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A NOTE ON LEBESGUE SPACES

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Abstract: A simple proof of the isomorphism theorem for Lebesgue spaces is presented and the restriction of the Lebesgue measure to Borel sets is characterized.

Key words: Lebesgue space, isomorphism of probability spaces.

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In this note we present a simple proof of the isomorphism theorem for Lebesgue spaces. Simultaneously we characterize the restriction of the Lebesgue measure to Borel sets.

First some fixed notations. By I we denote the unit interval on the line, B or L resp. the family of all Borel subsets of I or all Lebesgue measurable subsets of I resp., λ the Lebesgue measure on L, ν its restriction to B. Further put $Y = \{0,1\}^N$, where N is the set of all positive integers and denote by T the σ-algebra generated by the family of all cylinders in Y.

A basic step in our proof gives the following lemma.

Lemma. Let μ be a non-atomic probability measure on T. Then (Y,T,μ) and (I,B,ν) are isomorphic.

Proof. Put $B_n = \{y \in Y;\ y_n = 1\}$. We construct $C_n \subset I$,

C_n being union of finite number of intervals such that

$$\mu(B_1^{i_1} \cap B_2^{i_2} \cap \ldots \cap B_n^{i_n}) = \nu(C_1^{i_1} \cap C_2^{i_2} \cap \ldots \cap C_n^{i_n})$$

for every sequence $(i_1,\ldots,i_n)$ of 0 and 1. (Here $B_k^1 = B_k$, $B_k^0 = Y - B_k$ and similarly for C_k^i.) It can be easily constructed by

$$C_1 = \langle 0, \mu(B_1)),$$

$$C_2 = \langle 0, \mu(B_1 \cap B_2)) \cup \langle \mu(B_1), \mu(B_1) + \mu(B_1' \cap B_2))$$

etc. The sets $B_1,\ldots,B_n$ generate a decomposition ξ_n consisting of all non-empty intersections $B_1^{i_1} \cap B_2^{i_2} \cap \ldots \cap B_n^{i_n}$ ($i_k \in \{0,1\}$, $k = 1,\ldots,n$). Similarly let η_n be the decomposition generated by $C_1,\ldots,C_n$. If we put

$$\|\xi\| = \max_{C \in \xi} \mu(C),$$

then evidently $\|\xi_n\| = \|\eta_n\|$ $(n = 1,2,\ldots)$. Since $(\xi_n)_{n=1}^{\infty}$ generates T and μ is non-atomic, we obtain

$$\lim_{n\to\infty} \|\xi_n\| = 0$$

(see [3], § 41, Theorem A). Let K be the set of end-points of all η_n. Then the relation $\lim_{n\to\infty} \|\eta_n\| = \lim_{n\to\infty} \|\xi_n\| = 0$ implies that K is a dense subset of I (see [3], § 41, theorem B).

Now we can construct a mapping $\psi : I - K \longrightarrow Y$ by the following way:

$$(\psi(z))_n = \begin{cases} 1, & \text{if } z \in C_n \\ 0, & \text{if } z \notin C_n \end{cases}.$$

Denote by $Z^{(k)}$ the union of all intersections $B_1^{i_1} \cap \ldots \cap B_k^{i_k}$, where $\mu(B_1^{i_1} \cap \ldots \cap B_k^{i_k}) = 0$. Further let $Z^{(o)}$ be the set of all $y \in Y$, for which $y_n = 0$ for only finitely many indices n and all $y \in Y$ for which $y_n = 1$ for only finitely many indices n. Since μ is non-atomic, every singleton has measure zero; hence also $\mu(Z^{(o)}) = 0$. Therefore, if we put $Z = \bigcup_{i=0}^{\infty} Z^{(i)}$, then $\mu(Z) = 0$.

We now prove that $\psi : I - K \longrightarrow Y$ is a bijection between $I - K$ and $Y - Z$.

Evidently ψ is injective, since $z_1 \neq z_2$ implies the existence of such n that e.g. $z_1 \in C_n$ and $z_2 \notin C_n$ (K is dense and therefore $(C_n)_{n=1}^{\infty}$ separates points), hence $(\psi(z_1))_n = 1$, $(\psi(z_2))_n = 0$ and therefore $\psi(z_1) \neq \psi(z_2)$.

