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ISOMORPHISMS OF PRODUCTS OF INFINITE GRAPHS

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Abstract: We prove that every countable commutative semigroup can be represented by normal products or cartesian products or cartesian sums of countable simple graphs.

Key words: Products of graphs, representation, commutative semigroup.

AMS: 05C25, 06A50, 08A10, 20M30

By a graph $G = (V,E)$ we mean an undirected simple graph, i.e. V is a set, $E \subset \exp V$ and $e \in E$ implies card $e = 2$.

Given two graphs, say $G = (V,E)$, $G' = (V',E')$, the following products are investigated (in the terminology of [B]).

Cartesian product $G \overset{1}{\times} G' = (W,E_1)$

Cartesian sum $G \overset{2}{\times} G' = (W,E_2)$

Normal product $G \overset{3}{\times} G' = (E,E_3)$.

In all these cases, the set of vertices W is equal to $V \times V'$ and the sets of edges E_1, E_2, E_3 are defined as follows.

$(\langle v,v'\rangle, \langle u,u'\rangle) \in E_1$ iff $(v,u) \in E$ and $(v',u') \in E'$;

$(\langle v,v'\rangle, \langle u,u'\rangle) \in E_2$ iff either $v = u$ and $(v',u') \in E'$
or $v' = u'$ and $(v,u) \in E$;

and $E_3 = E_1 \cup E_2$.

(In [B], $\overset{1}{\times}$ is denoted by $\times$, $\overset{2}{\times}$ by $+$, $\overset{3}{\times}$ by $\bullet$.)

Clearly, each of the three types of products $\overset{i}{\times}$, $i = 1,2,3$, defines a commutative and associative operation on the class of all isomorphism types of graphs, i.e. the class $\mathfrak{G}$ of all isomorphism types of graphs endowed with $\overset{1}{\times}$ (or $\overset{2}{\times}$ or $\overset{3}{\times}$, respectively) forms a "large" commutative semigroup. Which commutative semigroups can be embedded in it? By [KNR] and [Tr_1], every commutative seimgroup can be embedded in $(\mathfrak{G}, \overset{1}{\times})$. In the present paper, we show that it can be embedded also in $(\mathfrak{G}, \overset{2}{\times})$ and $(\mathfrak{G}, \overset{3}{\times})$. Moreover, we show that the embedding can be in a sense uniform with respect to all these three types of products and that countable commutative semigroups can be embedded into isomorphism types of countable graphs (which is a new result also for $\overset{1}{\times}$). More precisely, the aim of the present paper is to prove the following

<u>Theorem</u>. For every commutative semigroup $(S,+)$ there exist collections $\{G_i(x) \mid x \in S,\ i = 1,2,3\}$ and $\{\varphi_{x,y} \mid x,y \in S\}$ such that

(a) $G_i(x) = (V(x), E_i(x))$, $i = 1,2,3$, are graphs with the same set of vertices $V(x)$; $E_1(x) \cap E_2(x) = \emptyset$ and $E_1(x) \cup E_2(x) \subset E_3(x)$;

(b) for every $x, y \in S$, $\varphi_{x,y}$ is a bijection of $V(x) \times V(y)$ onto $V(x + y)$ such that it is an isomorphism of $G_i(x) \overset{i}{\times} G_i(y)$ onto $G_i(x + y)$ for $i = 1,2,3$;

(c) if $\langle x,i \rangle, \langle x',i' \rangle \in S \times \{1,2,3\}$, $\langle x,i \rangle \neq \langle x',i' \rangle$, then $G_i(x)$ is not isomorphic to $G_{i'}(x')$;

(d) if S is countable, then all the sets $V(x)$, $x \in S$, are countable.

We give the detail proof of the theorem in the case that the given semigroup S is countable. If S is not countable (and the cardinality of the sets $V(x)$ is not restricted), the whole construction can be essentially simplified. This is sketched in 15. at the end of the proof.

