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LEAST AND LARGEST INITIAL COMPLETIONS – I

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Abstract: Necessary and sufficient conditions are exhibited for a concrete category to have an initial completion and to have a fibre-small one. Special types of completions are considered (e.g. least, largest, universal, and cartesian closed ones). Many examples illuminating the theory are also provided.

Key words: Initial completion of a concrete category, Mac Neille completion, fibre small category.

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Introduction. We study full extensions $\mathcal{L}$ of a given concrete category $\mathcal{K}$ such that in $\mathcal{L}$ each source of maps has an initial [He$_1$] (or projective [Hu] or optimal [Ma]) lift. Then $\mathcal{L}$ is called an initial (resp. final) completion of $\mathcal{K}$ provided that its objects are initial lifts of sources in $\mathcal{K}$ (resp. final lifts of sinks in $\mathcal{K}$). A category $\mathcal{K}$ can fail to have an initial completion although a number of authors, disregarding set theoretical difficulties, "proved" that all categories have initial completions. This suggests a cautious set-theoretical treatment. In this paper we characterize concrete categories having initial completions at all and having initial completions of various types, as well as those having corresponding fibre-small initial completions. The latter characterizations involve <u>smallness conditions</u> requiring that certain classes be sets (Bernays-Gödel terminology). On the other hand the former characterizations involve <u>legitimacy conditions</u> requiring that certain conglomerates be codable by classes; i.e., be in one-to-one correspondence with classes (§ 1).

Initial completions of a given concrete category are naturally related by: $\mathcal{L}_1 \leq \mathcal{L}_2$ iff $\mathcal{L}_1$ can be extended to $\mathcal{L}_2$. We study existence and "small-fibered existence" of:

1) a smallest initial completion, called the Mac Neille completion (§ 2);

2) a universal initial completion, which is a largest initial completion preserving initial lifts (§ 3);

3) a largest initial completion (§ 4).

Natural constructions of these completions for small concrete categories have been obtained by Herrlich [He_1]. It turns out that these constructions serve for large categories as well, if they are legitimate. <u>Our main result is that if these constructions fail to be legitimate, then the completions do not exist</u>.

By a result of Kučera and Pultr [KP] all "reasonable" categories over sets have a fibre-small Mac Neille completion, and by results of Herrlich, Nakagawa, Strecker and Titcomb, [HS_2],[HNST] (cf. also Börger and Tholen [BT]) many also have universal initial completions. Surprisingly, it turns out that "everyday" categories often fail to have a largest completion; e.g., topological spaces - even compact ones!, semigroups, lattices, rings, and graphs.

The smallest and largest completions over sets have been considered by Antoine [An_2]. He exhibited the natural constructions but made the incorrect set-theoretical conclusion that each category has a largest initial completion. In a paper of Adámek and Koubek [$AK_{1,2}$] categories with a cartesian closed fibre-small initial completion are described via a smallness condition. We discuss the interrelationships between conditions considered in this paper and their condition (§ 5 and § 6). Our characterization of categories having a fibre-small initial completion is closely related to results of Kučera and Pultr [KP]. These results will be generalized and reflective initial completions will be studied in [AHS].

The first part of the present paper is devoted to the least initial completion; in the second part, published in the same issue of this journal, the universal, largest and cartesian closed initial extensions are discussed.

§ 1. Preliminaries

1.1 As in [HS_1], we will work within a framework of sets, classes, and conglomerates, where every set is a class and every class is a conglomerate. We will say that a conglomerate H is codable by a class K iff there exists an injection from H to K. (Note that set-theorists use a wider notion of codability; but that for our purposes the notion defined above will be satisfactory.) Any conglomerate codable by a class will be called legitimate whereas any codable by a set will be called small. No extensive use of set theory will be employed. Indeed, we will essentially use only two facts: The conglomerate of all sets is a class; and the conglomerate of all subclasses of a proper class is not legitimate.

