$ is a morphism. The dual notion is weakly-closed sink; i.e., a sink closed with respect to composition with morphisms from the right. If $\mathcal{X}$ is a small category, we can define the category $\mathcal{Z}$ of all weakly-closed sources and source maps. This is the largest initial completion of $\mathcal{X}$. If $\mathcal{X}$ is a large category,

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(+) Antoine and Day worked with the dual notion, obtaining the largest final completion.

the same is true provided that $\bar{\mathfrak{L}}$ is legitimate. Similarly with the cases of the Mac Neille and universal initial completions, if $\bar{\mathfrak{L}}$ is not legitimate, then $\mathcal{K}$ does not have a largest initial completion. But now many usual categories over Set fail to yield $\bar{\mathfrak{L}}$ legitimate.

4.2 Theorem. For a concrete category, $\mathcal{K}$, the following are equivalent:

(i) $\mathcal{K}$ has a largest initial completion;

(ii) the conglomerate of weakly-closed sources in $\mathcal{K}$ is legitimate.

If these conditions hold, then the largest initial completion is the category of weakly-closed sources.

Proof: (ii) $\Longrightarrow$ (i). This is a straightforward analogue of the proof when $\mathcal{K}$ is small; see [He_1].

(i) $\Longrightarrow$ (ii). Let $(\phi, \mathcal{L})$ be the largest initial completion (where ϕ is considered as the inclusion of $\mathcal{K}$ into $\mathcal{L}$). For each weakly-closed source S we have its initial lift P_S in $\mathcal{L}$. We need to show that $S \neq S'$ implies $P_S \neq P_{S'}$. Given a weakly-closed source $S = (X \xrightarrow{f_i} |V_i|)$ we define a new completion $\mathcal{L}_S$ of $\mathcal{K}$. This is analogous to the category $\mathcal{L}_S$ in 3.2 except that objects (H,W,T) have the additional property that the source $T = (W \xrightarrow{h_j} \phi(U_j))$ is initial in $\mathcal{L}$. Again, $\mathcal{L}_S$ is a (legitimate) initially complete category. Define a concrete functor $\psi : \mathcal{L} \longrightarrow \mathcal{L}_S$ by:

$$\psi(W) = (H_W, W, T_W) \text{ where } T_W = \bigcup \hom_{\mathcal{L}}(W, \phi(U))_{U \in \mathcal{K}} \text{ and}$$

$$H_W = \{ f : P_S \longrightarrow W \in \mathcal{L} \mid W \xrightarrow{g} \phi(U) \in T_W \text{ implies } X \xrightarrow{g \cdot f} |U| \in S \}$$

Using the fact that ϕ is initially dense and that ψ is a full embedding, one can verify that $\psi(W)$ is indeed an object of $\mathcal{L}_S$. Furthermore each object (H,W,T) is clearly the initial lift of the source of all

$$|W| \xrightarrow{g} |\psi(U)| \text{ with } W \xrightarrow{g} \phi(U) \text{ in } T.$$

Thus ψ is an initially dense full embedding. Since $(\phi, \mathcal{L})$ is the largest completion, it follows that ψ is an isomorphism, which implies that any $\mathcal{L}_S$-object (H,W,T) is identical with the object $\psi(W) = (H_W, W, T_W)$. Since, in particular, (H_0, P_S, S) with

$$H_0 = \{X \xrightarrow{h} X \mid X \xrightarrow{f_i} |V_i| \text{ in } S \text{ implies } X \xrightarrow{f_i \cdot h} |V_i| \in S\}$$

is an $\mathcal{L}_S$-object, we conclude $S = T_{P_S}$.

Thus $S \neq S'$ implies $P_S \neq P_{S'}$.

4.3 Example. The largest initial completion of the category Set over itself ($\mathcal{K} = \mathcal{X} =$ Set) is $(\phi, \mathcal{L})$ described as follows: The objects of $\mathcal{L}$ are all pairs (X,D) where X is a set and D is a subset of the partially-ordered set of all equivalence relations on X satisfying: $d \in D$, $d \leq d_1$ implies $d_1 \in D$. The morphisms $f:(X,D) \longrightarrow (X',D')$ of $\mathcal{L}$ are maps $f:X \longrightarrow X'$ satisfying: $d \in D'$ implies $(f \times f)^{-1}(d) \in D$. The embedding ϕ carries X to $(X, \emptyset)$ and $X \xrightarrow{f} X'$ to $(X,\emptyset) \xrightarrow{f} (X',\emptyset)$.

