Commentationes Mathematicae Universitatis Carolinae

Jan Čerych

A generalized Bishop-Gončar construction

Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 4, 775--781

Persistent URL: http://dml.cz/dmlcz/105892

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

19,4 (1978)

A GENERALIZED BISHOP - GONČAR CONSTRUCTION

Jan ČERYCH, Praha

Abstract: If A is a closed subalgebra of C(X) and F is a closed subset of X, this note gives a sufficient condition in order that F is an intersection of peak sets for A.

Key words: C(X), Banach function algebra, peak point and set, p-point and set.

AMS: 46J10

The purpose of this paper is to apply the well-known Bishop-Gončar construction (Propositions 1 and 2) of the peak point of a Banach function algebra to the case of the peak set which is not necessarily one-point one, and to prove an essential generalization of this construction (Theorem).

In the whole text A will be the Banach function algebra on X, i.e. the Banach algebra of continuous complex-valued functions defined on the compact Hausdorff space X, with usual algebraic operations and the sup-norm $|\cdot|$, containing constant functions and separating the points of X.

A peak set (for A) is any closed nonvoid subset F of X such that there is an f in A satisfying

$f/_F = 1$, $|f(x)| < 1$ for every x in X - F,

where $f/_F$ denotes the restriction of f to F.

A p-set (for A) is a nonvoid intersection of an arbit-

rary system of peak sets for A.

It is obvious that a p-set is a peak set if (and only if) it is a G_δ-set.

A <u>peak point</u> (<u>p-point</u>) is a one-point peak set (p-set, respectively).

In [2] Gončar has proven the following

<u>Proposition 1.</u> Let X be a metric space, x a point of X. Let $0 < a < b < 1$. Suppose that for any open neighbourhood U of x in X there is an f in A satisfying

$$|f| < 1, \quad |f(x)| > b, \quad |f|_{X-U} < a,$$

where $|f|_Y$ means $\sup \{ |f(y)| : y \text{ in } Y \}$.

Then x is a peak point for A.

Gončar's construction is a beautiful generalization of the well-known Bishop's one [1]; Bishop came to the same conclusion for a special choice $a = 1/4$, $b = 3/4$. Curtis in [3] has proved Gončar's theorem for non-metric space X (he required only the singleton x to be G_δ), and for the cosed subspace A of X(X) which is not necessarily an algebra. Curtis' proof is rather simpler than the Gončar's one.
All these proofs remain valid without any change if we omit the condition "X is metric" or "x is G_δ" and substitute "a p-point" for "a peak point" (cf. Gamelin [4], Chap. II, Sec. 12).

Another, non-constructive, and may be sometimes more fruitful way of researching peak (and interpolate) sets is the way of estimating orthogonal measures to A. Here the classical paper is the Glicksberg's one [5]. Glickberg's results were generalized, for the case of mere subspaces of C(X), by

Bernard [6], Briem [7, 8], Briem and Rao [9]. Bernard in [6] follows both constructive and "measure" ways: his nice constructive result is (in translation from the "interpolation" language to the "peak" one) a precursor of the mentioned Curtis' construction in [3] (via our Proposition 2).

An immediate consequence of Proposition 1 is the following

Proposition 2: Let $0 < a < b < 1$. Let F be a closed nonvoid subset of X. Suppose that for any open neighbourhood U of F in X there is an f in A satisfying

$$|f| < 1, \quad f/_F = b, \quad |f|_{X-U} < a.$$

Then F is a p-set for A.

Proof: Let Y be the tppological space arisen from X by means of identifying all points of F. More precisely, Y is the quotient space of X in accordance with the pairwise disjoint closed covering of X formed from all singletons y, y in X - F, and the set F. It is rather simple to realize that Y is a Hausdorff compact, too. Let B be the subalgebra of A comprised of all functions in A which are constant on F. B may be regarded as a Banach function algebra on Y, and then it satisfies the hypotheses of Proposition 1.

Our aim is to generalize Proposition 2 in the following manner:

Theorem. Let F be a closed nonvoid subset of X, and let $0 < a < 1 \leq b$. Suppose that for any open neighbourhood U of F in X and for any $e > 0$ there is an f in A satisfying $|f| < b$, $|f - 1|_F < e$, $|f|_{X-U} < a$.

Then F is a p-set for A.

Before proceeding to the proof of Theorem, we shall state two lemmas.

Lemma 1. Let U be an open neighbourhood of F in X, and let $0 < e < 1$. Under the hypotheses of Theorem, there is an f in A satisfying

$$|f| < 2b, \quad |f - 1|_F < e, \quad |f|_{X-U} < e.$$

Proof: We shall construct, by induction, functions f_n in A, $n = 1,2,\dots$ such that

(1) $\quad |f_n| < 2b, \quad |f_n - 1|_F < e, \quad |f_n|_{X-U} < a^n.$

The existence of f_1 satisfying (1) follows immediately from the hypotheses. Suppose now $f_1,\dots,f_n$ have been constructed. There is a positive number q for which

(2) $\quad (1 + q)|f_n - 1|_F + q < e.$

Setting

$$V = \{x \text{ in } X: |f_n(x) - 1| < e\} \cap U,$$

V is an open neighbourhood of F in X. By the hypotheses, there is a function g in A satisfying

$$|g| < b, \quad |g - 1|_F < q, \quad |g|_{X-V} < a.$$

Put $f_{n+1} = f_n.g$. Then f_{n+1} is in A and satisfies the induction conditions (2). Indeed, $|f_{n+1}| < 2b$, because

$|f_{n+1}|_V \leq |f_n|_V |g| \leq (1 + e)b < 2b$, and

$|f_{n+1}|_{X-V} \leq |f_n| . |g|_{X-V} \leq 2ba < 2b;$

$|f_{n+1} - 1|_F = |gf_n - g + g - 1|_F \leq |g|_F\,|f_n - 1|_F +$

$\quad + |g - 1|_F < (1 + q)|f_n - 1|_F + q < e$ by (2);

$$|f_{n+1}|_{X-U} \leqq |f_n|_{X-U} \cdot |g|_{X-V} < a^n \cdot a = a^{n+1}.$$

Finally, let m be a positive integer such that $a^m < e$. Putting $f = f_m$, we are done.