1.2 A concrete category over a (base) category $\mathfrak{X}$ is a category $\mathcal{K}$ equipped with a faithful amnestic[(+)] functor $\mathcal{X} \longrightarrow \mathfrak{X}$, which we will denote by

$$U \xrightarrow{f} V \quad \longmapsto \quad |U| \xrightarrow{f} |V|$$

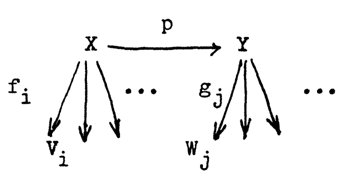
($\mathfrak{X}$-morphisms are called <u>maps</u> for distinction.)
We will assume throughout that such $\mathcal{X}$, $\mathfrak{X}$, and functor are given. The concrete category will be called <u>fibre-small</u> if for each object X in $\mathfrak{X}$ there exists only a set of objects V in $\mathcal{X}$ with $|V| = X$.

1.3 A <u>structured map from an object</u> X of $\mathfrak{X}$ is a pair (f,V) where $X \xrightarrow{f} Y$ is a map and V is a $\mathcal{X}$-object with $|V| = Y$. We will use the concise notation $X \xrightarrow{f} |V|$.

A class-indexed family S of structured maps from X is called a <u>source</u> from X . Notationally :

$$S = (X \xrightarrow{f_i} |V_i|)_{i\in I} \text{ or, simply } (X \xrightarrow{f_i} |V_i|)$$

Let $S = (X \xrightarrow{f_i} |V_i|)$ and $T = (Y \xrightarrow{g_j} |W_j|)$ be sources. Then a <u>source map</u> $p:S \longrightarrow T$ is a map $X \longrightarrow Y$ such that for each j there exists some i with $(f_i,V_i) = (g_j \cdot p,W_j)$.



It should be noted that if each $\mathcal{X}$-object V is identified with the source of all structured maps from $|V|$ that are $\mathcal{X}$-morphisms, then source maps between $\mathcal{X}$-objects are precisely $\mathcal{X}$-morphisms.

(+) Amnesticity means that whenever the image of an isomorphism f is an identity map, then f must be an identity morphism.

Dual notions are: <u>structured map into an object</u> X (i.e., a pair (V,f) with $|V| \xrightarrow{f} X$); <u>sink</u> to X (i.e., a class-indexed family $(|V_i| \xrightarrow{f_i} X)_{i \in I}$); and <u>sink map</u>.

1.4 There are two reasons why we cannot speak about the category of all sources in $\mathcal{K}$ (as objects) and all source maps (as morphisms):

(i) There may be too many sources to constitute a class.

(ii) An individual source, when indexed by a proper class cannot be a member of a class.

Only the first obstacle is serious. Once we restrict ourselves to a "class of sources", we "have" a category. More precisely: <u>Convention</u>. Let $\mathcal{L}$ be a conglomerate of (some) sources for a concrete category $\mathcal{K}$. Let $\mathcal{L}$ be codable by a class $\mathcal{L}'$ (i.e., there is an injection $\psi : \mathcal{L} \longrightarrow \mathcal{L}'$). Then $\mathcal{L}$ will be considered as a concrete category whose objects are the images of $\mathcal{L}$ in $\mathcal{L}'$ and whose morphisms are [in a bijective correspondence to] all source maps between sources in $\mathcal{L}$. The forgetful functor sends $\psi(X \xrightarrow{f_i} |V_i|)$ to X.

Notice that if $\mathcal{L}$ contains all $\mathcal{K}$ -objects (in the above sense) then $\mathcal{L}$ is a full concrete extension of $\mathcal{K}$.

Dual considerations, concerning "categories of sinks", are analogous.