That this is the largest initial completion can be seen from the fact that each complete source $(X \xrightarrow{f_i} X_i)_I$ in Set is determined by the set D of all equivalence relations induced by these maps; $D = \{\text{Ker } f_i \mid i \in I\}$.

4.4 Example. A category need not have a largest completion over itself! Let $\mathfrak{L} = \mathfrak{K}$ be the category $\mathfrak{L}$ of Example 3.7. For each subclass $Q \subseteq$ Ord we have a weakly-closed source $S_Q = (X \xrightarrow{f_i} i)_{i \in Q}$ and $Q \neq Q'$ implies $S_Q \neq S_{Q'}$, Hence the conglomerate of all weakly-closed sources fails to be legitimate. On the other hand, each category is initially complete over itself, hence it is identical with its universal initial completion.

4.5 Example. The largest final completion of Set over itself is $(\phi, \mathcal{L})$ described as follows: The objects of $\mathcal{L}$ are all pairs (X,D) where X is a set and D is a subset of the power set of X satisfying: $A \in D$ and $A_1 \subseteq A$ implies $A_1 \in D$. The morphisms $f:(X,D) \longrightarrow (X',D')$ of $\mathcal{L}$ are all maps $f:X \longrightarrow X'$ satisfying: $A \in D$ implies $f[A] \in D'$. The embedding ϕ carries $X \xrightarrow{f} X'$ to $(X,\emptyset) \xrightarrow{f} (X',\emptyset)$.

4.6 Unlike the situation for Mac Neille or universal completions, "everyday" concrete categories often fail to have largest initial completions or largest final completions, as the following theorem and examples show. To aid in what follows, we will call a proper class P of objects of a concrete category nearly rigid iff there is a cardinal number α such that:

(i) $V \in P$ implies card $|V| > \alpha$; and

(ii) for each morphism $U \xrightarrow{f} V$ between distinct objects of P, card $f[|U|] \leq \alpha$.

4.7 Theorem. If a concrete category $\mathfrak{K}$ over Set has a nearly rigid class of objects, then it has neither a largest initial completion nor a largest final completion.

Proof: Let P be a nearly rigid class in $\mathfrak{K}$ with respect to a cardinal α. For each subclass $Q \subseteq P$ we shall define a weakly-closed source S_Q and a weakly-closed sink T_Q such that $Q \neq Q'$ implies both $S_Q \neq S_{Q'}$ and $T_Q \neq T_{Q'}$. Thus neither the conglomerate of weakly-closed sources nor the conglomerate of weakly-closed sinks is codable by a class. Choose a set X with cardinality the least cardinal larger than α. For $Q \subseteq P$, let S_Q be the weakly-closed source of all $X \xrightarrow{h \cdot f} |V|$ where $U \xrightarrow{h} V$ is a morphism, $U \in Q$, and $X \xrightarrow{f} |U|$ is a one-to-one map. If Q' is a distinct subclass of P, say with $U_0 \in Q \setminus Q'$, then there is a one-to-one map $X \xrightarrow{f_0} |U_0|$ which is in $S_Q \setminus S_{Q'}$. Analogously let T_Q be the weakly-closed sink of all $|V| \xrightarrow{g \cdot q} X$ where $V \xrightarrow{q} U$ is a morphism, $U \in Q$, and $|U| \xrightarrow{g} X$ is a surjective map.

4.8 Examples. In the set theory satisfying the axiom:

(M) There exists only a set of measurable[+] cardinals; many categories are known to have a nearly rigid class. Each of the following categories has a nearly rigid class with $\alpha = 1$:

(i) lattices [S];

(ii) compact Hausdorff space [T_1] ;

(iii) metrizable spaces [T_2] .

Each of the following categories has a nearly rigid class with $\alpha = 0$:

(iv) semigroups [HL] ;

(v) rings and integral domains with unit [FS] ;

(vi) symmetric graphs [HP_2] ;

(vii) 0-1 lattices [GS] ;

(viii) unary algebras with two operations [HP_1]-even idempotent ones [PS] .