Let us notice that the representation of commutative semigroups by products of graphs in the above sense generalizes the non-validity of cancellation, square root property, Cantor-Bernstein property and some similar properties. Products of graphs and relational structures with respect to these properties have been investigated in a number of papers, let us mention at least [L], [McK] and [Ch] for the older references.

1. First, let us recall how these three types of products are defined in the infinite case. Let $\{G(\alpha) \mid \alpha \in A\}$ be a collection of graphs, $G(\alpha) = (V(\alpha), E(\alpha))$. Then $\prod_{\alpha \in A}^{i} G(\alpha) = (V, E_i)$, $i = 1,2,3$ are graphs defined on the set $V = \prod_{\alpha \in A} V(\alpha)$ as follows ($\pi_\alpha : V \longrightarrow V(\alpha)$ denotes the α-th projection)

$(u,v) \in E_1$ iff $(\pi_\alpha(u), \pi_\alpha(v)) \in E(\alpha)$ for all $\alpha \in A$;

$(u,v) \in E_2$ iff there exists $\beta \in A$ such that $(\pi_\beta(u), \pi_\beta(v)) \in E(\beta)$ and $\pi_\alpha(u) = \pi_\alpha(v)$ for all $\alpha \in A \setminus \{\beta\}$;

$(u,v) \in E_3$ iff $u \neq v$ and for every $\alpha \in A$ either $\pi_\alpha(u) = \pi_\alpha(v)$ or $(\pi_\alpha(u), \pi_\alpha(v)) \in E(\alpha)$.

2. Denote by N the set of all non-negative integers. Let p_n be the $(n + 3)$-th prime (i.e. $p_0 = 5$), $q_n = p_n - 1$. For $n \in N$ define

$V(n) = \{0,1,a\} \cup (N \times q_n)$,

$E(n) = \{(0,1),(1,a)\} \cup \{(a, \langle 0,z \rangle) \mid z \in q_n\} \cup \{(\langle m,z \rangle, \langle m+1, z \rangle) \mid m \in N, z \in q_n\}$,

$G(n) = (V(n),E(n))$.

3. Let us consider N as an additive semigroup. Denote by N^{N} the set of all functions on N with the values at $\mathbb{N}$ (and with the addition defined by $(f+g)(m) = f(m) + g(n)$ for all $n \in N$). Denote by $\mathbb{O}$ the constant zero. For $f \in N^{\mathbb{N}} \setminus \{\mathbb{O}\}$ put

$$L(f) = \{\langle j,n\rangle \mid n \in N,\ 0 < j \leq f(n)\}$$

Since $f \neq \mathbb{O}$, $L(f)$ is non-empty. If $\ell = \langle j,n\rangle \in L(f)$, put $\bar{\ell} = n$. For every $f \in N^{\mathbb{N}} \setminus \{\mathbb{O}\}$ denote

$$V(f) = \prod_{\ell \in L(f)} V(\bar{\ell})$$

and denote by $\pi_{\ell} : V(f) \longrightarrow V(\bar{\ell})$ the ℓ-th projection. For every pair $f,\ g \in N^{N} \setminus \{\mathbb{O}\}$ define

$$\psi_{f,g} : L(f) \vee L(g) \longrightarrow L(f+g)$$

(where $\vee$ denotes the disjoint union) by

$$\psi_{f,g}\langle j,n\rangle = \langle j,n\rangle \text{ for } \langle j,n\rangle \in L(f),$$

$$\psi_{f,g}\langle j,n\rangle = \langle f(n)+j,n\rangle \text{ for } \langle j,n\rangle \in L(g).$$

Then $\psi_{f,g}$ is a bijection, defining a bijection

$$\wp_{f,g} : V(f) \times V(g) \longrightarrow V(f+g)$$

by the rule $\pi_{\ell} \circ \wp_{f,g} = \pi_{\psi_{f,g}^{-1}(\ell)}$ for all $\ell \in L(f+g)$.