1.5 An <u>initial lift</u> of a source $(X \xrightarrow{f_i} |V_i|)$ is an object V with $|V| = X$ such that:

(i) each $V \xrightarrow{f_i} V_i$ is a morphism;

(ii) for every structured map into X, $|W| \xrightarrow{g} X$ such that each $W \xrightarrow{f_i \cdot g} V_i$ is a morphism, it follows that $W \xrightarrow{g} V$ is a morphism.

Such a source $(V \xrightarrow{f_i} V_i)$ is said to be initial. In the terminology of Hušek [Hu] (see also [$AK_{1,2}$]) the object V is said to be projectively generated by the source $(X \xrightarrow{f_i} |V_i|)$. A concrete category is called initially complete if each source has an initial lift.

Dual notions are: final lift of a sink; final sink; inductively generated object; finally complete (a concrete category is known to be finally complete iff it is initially complete).

1.6 Let $\Phi : \mathcal{K} \longrightarrow \mathcal{L}$ be a functor. Then:

(i) Φ is called concrete if for objects

$$|\Phi(V)| = |V|$$

and for morphisms

$$\Phi f = f.$$

(ii) Φ is said to preserve initiality if it is concrete and for each initial cousrce $(V \xrightarrow{f_i} V_i)$ in $\mathcal{K}$, the source $(\Phi(V) \xrightarrow{f_i} \Phi(V_i))$ is initial in $\mathcal{L}$.

(iii) Φ is said to be initially dense if it is concrete and each object in $\mathcal{L}$ is the initial lift of some source of the form $(X \xrightarrow{f_i} |\Phi(V_i)|)$.

(iv) The pair $(\Phi, \mathcal{L})$ (or sometimes just $\mathcal{L}$) is called an initial completion of $\mathcal{K}$ is Φ is an initially dense concrete full embedding and $\mathcal{L}$ is initially complete.

Dual notions are: preserve finality; finally dense; final completion.

Antoine [An_2] has observed that if Φ is initially dense then Φ must preserve finality; and of course, Φ finally dense implies that Φ preserves initiality.

If $(\Phi, \mathcal{L})$ and $(\Phi', \mathcal{L}')$ are each initial (or final) completions of $\mathcal{K}$ then we say that $(\Phi, \mathcal{L})$ is smaller than $(\Phi', \mathcal{L}')$

if there exists a full concrete embedding $\psi : \mathcal{L} \longrightarrow \mathcal{L}'$ with $\phi' = \psi\phi$. Note that "smaller than" is not a partial order since it lacks antisymmetry. However in this context when we speak of a "smallest" (resp. "largest") initial completion we will mean an initial completion that is both smaller (resp. larger) than any other and such that there is no properly smaller (resp. larger) initial completion. Dual notion: "smallest" (resp. "largest") final completion.

1.7 Example. Let $\mathfrak{X}$ be the terminal (single morphism) category. Then concrete categories are just partially ordered classes (amnesticity gives antisymmetry!) and initially complete categories are just large complete lattices. Here initiality preservation means preservation of infima. For two partially ordered classes $\mathcal{K} \subseteq \mathcal{L}$ the class $\mathcal{K}$ is initially dense in $\mathcal{L}$ iff each $V \in \mathcal{L}$ is the infimum of all larger $V_i \in \mathcal{K}$.

1.8 Theorem. Let $\phi_0 : \mathcal{K} \longrightarrow \mathcal{L}_0$ be a full concrete embedding with $\mathcal{L}_0$ initially (= finally) complete. Then the following are equivalent:

(i) ϕ_0 is both initially and finally dense (and preserves both initiality and finality).

(ii) $(\phi_0, \mathcal{L}_0)$ is the smallest initial completion of $\mathcal{K}$.

(iii) $(\phi_0, \mathcal{L}_0)$ is the smallest final completion of $\mathcal{K}$.