Let $y \in Y - Z$. Since $y \notin Z^{(k)}$, we have $\nu(\bigcap_{n=1}^{k} C_n^{y_n}) = \mu(\bigcap_{n=1}^{k} B_n^{y_n}) > 0$ and hence $\emptyset \neq \bigcap_{n=1}^{k} C_n^{y_n} \subset \bigcap_{n=1}^{k} \overline{C_n^{y_n}}$. Since $(\bigcap_{n=1}^{k} \overline{C_n^{y_n}})_{k=1}^{\infty}$ is a sequence of non-empty closed sets, whose diameters converge to 0, there is exactly one $z \in I$, for which

$$z \in \bigcap_{n=1}^{\infty} \overline{C_n^{y_n}}.$$

The point z is not an end-point for any C_n. Namely, if $z \in K$, then either $y_n = 0$ for almost all n, or $y_n = 1$ for almost all n, i.e. $y \in Z^{(o)} \subset Z$, what is impossible. Since z is not an end-point for $C_n^{y_n}$, but $z \in \overline{C_n^{y_n}}$, we obtain $z \in C_n^{y_n}$ ($n = 1,2,\ldots$). But it means that $(\psi(z))_n = 1$, if $y_n = 1$, $(\psi(z))_n = 0$, if $y_n = 0$, hence $\psi(z) = y$ and $\psi : I - K \longrightarrow Y - Z$ is surjective.

Take $z \in C_n - K$. The relation holds iff z is not an end-point of C_n and lies in the left part under the n-th partition. But it holds iff $(\psi(z))_n = 1$, $\psi(z) \notin Z$. We have proved $\psi(C_n - K) = B_n - Z$. Evidently, $\mu(B_n) = \mu(B_n - Z) = \nu(C_n - K) = \nu(C_n)$. Since these relations hold also for the sets belonging to the rings generated by $(B_n)_{n=1}^{\infty}$ or $(C_n)_{n=1}^{\infty}$ resp., we see that ψ and ψ^{-1} are measurable and measure preserving. Hence ψ is an invertible transformation, (Y,T,μ), (I,B,ν) are isomorphic.

Definition 1. A sequence $(A_n)_{n=1}^{\infty}$ of measurable subsets of a measurable space (X,S) is called separating, if to every $x,y \in X$, $x \neq y$ there is n such that A_n contains exactly one of the points. A separating sequence is called a separating base, if it generates S.

Definition 2. For any separating base $(A_n)_{n=1}^{\infty}$ define $i:X \longrightarrow Y$ by the formula $(i(x))_n = 1$, if $x \in A_n$, $(i(x))_n = 0$, if $x \notin A_n$. We say that $(A_n)_{n=1}^{\infty}$ is a quasicomplete base, if $i(X) \in T$.

Theorem 1. Let (X,S,P) be a non-atomic probability space having a separating quasicomplete base. Then (X,S,P) is isomorphic with (I,B,ν).

Proof. Let $(A_n)_{n=1}^{\infty}$ be a separating quasicomplete base, $i:X \longrightarrow Y$ be the imbedding induced by the base. Put $\mu = Pi^{-1}$, i.e. $\mu(E) = P(i^{-1}(E))$, $E \in T$. Since $i(X) \in T$, $\mu(Y - i(X)) = 0$, evidently (X,S,P) is isomorphic with (Y,T,μ). But Lemma states the isomorphism between (Y,T,μ) and (I,B,ν).

Definition 3. Denote by Pi^{-1} the measure defined on T by the formula $Pi^{-1}(E) = P(i^{-1}(E))$ and by T_c the family of all $(Pi^{-1})^*$ -measurable subsets of Y. A separating sequence is called an almost complete base, if $i(X) \in T_c$ and S is the σ-algebra generated by this base and the family $\{E \subset X;\ P^*(E) = 0\}$.

Theorem 2. Let (X,S,P) be a complete non-atomic probability space having a separating almost complete base. Then (X,S,P) and (I,L,λ) are isomorphic.

Proof. As before, i is one-to-one, $i: X \longrightarrow i(X)$. Let μ_c be the restriction of $(Pi^{-1})^*$ to the σ-algebra T_c of all measurable sets. We see that $\mu_c(Y - i(X)) = 0$, i maps A_n on $\{y;\ y_n = 1\} \cap i(X)$ and these sets generate (after completions) σ-algebras in their spaces. Therefore (X,S,P) and (Y,T_c,μ_c) are isomorphic. Put $\mu = Pi^{-1}: T \longrightarrow R$. Then by Lemma (Y,T,μ) is isomorphic with (I,B,ν). Evidently their completions (Y,T_c,μ_c), $(I,B_c,\nu_c) = (I,L,\lambda)$ are isomorphic, too. Therefore (X,S,P) is isomorphic with (I,L,λ).

R e f e r e n c e s

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