Denote by

$$p_1 : V(f) \times V(g) \longrightarrow V(f)$$

$$p_2 : V(f) \times V(g) \longrightarrow V(g)$$

the first and the second projections.

4. Let a countable commutative semigroup $(S,+)$ be given. Denote by $\exp N^{\mathbb{N}}$ the commutative semigroup of all subsets of N^{N} (where the addition is given by the usual formula

$A + B = \{f + g \mid f \in A, g \in B\}$). By [$Tr_2$], there exists a homomorphism

$$g : (S,+) \longrightarrow \exp N^N$$

such that

(i) for every $x \in S$, $h(x)$ is infinite and countable and, for all $f \in h(x)$, $f(n) \neq 0$ for infinitely many $n \in N$;

(ii) for $x, x' \in S$, $x \neq x'$, the sets $h(x)$ and $h(x')$ are disjoint.

For every $x \in S$, $f \in h(x)$, we define by induction

$X_0(x,f) = \{v \in V(f) \mid \pi_\ell(v) = 0$ for all $\ell \in L(f)$ except a finite number$\} \cup \{v \in V(f) \mid \pi_\ell(v) = 1$ for all $\ell \in L(f)$ except a finite number$\}$,

$$X_{n+1}(x,f) = \bigcup_{\substack{x_1, x_2 \in S,\, x_1 + x_2 = x \\ f_1 \in h(x_1),\, f_2 \in h(x_2),\, f_1 + f_2 = f}} \varphi_{f_1,f_2}(X_n(x_1,f_1) \times$$

$$\times X_n(x_2,f_2)) \cup \bigcup_{\substack{y \in S \\ g \in h(y)}} (p_1 \varphi^{-1}_{f,g}(X_n(x + y, f + g)) \cup$$

$$\cup\, p_2 \varphi^{-1}_{g,f}(X_n(x + y, f + g)),$$

$$X(x,f) = \bigcup_{n=0}^{\infty} X_n(x,f).$$

(Let us notice that if $f \in \bigcup_{x \in S} h(x)$, then there is unique $x \in S$ such that $f \in h(x)$. Hence we could write only $X(f)$ instead of $X(x,f)$, but we prefer the more expressive notation $X(x,f)$.

5. <u>Lemma</u>. For every $x \in S$, $f \in h(x)$, $X(x,f)$ is a countable subset of $V(f)$. For every $x, x' \in S$, $f \in h(x)$, $f' \in h(x')$, $\varphi_{f,f'}$ maps $X(x,f) \times X(x',f')$ bijectively onto $X(x + x', f + f')$.

<u>Proof</u>. Since all the sets $V(n)$ are countable, the set

$X_o(x,f)$ is countable. Since S and all the sets $h(y)$, $y \in S$, are countable, any $X_n(x,f)$ is countable. Hence $X(x,f)$ is countable. Since $\varphi_{f_1,f_2}(X_o(x_1,f_1) \times X_o(x_2,f_2)) \supset X_o(f_1 + f_2)$, we obtain $X_o(x,f) \subset X_1(x,f)$; then $X_n(x,f) \subset X_{n+1}(x,f)$ for all x,f,n, hence, clearly, $\varphi_{f,f'}(X(x,f) \times X(x',f')) \supset X(x + x', f + f')$. Conversely, if $u \in X(x + x', f + f')$, put $v = p_1 \varphi^{-1}_{f,f'}(u)$, $v' = p_2 \varphi^{-1}_{f,f'}(u)$. Then $v \in X(x,f)$, $v' \in X(x',f')$ and $\varphi_{f,f'}(\langle v,v' \rangle) = u$.