This is proved in [He$_1$] in the case where $\mathfrak{X}$ and $\mathcal{K}$ are small categories, but the result holds for large categories as well (see [P]). Moreover:

(1) If a category $\mathcal{K}$ can be concretely fully embedded in an initially complete category $\mathcal{L}$, then $\mathcal{K}$ has an initial completion (e.g. the initial

hull[(+)] of $\mathcal{K}$ in $\mathcal{L}$).

(2) If has an initial completion $\mathcal{L}$, then it has a smallest initial completion, namely the final full[(+)] of $\mathcal{K}$ in $\mathcal{L}$. The smallest initial completion is unique (up to isomorphism) if it exists, and by the above theorem it is self-dual. The smallest initial (final) completion is called the <u>Mac Neille</u> completion.

1.9. <u>Example</u>. The Mac Neille completion of the category of compact spaces (and continuous functions) is the category of compactly generated spaces. That this is true can be seen from the following and Theorem 1.8. The category of compactly generated spaces is well-known to be initially complete and the category of compact spaces is clearly a full concrete subcategory of it. Also since every space can be embedded in a compact one, each space is the initial lift of a singleton source with codomain compact. Hence the inclusion functor is initially dense. Finally a space is compactly generated iff it is the final lift of the sink consisting of all embeddings of its compact subspaces. Thus the inclusion functor is also finally dense.

1.10. <u>Theorem</u>. Let $\phi^*: \mathcal{K} \longrightarrow \mathcal{L}^*$ be an initial completion. Then the following are equivalent:

(i) ($\phi^*, \mathcal{L}^*$) is the largest initiality (and finality) preserving initial completion of $\mathcal{K}$.

(ii) Any initiality preserving concrete functor from into some initially complete category has a unique initiality preserving concrete extension to $\mathcal{L}^*$.

(+) The <u>initial</u> <u>hull</u> of $\mathcal{K}$ in $\mathcal{L}$ is the full subcategory over objects that are initial lifts of sources of the form $(X \xrightarrow{f_i} |\phi(V_i)|)$. The dual notion is <u>final</u> <u>hull.</u>

Again, this is a result in [He_1] which is true for large as well as for small categories. Initial completions with the above properties are uniquely determined (up to isomorphism) and are called <u>universal initial completions</u>. The dual notion is: <u>universal final completions</u>.

1.11 <u>Example</u>. The category <u>Comp Haus</u> of compact Hsudorff spaces has neither a largest initial completion nor a largest final completion (see 4.8 (ii)). However it does have a universal initial completion: namely, the category of all proximity spaces and proximal maps (see [HS_2] or [Ho]).

1.12 Given a concrete category $\mathcal{X}$, we can ask:

(a) Does $\mathcal{X}$ have any initial completion? (equivalently, does it have a Mac Neille completion?).

(b) Does $\mathcal{X}$ have a special initial (or final) completion, universal or largest, etc.?

(c) Which (if any) of these completions is fibre-small?

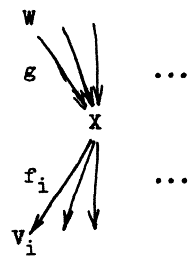
The remainder of the paper is devoted to answering these questions.

§ 2. <u>Mac Neille Completion</u>. For a small concrete category the least (Mac Neille) completion can be described by means of special sources (see [He_1]). We introduce these sources now for arbitrary concrete categories.

2.1 A source $(X \xrightarrow{f_i} |V_i|)$ is said to be <u>closed</u> if it contains each structured map $X \xrightarrow{f} |U|$ having the following property: given a structured map $|W| \xrightarrow{q} X$ such that each $W \xrightarrow{f_i \cdot q} V_i$ is a morphism, it follows that $W \xrightarrow{f \cdot q} U$ is also a morphism.

The smallest closed source, containing a given source S, is called the closed hull of S.
Dual notions are: closed sink; closed hull (of a sink).