(+) A cardinal number β is called measurable iff there exists a σ-additive $\{0,1\}$-valued measure on a set of cardinality β.

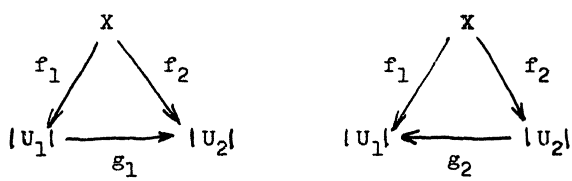
Without any set-theoretical assumptions a nearly rigid class with $\alpha = 0$ exists in the category of paracompact T_2 spaces [Ko] and in hypergraphs [Ku].

Thus, by Theorem 4.7, all of the above mentioned categories fail to have a largest initial or a largest final completion.

4.9 Next, we consider fibre-small largest initial completions. Two structured maps from X

$$X \xrightarrow{f_1} |U_1| \text{ and } X \xrightarrow{f_2} |U_2|$$

are said to be $\approx$ -equivalent, $(f_1,U_1) \approx (f_2,U_2)$, iff there exist morphisms $g_1:U_1 \longrightarrow U_2$ and $g_2:U_2 \longrightarrow U_1$ such that



commute; i.e., iff the weakly-closed hulls of the singleton sources are equal.

4.10 Definition. A concrete category is dalled extremely strongly fibre-small iff for each object X in $\mathfrak{X}$ the conglomerate of all weakly-closed sources from X is small; equivalently iff the conglomerate of all $\approx$ -equivalence classes of structured morphisms $X \xrightarrow{f} |U|$ is small.
Dual notion: extremely strongly co-fibre small.

4.11 Theorem. A concrete category has a fibre-small initial comple tion iff it is extremely strongly fibre-small.

Proof: Analogous to the proof of Theorem 3.5.

§ 5. Cartesian Closed Completion. A basic feature of the largest (final) completion $\bar{\mathcal{L}}$ of Antoine (whose base category $\mathfrak{X}$ = Set) is that $\bar{\mathcal{L}}$ is a cartesian closed category. In this section we will consider cartesian closed completions assuming that the base category $\mathfrak{X}$

1) is cartesian closed;

2) is complete and co-well powered; and

3) has a separator.

Categories with a cartesian closed fibre-small initial completion were characterized in [$AK_{1,2}$]. The proof there is formulated for $\mathfrak{X}$ = Set, but it is easily verified to hold in the generality below.

5.1 Let $\mathfrak{X}$ be a concrete category with finite concrete products (= products preserved by the forgetful functor). Two structured maps $X \xrightarrow{f_1} |U_1|$ and $X \xrightarrow{f_2} |U_2|$ are called productive-structurally equivalent (denoted by $(f_1,U_1) \sim^* (f_2,U_2)$) iff for each object W the maps:

$$X \times |W| \xrightarrow{f_1 \times 1} |V_1 \times W| \text{ and } X \times |W| \xrightarrow{f_2 \times 1} |V_2 \times W|$$

are $\sim$-equivalent.

5.2 Definition (see [$AK_{1,2}$]). A concrete category with finite concrete products is called strictly fibre-small if for every X in $\mathfrak{X}$ the conglomerate of all $\sim^*$-equivalence classes of structured morphisms $X \xrightarrow{f} |U|$ is small.

5.3 Theorem. [$AK_{1,2}$] If $\mathfrak{X}$ is a concrete category with finite concrete products, then the following are equivalent:

(i) $\mathfrak{X}$ has a cartesian closed fibre-small initial completion that preserves finite products;

(ii) $\overline{\mathcal{X}}$ has a cartesian closed fibre-small initiality-preserving initial completion.

(iii) $\mathcal{X}$ is strictly fibre-small.

5.4 Theorem. For each extremely strongly fibre-small concrete category the (fibre-small) largest initial completion is cartesian closed but, in general, is not initiality preserving.