6. We recall (see 3.) that for $f \in N^N \setminus \{0\}$, $\ell = \langle i,n \rangle \in L(f)$, $\bar{\ell}$ is defined as n and the graphs $G(n)$ are defined in 2. Now, put

$$G_i(f) = \prod^i_{\ell \in L(f)} G(\bar{\ell})$$

for $i = 1,2,3$. Then, clearly, $V(f)$ is the set of vertices of all $G_1(f)$, $G_2(f)$, $G_3(f)$. Moreover, for every f, $g \in N^N \setminus \{0\}$, $\varphi_{f,g}$ is an isomorphism of $G_i(f) \overset{i}{\times} G_i(g)$ onto $G_i(f + g)$ for all $i = 1,2,3$. For every $x \in S$, $f \in h(x)$ denote by $H_i(x,f)$ the full subgraph of $G_i(f)$ generated by the set $X(x,f)$. Then the domain-range restriction of $\varphi_{f,f'}$ is an isomorphism of $H_i(x,f) \overset{i}{\times} H_i(x',f')$ onto $H_i(x + x', f + f')$ for all $i = 1,2,3$. For $x \in S$ define

$$G_i(x) = \coprod_{\substack{f \in h(x) \\ n \in N}} (H_i(x,f))_n .$$

More in detail, consider the set $V(x) = \bigcup_{\substack{f \in h(x) \\ n \in N}} X(x,f) \times \{f\} \times \{n\}$ and define the graph $G_i(x) = (V(x), E_i(x))$ such that for every $f \in h(x)$, $n \in N$, the mapping $z \rightsquigarrow \langle z,f,n \rangle$ is an isomorphism of $H_i(x,f)$ onto a full subgraph of $G_i(x)$ and there are no other edges in $G_i(x)$ than edges obtained by this way. Then, clearly, $V(x)$ is the set of all vertices of $G_1(x)$, $G_2(x)$,

$G_3(x)$ and $E_1(x) \cap E_2(x) = \emptyset$, $E_1(x) \cup E_2(x) \subset E_3(x)$, i.e. the system $\{G_i(x) \mid x \in S,\ i = 1,2,3\}$ has all the properties, required in (a) of the Theorem.

7. For every $x, x' \in S$ choose a bijection

$$\psi_{x,x'} : (h(x) \times N) \times (h(x') \times N) \longrightarrow h(x + x') \times N$$

such that always $\psi_{x,x'}(\langle f,n\rangle, \langle f',n'\rangle) = \langle f + f', m\rangle$ (this is possible because $h(x) + h(x') = h(x + x')$). Now, define

$$\varphi_{x,x'} : V(x) \times V(x') \longrightarrow V(x + x')$$

by $\varphi_{x,x'}(\langle z, \langle f,n\rangle\rangle, \langle z', \langle f',n'\rangle\rangle) = \langle \wp_{f,f'}(z,z'),$ $\psi_{x,x'}(\langle f,n\rangle, \langle f',n'\rangle)\rangle$. Thus $\varphi_{x,x'}$ maps the product of the n-th copy of $X(x,f)$ and the n'-th copy of $X(x',f')$ onto the n-th copy of $X(x + x', f + f')$ as $\wp_{f,f'}$. Since $\psi_{x,x'}$ is a bijection, $\varphi_{x,x'}$ is an isomorphism of $G_i(x) \overset{i}{\times} G_i(x')$ onto $G_i(x + x')$. Hence the system $\{\varphi_{x,x'} \mid x, x' \in S\}$ has the properties required in (b) of the Theorem.

8. It remains to prove (c). First, let us notice that for every $x \in S$, $G_1(x)$ contains vertices of the degree 1 (namely the points of all copies of $H_1(x,f)$, having all coordinates equal to 0) but neither $G_2(x)$ nor $G_3(x)$ contain such vertices (by (i) in 4., $L(f)$ is infinite for every $f \in H(x)$, hence all vertices of $G_2(x)$ and $G_3(x)$ have infinite degrees). Hence $G_1(x)$ is never isomorphic to $G_2(x')$ or $G_3(x')$. For every $x \in S$, $G_3(x)$ contains triangles (the points of $X(x,f)$, having all coordinates in $\{0,1\}$, form a complete graph in $H_3(x,f)$), but no $G_2(x)$ contains a triangle (because no $G(n)$ contains a triangle). Hence $G_3(x)$ is never isomorphic to $G_2(x')$. Thus, it suffices to prove that if $x \neq x'$, then $G_i(x)$

is not isomorphic to $G_i(x')$ for $i = 1,2,3$. First, let us prove a lemma, suitable for all the three cases.