2.2 For an arbitrary source $S = (X \xrightarrow{f_i} |V_i|)$ we denote by S^{op} the sink of all structured maps $|W| \xrightarrow{g} X$ such that



each $W \xrightarrow{f_i \cdot g} V_i$ is a morphism. Dually, for each sink T we have its opposite source T^{op}. This defines a Galois correspondence between sources from X and sinks into X (both ordered by inclusion). Closed elements of this correspondence are precisely the closed sources and sinks defined above. Specifically:

(a) for each source S the sink S^{op} is closed; for each sink T, the source T^{op} is closed;

(b) $S^{op\ op}$ is the closed hull of S; $T^{op\ op}$ is the closed hull of T.

2.3 If $\mathcal{X}$ is a small category then all of its sources form a class (because for any object X in $\mathcal{X}$, all sources from X form a set). Hence we can define $\mathcal{L}_0$ to be the category of closed sources and source maps (see 1.4). If we let $\phi_0 : \mathcal{X} \rightarrow \mathcal{L}_0$ be defined by:

$\phi_0(V)$ = the source of all $|V| \xrightarrow{g} |W|$ which are morphisms $V \xrightarrow{g} W$

$\phi_0(f) = f$;

then $\mathcal{L}_0$ is easily verified to be initially complete and ϕ_0 to be an initially and finally dense full embedding. Thus $(\phi_0, \mathcal{L}_0)$ is the Mac Neille completion of $\mathcal{K}$. If $\mathcal{K}$ is not small, all closed sources might still be codable by a class. If so, they form a category (in the sense of 1.4), which is easily seen to be the Mac Neille completion of $\mathcal{K}$. On the other hand, if all closed sources fail to be codable by a class, we shall prove that a Mac Neille completion does not exist.

2.4 Theorem. For a concrete category, $\mathcal{K}$, the following are equivalent:

(i) $\mathcal{K}$ has an initial completion;

(ii) $\mathcal{K}$ has a final completion;

(iii) $\mathcal{K}$ has a Mac Neille completion;

(iv) the conglomerate of closed sources in $\mathcal{K}$ is legitimate;

(v) the conglomerate of closed sinks in $\mathcal{K}$ is legitimate.

If these conditions hold, then the Mac Neille completion of $\mathcal{K}$ is the category of closed sources.

Proof: Clearly (iii) $\Longrightarrow$ (i), and by 2.3 (iv) $\Longrightarrow$ (iii).

To see that (i) $\Longrightarrow$ (iv) assume that $(\phi, \mathcal{L})$ is some initial completion of $\mathcal{K}$. Without loss of generality we consider $\mathcal{K}$ as a full concrete subcategory of $\mathcal{L}$; i.e. ϕ is the inclusion. For each closed source $S = (X \xrightarrow{f_i} |V_i|)$ we have its initial lift (an object) P_S in $\mathcal{L}$. It suffices to verify that distinct closed sources S and S' have distinct initial lifts $P_S \neq P_S'$. Then the conglomerate of all closed sources will be codable by the class of objects of $\mathcal{L}$; hence it will be legitimate.

If $S \neq S'$, without loss of generality, there is some

$X \xrightarrow{f} |U|$ in $S' \setminus S$. Since S is closed, $(f,U) \notin S$ implies that there is some structured map $|W| \xrightarrow{g} X$ such that

(a) $|W| \xrightarrow{f \cdot g} |U|$ is not a morphism in $\mathcal{K}$, yet

(b) for each $X \xrightarrow{f_i} |V_i|$ in S, $|W| \xrightarrow{f_i \cdot g} |V_i|$ is a morphism in $\mathcal{K}$.

By (b) (and the definition of initiality) $W \xrightarrow{g} P_S$ is a morphism in $\mathcal{L}$. Therefore $P_S \neq P_{S'}$, since $W \xrightarrow{g} P_{S'}$ is not a morphism in $\mathcal{L}$. If it were, then $W \xrightarrow{f \cdot g} U$ would be in $\mathcal{L}$ (since $(f,U) \in S'$), in contradiction to (a) and the fact that $\mathcal{K}$ is full in $\mathcal{L}$.