Proof: Let $\overline{\mathcal{L}}$ be the largest initial completion of $\mathcal{X}$. Then $\overline{\mathcal{L}}$ is fibre-small (4.11) and also inherits all of the following properties from $\mathcal{X}$ (see [He_2]): co-well-powered, cocomplete, has finite products and a separator. Thus, we can use Freyd's special adjoint functor theorem in its dual form. It suffices to show that the functors:

$$S\times\_\_ : \overline{\mathcal{L}} \longrightarrow \overline{\mathcal{L}} \quad (S \text{ in } \overline{\mathcal{L}})$$

all preserve colimits. Then they will all be left adjoints; i.e., $\overline{\mathcal{L}}$ will be cartesian closed.

By Theorem 4.2 $\overline{\mathcal{L}}$ is (isomorphic to) the category of weakly-closed sources. It is readily verified that for each weakly-closed source S the functor $S\times\_\_$ does preserve colimits. Both finite products and colimits in $\overline{\mathcal{L}}$ are hence "natural". If T is another weakly-closed source, then $S\times T$ is the weakly-closed source of all

$$X\times Y \xrightarrow{f\cdot\pi_X} |V| \text{ and } X\times Y \xrightarrow{g\cdot\pi_X} |W|$$

with $X \xrightarrow{f} |V|$ in S, $Y \xrightarrow{g} |W|$ in T and π_X, π_Y projections. Furthermore given a small functor $D: \mathcal{D} \longrightarrow \overline{\mathcal{L}}$ with $|D(d)| = Y_d$ (d an object of $\mathcal{D}$) let $(Y_d \xrightarrow{\nu_d} Y)_{d\in\mathcal{D}}$ be the colimit of the underlying functor D_0 ($= |\,|\cdot D$) in $\mathcal{X}$. Then colim D is the source of all $Y \xrightarrow{g} |W|$ subject to the condition that:

for each d, $Y_d \xrightarrow{g\cdot\nu_d} |W|$ is in the source D(d).

Since $\mathfrak{X}$ is cartesian closed, we have

$$X \times \operatorname{colim} D_0 = \operatorname{colim}\,(X \times D_0)$$

from which it easily follows that:

$$S \times \operatorname{colim} D = \operatorname{colim}\,(S \times D).$$

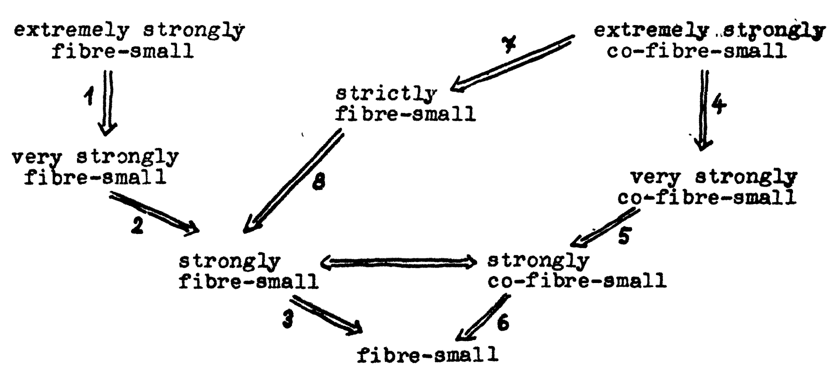
5.5 Theorem. The largest final completion is cartesian closed (and small fibred and initiality preserving) for every extremely strongly co-fibre-small category.

Proof: Analogous to the proof of Theorem 5.4 - but not dual, since the dual for $\mathfrak{X}$ need not be cartesian closed.

5.6 Corollary. Each extremely strongly co-fibre-small category is strictly fibre-small.

Clearly, by 5.3 (i) and 2.7, each strictly fibre-small category is strongly fibre-small.

§ 6. Implications Among Fibre-smallness Conditions. The following diagram summarizes some of the above results concerning fibre-small completions:



None of these implications can be reversed as the following counter-examples show:

1: compact Hausdorff spaces (see 4.8 (i))

4: dual to 1

2: Example 3.7

5: dual to 2

3: [He_1, 3.1 c]

6: dual to 3

7: The cartesian closed category of compactly generated Hausdorff spaces has no largest initial completion (see 4.8 (ii))

8: An example of an initially complete fibre-small category (hence very strongly fibre-small and very strongly co-fibre-small) which fails to be strictly fibre-small is exhibited in [$AK_{1,2}$].

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