9. Lemma. (α) If $u \in X(x,f)$, then there exists $F \subset L(f)$ finite such that $\pi_\ell(u) \in \{0,1\}$ for all $\ell \in L(f) \setminus F$;

(β) Let $u \in X(x,f)$, $v \in V(f)$; let there exist $F \subset L(f)$ finite such that for all $\ell \in L(f) \setminus F$ both $\pi_\ell(u)$, $\pi_\ell(v)$ are in $\{0,1\}$ and $\pi_\ell(u) \neq \pi_\ell(v)$; then $v \in X(x,f)$.

Proof. $X(x,f)$ is defined as $\bigcup_{n=0}^{\infty} X_n(x,f)$. $X_0(x,f)$ fulfils (α) and (β). Then proceed by induction (simultaneously for all $x \in S$, $f \in h(x)$) by n.

10. We recall (see 2.) that p_n is the $(n + 3)$-th prime.

Lemma. Let u be a vertex of $H_1(x,f)$. Then its degree is equal to the prime p_n iff there exists $t = \langle j,n \rangle \in L(f)$ such that $\pi_\ell(u) = 0$ for all $\ell \in L(f) \setminus \{t\}$ and $\pi_t(u) = a$.

Proof. Let a vertex u of $H_1(x,f)$ be given. Denote by L the set of all $\ell \in L(f)$ such that $\pi_\ell(u) = 1$. If L is infinite, then the degree of u is infinite. For, we can find a vertex u_ℓ joined by an edge with u for every $\ell \in L$ such that $\pi_\ell(u_\ell) = a$, $\pi_k(u_\ell) = 0$ for all $k \in L \setminus \{\ell\}$, $\pi_k(u_\ell) = 1$ whenever $\pi_k(u) = 0$. Let us suppose that L is finite. Denote by F the set of all $\ell \in L(f)$ such that $\pi_\ell(u) \neq 0$, so F is finite. For every $\ell = \langle j,m \rangle \in F$ denote by d_ℓ the degree of $\pi_\ell(u)$ in $G(n)$. Then the degree of u in $H_1(x,f)$ is equal to $\prod_{\ell \in F} d_\ell$. Since $\pi_\ell(u) \neq 0$ whenever $\ell \in F$, $\prod_{\ell \in F} d_\ell$ is a prime iff F has precisely one element, say $F = \{t\}$. Moreover, $d_t = p_n$ iff $t = \langle j,n \rangle$ and $\pi_t(u) = a$.

11. Proposition. If $x \neq x'$, then $G_1(x)$ is not isomorphic to $G_1(x')$.

Proof. By the previous lemma, f can be recognized from the graph $H_1(x,f)$. For, $f(n)$ is the number of vertices with the degree p_n. Let us mention that all these vertices are contained in the same component of $H_1(x,f)$, namely the component containing the unique vertex u of $H_1(x,f)$ with the degree 1 (i.e. the vertex with $\pi_\ell(u) = 0$ for all $\ell \in L(f)$. If $x \neq x'$, choose $f \in h(x) \setminus h(x')$. Then $G_1(x)$ contains a component with a vertex with the degree 1 and exactly f(n) verzices with the degree p_n for every $n \in N$, but $G_1(x')$ contains no such component.

12. Given a graph $G = (V,E)$, and $u \in V$, denote $b(u) = \{v \in V \mid (u,v) \in E\}$. Denote by c(u) the supremum of cardinalities of all sets $C \subset b(u)$ such that any pair of elements of C is not joined by an edge.

Lemma. Let u be a vertex of $H_3(x,f)$. Then c(u) is equal to the prime p_n iff there exists $t = \langle j,n\rangle \in L(f)$ such that $\pi_\ell(u) = 0$ for all $\ell \in L(f) \setminus \{t\}$ and $\pi_t(u) = a$.