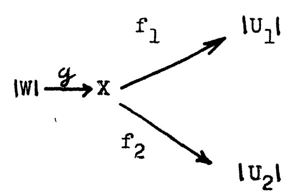
Thus conditions (i),(iii) and (iv) are equivalent. The equivalence of (ii),(iii) and (v) is dual.

2.5 Next, we consider fibre-small Mac Neille completions. Two structured maps from X

$$X \xrightarrow{f_1} |U_1| \text{ and } X \xrightarrow{f_2} |U_2|$$

are said to be $\sim$ -equivalent if for each structured map into X, $|W| \xrightarrow{g} X$, we have:

$$W \xrightarrow{f_1 \cdot g} U_1 \text{ is a morphism iff } W \xrightarrow{f_2 \cdot g} U_2 \text{ is a morphism.}$$



Note that this is equivalent to saying that the two singleton sources $X \xrightarrow{f_1} |U_1|$ and $X \xrightarrow{f_2} |U_2|$ have the same closed hull. We denote this by: $(f_1,U_1) \sim (f_2,U_2)$. Dually, two structured maps into X

$$|U_1| \xrightarrow{f_1} X \text{ and } |U_2| \xrightarrow{f_2} X$$

are $\sim$-equivalent. $(U_1,f_1)\sim(U_2,f_2)$, iff as singleton sinks they have the same closed hull.

2.6 Definition. A concrete category is called strongly fibre-small iff for each object X in $\mathfrak{X}$ the conglomerate of all closed sources from X is small; equivalently, iff the conglomerate of all $\sim$-equivalence classes of structured **maps** $X \xrightarrow{f} |U|$ is small. Notice that (by the Galois correspondence (2.2)) this property is self-dual; i.e., a category is strongly fibre-small iff for each X in $\mathfrak{X}$ the conglomerate of all closed sinks into X is small; equivalently, iff the conglomerate of all $\sim$-equivalence classes of structured **maps** $|U| \xrightarrow{f} X$ is small.

2.7 Theorem. For a concrete category, $\mathfrak{X}$, the following are equivalent:

(i) $\mathfrak{X}$ has a fibre-small initial completion;

(ii) $\mathfrak{X}$ has a fibre-small final completion;

(iii) $\mathfrak{X}$ has a fibre-small Mac Neille completion;

(iv) $\mathfrak{X}$ is strongly fibre-small.

Proof: (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii) is clear. To see that (iii)$\Longrightarrow$(iv) suppose that $\mathfrak{X}$ has a fibre-small Mac Neille completion. Then the category of closed sources must be fibre-small as well (see 2.4). Hence for each X in $\mathfrak{X}$ the conglomerate of distinct closed sources from X is codable by a set. To show that (iv)$\Longrightarrow$(iii) assume that for each object X in $\mathfrak{X}$ **there is a** representative set A_X of structured maps from X, **with respect to** $\sim$. If a source is closed, then, with each $X \xrightarrow{f} |U|$, **it** contains all of its closed hull . Hence the conglome**rate of closed** sources from X is codable by the set of all subsets of A_X.

2.8 Remark. Herrlich [He_1] exhibits subcategories $\mathcal{X}_1$ and $\mathcal{X}_2$ of Set (considered as concrete via their inclusion into Set) which have the following properties:

(a) $\mathcal{X}_1$ has no initial completion.

(b) $\mathcal{X}_2$ has a Mac Neille completion which fails to be fibre-small.

2.9. Mac Neille completion of a linearly ordered class. Let $\mathcal{X} = (K, \leq)$ be a linearly ordered class (concrete in the sense of 1.6). The closed sources (considered instead of $(X \xrightarrow{1_X} |V_i|)_{i\in I}$, simply as subclasses $(V_i)_{i\in I} \subseteq K$) are precisely cuts; i.e., subclasses $C \subsetneq K$ such that

(i) $\inf C \in C$, if the infimum exists; and

(ii) $a \in C,\ b \in K \setminus C \Longrightarrow b < a$.