<u>Lemma 2</u>. Let f be in A, $K > 0$, and let $|f|_F < K$. Then, for an arbitrary e, $0 < e < 1$, there is a function g in A satisfying

$$|g| < 2bK, \quad |f - g|_F < e,$$

provided all the hypotheses of Theorem are fulfilled.

<u>Proof:</u> Let

$$U = \{x \text{ in } X: |f(x)| < K\}.$$

Obviously U is an open neighbourhood of F in X. Then there exists, by Lemma 1, an h in A such that

$$|h| < 2b, \quad |h - 1|_F < eC, \quad |h|_{X-U} < eC,$$

where C is equal to $(K + |f|)^{-1}$. Putting $g = fh$ we have

$$|g - f|_F \leqq |f| \cdot |h| - 1|_F < e, \text{ and } |g| < 2bK.$$

Actually,

$$|g|_{X-U} \leqq |f| \cdot |h|_{X-U} < |f| \cdot eC \leqq e, \text{ and}$$

$$|g|_U \leqq |f|_U \cdot |h| < K \cdot 2b.$$

<u>Proof of Theorem</u>: Let U be an arbitrary open neighbourhood of F in X. We shall construct, by induction, functions f_n in A, $n = 1,2,\ldots$ satisfying the conditions

(3) $|f_n| < 8(1 - 2^{-n})b^2$,

(4) $|f_n - 1|_F < 2^{-n}$,

(5) $|f_n|_{X-U} < 2^{-1}(1 - 2^{-n})$,

(6) $|f_n - f_{n+1}| < 2^{2-n}b^2$.

By Lemma 1, we have an f_1 in A such that

$$|f_1| < 2b < 4b^2, \quad |f_1 - 1|_F < 2^{-2} < 2^{-1}, \quad |f_1|_{X-U} < 2^{-2}.$$

Having the functions $f_1, \dots, f_n$ constructed, take a g in A, by Lemma 1, such that

$$|g| < 2b, \quad |g - 1|_F < (8b)^{-1}, \quad |g|_{X-U} < (8b)^{-1},$$

and, by Lemma 2, an h in A satisfying

$$|h| < 2^{1-n}b, \quad |f_n - 1 - h| < 2^{-2-n}.$$

Put $f_{n+1} = f_n - gh$. Then f_{n+1} is in A and

$$|f_{n+1}| \leq |f_n| + |g| . |h| < 8(1 - 2^{-n})b^2 + 2^{2-n} b^2 =$$
$$= 8(1 - 2^{-1-n}) b^2,$$

$$|f_{n+1} - 1|_F = |f_n - gh - h + h = 1|_F \leq |f_n - 1 - h|_F +$$
$$+ |h| . |g - 1|_F < 2^{-2-n} + 2^{1-n}b(8b)^{-1} = 2^{1-n},$$

$$|f_{n+1}|_{X-U} \leq |f_n|_{X-U} + |g|_{X-U} |h| < 2^{-1}(1 - 2^{-n}) +$$
$$+ (8b)^{-1} . 2^{1-n}b = 2^{-1}(1 - 2^{-1-n}),$$

$$|f_{n+1} - f_n| \leq |g| \, |h| < 2^{2-n}b^2.$$

This shows that all conditions (3 - 6) are fulfilled.

By (6), the f_n have a limit in A, say f. By (3), $|f| \leq 8b^2$; by (4), $f/_F = 1$, and, finally, $|f|_{X-U} \leq 2^{-1}$ by (5).

The assertion of Theorem now follows from Proposition 2.

The author would like to express here his deep gratitude to RNDr. Jaroslav Fuka CSc for a whole range of fruitful conversations on the subject of this paper and many other questions of Banach function algebras theory.

R e f e r e n c e s

[1] E. BISHOP: A Minimal Boundary for Function Algebras, Pacific J. Math. 9(1959), 629-641.

[2] A.A. GONČAR: O minimalnoj granice algebry A(E), Izvěstija Akad. Nauk SSSR 27(1963), 949-955.

[3] P.C. CURTIS, Jr.: Peak Points for Algebras of Analytic Functions, J. Funct. Anal. 3(1969), 35-47.

[4] T.W. GAMELIN: Function Algebras, Prentice-Hall, 1969.

[5] I. GLICKSBERG: Measures orthogonal to algebras and sets of antisymmetry, Trans. Amer. Math. Soc. 105 (1962), 415-435.

[6] A. BERNARD: Caractérisations de certaines parties d'un espace compact muni d'un espace vectoriel ou d'une algèbre de fonctions continues, Ann. Inst. Fourier, Grenoble, 17(1967), 359-382.

[7] E. BRIEM: Restrictions of subspaces of C(X), Inventions Math. 10(1970), 288-297.

[8] - " - : Interpolation in subspaces of C(X), J. Funct. Anal. 12(1973), 1-12.

[9] E. BRIEM and M. RAO: Normpreserving extensions in subspaces of C(X), Pacific J. Math. 42(1972), 581-587.

Matematicko-fyzikální fakulta

Universita Karlova

Sokolovská 83, 18600 Praha 8

Československo

(Oblatum 31.5.1978)