Proof. Let a vertex $u \in H_3(x,f)$ be given. Denote by L the set of all $\ell \in L(f)$ such that $\pi_\ell(u) = 1$. If L is infinite, then $c(u) = \aleph_0$. For, we can find a vertex u_ℓ joined by an edge with u, for every $\ell \in L$, such that u_ℓ and $u_{\ell'}$ are not joined by an edge whenever $\ell, \ell' \in L, \ell \neq \ell'$ (it is sufficient to put $\pi_\ell(u_\ell) = a$, $\pi_k(u_\ell) = 0$ for all $k \in L \setminus \{\ell\}$, $\pi_k(u_\ell) = \pi_k(u)$ otherwise). Let us suppose that L is finite. Denote by F the set of all $\ell \in L(f)$ such that $\pi_\ell(u) \neq 0$, so F is finite. If v is a vertex of $H_3(x,f)$, joined by an edge with u, then, by the definition of Π_3, the vertex $\bar{v}$ such that $\pi_\ell(\bar{v}) = \pi_\ell(v)$ for all $\ell \in F$, $\pi_\ell(\bar{v}) = 0$ otherwise, is joined with u as well as with v. Hence, the number c(u) is determined by the subgraph generated by all vertices $\bar{v}$ with $\pi_\ell(\bar{v}) = 0$ for all $\ell \in L(f) \setminus F$. For every $\ell = \langle j,n \rangle \in F$ denote by c_ℓ the number $c(\pi_\ell(u))$ in $G(n)$. Then $c(u) = \prod_{\ell \in F} c_\ell$. This is a prime iff F has precisely one element. Clearly $c(u) = p_n$ iff $F = \{t\}$, $t = \langle j,n \rangle$ and $\pi_t(u) = a$.

13. Proposition. If $x \neq x'$, then $G_3(x)$ is not isomorphic to $G_3(x')$.

Proof. By the previous lemma, f can be recognized from the graph $H_3(x,f)$. The rest of the proof is quite analogous as in 11.

14. Given a graph $G = (V,E)$ and $u \in V$, consider the sets $A \subset b(u)$ (where $b(u)$ is as in 12.) such that

if $v, v' \in A$, $v \neq v'$, then $(v,v') \notin E$ and for no $w \in V$, distinct from u, (v,w) and (v',w) are in E.

Denote by $a(u)$ the supremum of cardinalities of all such sets A.

Lemma. Let u_0 be a vertex of $H_2(x,f)$. Then $a(u_0) = 1$ iff $\pi_\ell(u_0) = 0$ for all $\ell \in L(f)$.

Proof. If $\pi_\ell(u_0) \neq 0$ for some $\ell \in L(f)$, then $a(u_0) \geq 2$ because $\pi_\ell(u_0)$ has this property in $G(\bar{\ell})$. If $\pi_\ell(u_0) = 0$ for all $\ell \in L(f)$ and $v, v' \in b(u_0)$, $v \neq v'$, then there exist k, $k' \in L(f)$, $k \neq k'$, such that $\pi_k(v) = 1$, $\pi_{k'}(v') = 1$, $\pi_k(v') = 0$, $\pi_{k'}(v) = 0$ and $\pi_\ell(v) = \pi_\ell(v') = 0$ for $\ell \in L(f) \setminus \{k,k'\}$. Then w, defined by $\pi_k(w) = \pi_{k'}(w) = 1$, $\pi_\ell(w) = 0$ for $\ell \in L(f) \setminus \{k,k'\}$, is distinct from u_0 and is joined with both v and v'.

Lemma. Let u_0 be the unique vertex of $H_2(x,f)$ such that $a(u_0) = 1$. Let u be a vertex of $H_2(x,f)$. Then there exists

$t = \langle j,m \rangle \in L(f)$ such that $\pi_t(u) = a$, $\pi_\ell(u) = 0$ for all $\ell \in L(f) \setminus \{t\}$ if $a(u) = p_n$ and there exists $v \in b(u_0)$ such that $u \in b(v)$.