(The opposite sink is $K \setminus C$ or $(K \setminus C) \cup \{\sup (K \setminus C)\}$ if the supremum exists.) Thus a linearly ordered class has an initial completion iff its cuts are codable by a class.

Examples. In finite set theory (where "set" means finite set; "class" means countable set; and "conglomerate" means set) everyone knows an example of a linearly ordered "class" having no initial completion: namely, the rationals; since they cannot be embedded in any countable complete lattice. This example generalizes immediately to our usual set theory:

Let Ord be the class of all ordinals and let

$$L = \{f: \text{Ord} \longrightarrow \{0,1\}\}$$

ordered lexicographically. Then L is not legitimate. But K = $= \{f \in L \mid \text{either } f^{-1}(0) \text{ is a set or } f^{-1}(1) \text{ is a set}\}$ is legitimate and is sup-dense and inf-dense in L. Hence the linearly ordered class K has no Mac Neille completion; hence no (initial) completion.

2.10 Mac Neille completion of unary algebras. Let $\mathcal{K}$ be the category of unary algebras (X,d), $d:X \longrightarrow X$, concrete over Set. Then the Mac Neille completion of $\mathcal{K}$ is the category of graphs (X,ρ), $(\rho \subseteq X \times X)$ subject to (I) and (P) below, together with graph homomorphisms:

(I) For each $(x,y) \in \rho$ there exists $z \in X$ with $(y,z) \in \rho$

(P) $(x,y) \in \rho$ implies $(x',y') \in \rho$ whenever $x \sim x'$ and $y \sim y'$; where $\sim$ is the least equivalence on X for which $(x,y_1) \in \rho$, $(x,y_2) \in \rho$ implies $y_1 \sim y_2$.

Proof: The category of all graphs is an initially complete full extension of $\mathcal{K}$. In it, the final hull of $\mathcal{K}$ consists of graphs satisfying (I): (X,ρ) satisfies (I) iff $\rho = \bigcup_{j \in J} d_j$ where (X_j,d_j) are all unary algebras with $X_j \subseteq X$ and $d_j \subseteq \rho$. We need to show that the initial hull of $\mathcal{K}$ in the category $\mathcal{L}_I$ of graphs with (I) is the category of graphs with both (I) and (P). If a graph (X,ρ) is the initial lift of a source $(X \xrightarrow{f_j} |(X_j,d_j)|)$ where each (X_j,d_j) is a unary algebra, put $x_1 \approx x_2$ iff $f_j(x_1) = f_j(x_2)$ for each j. Then $\sim$ is clearly finer than $\approx$. Since $\approx$ has the property required in (P), so does $\sim$. Conversely, if a graph (X,ρ) has properties (I) and (P), then the quotient graph under $\sim$, $(\bar{X},\bar{\rho})$, satisfies:

$$[x]\bar{\rho}[y_1] \text{ and } [x]\bar{\rho}[y_2] \text{ implies } [y_1] = [y_2].$$

Hence there exists a unary algebra (Y,d) containing $(\bar{X},\bar{\rho})$. The natural map $(X,\rho) \longrightarrow (\bar{X},\bar{\rho}) \hookrightarrow (Y,d)$ constitutes an initial source.

2.11 Remark. For most usual concrete categories over Set there exists a fibre-small initial completion. This follows from results of Kučera and Pultr [KP]: if a transportable fibre-small category over Set has the property that each morphism $V \xrightarrow{f} W$ factors as $V \xrightarrow{g} U \xrightarrow{h} W$ with g onto and h one-to-one, then $\mathcal{X}$ is strongly fibre-small. A generalization of this result is considered in [AHS].

(Oblatum 2.5. 1978)