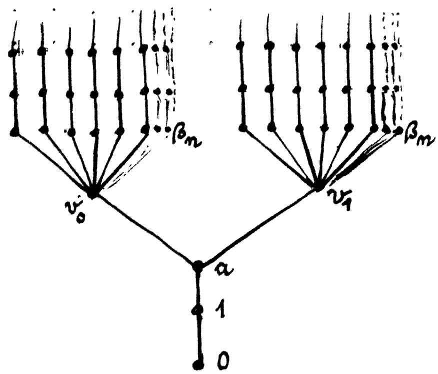
Proof. Clearly, if $\pi_t(u) = a$ for $t = \langle j,n \rangle \in L(f)$ and $\pi_\ell(u) = 0$ for all $\ell \in L(f) \setminus \{t\}$, then u has the property. Conversely, let $a(u) = p_n$ and there exists v such that $v \in b(u_0)$, $u \in b(v)$. Then necessarily there exists $t \in L(f)$ such that $\pi_t(v) = 1$, $\pi_\ell(v) = 0$ for all $\ell \in L(f) \setminus \{t\}$. Since $u \in b(v)$, either $\pi_t(u) \in \{0,a\}$ and $\pi_\ell(u) = 0$ for all $\ell \in L(f) \setminus \{t\}$ or $\pi_t(u) = 1$ and there exists $t' \in L(f)$, $t' \neq t$, such that $\pi_{t'}(u) = 1$ and $\pi_\ell(u) = 0$ for $\ell \in L(f) \setminus \{t,t'\}$. But the second case is impossible because this implies $a(u) = 2$. In the first case, $a(u) = p_n$ implies $\pi_t(u) = a$ and $t = \langle j,m \rangle$ for some $j \leq f(n)$.

Proposition. If $x \neq x'$, then $G_2(x)$ is not isomorphic to $G_2(x')$.

Proof. By the previous two lemmas, f can be recognized from the graph $H_2(x,f)$. Then proceed as in 11.

15. If the cardinality of the constructed graphs is not limited, the construction can be simplified and generalized as follows. Given an arbitrary semigroup $(S,+)$, there exists a homomorphism $h: S \longrightarrow \exp N^M$, where M is a set with card $M = \aleph_0 \cdot$ card S, such that card $h(x) = \aleph_0 \cdot$ card S, every $f \in h(z)$ is non-zero for infinitely many $m \in M$ and $h(x) \cap h(x') = \emptyset$ for $x \neq x'$, by [Tr_2]. Choose a collection $\{\beta_n \mid n \in M\}$ of distinct cardinals with $\beta_n > 2^{\text{card } M}$ for all $m \in M$ and define

$V(n) = \{0,1,a,v_0,v_1\} \cup$
$\cup (\{v_0,v_1\} \times \beta_n \times N)$
$E(n) = \{(0,1),(1,a),(a,v_0),$
$(a,v_1)\} \cup \{(v_j,\langle 0,z\rangle) \mid$
$\mid j = 0,1,\ z \in \beta_n\} \cup$
$\cup \{(\langle m,z\rangle,\langle m+1,z\rangle) \mid$
$\mid m \in N, z \in \beta_m\}$
$G(n) = (V(n),E(n))$



and put $G_i(f) = \prod_{\ell \in L(F)}^{i} G(\bar{\ell})$, $G_i(x) = \coprod_{\substack{f \in h(x) \\ m \in M}} (G_i(f))_m$.

Then for the collection $\{G_i(x) \mid i = 1,2,3,\ x \in S\}$ a system $\{\varphi_{x,y} \mid x,y \in S\}$ can be found such that (a),(b),(c) of the theorem are fulfilled, the proofs are analogous as the previous ones.

16. Remark. By the method of [AK], the presented theorem can be generalized (with straightforward modifications of the presented proof) to the sum-productive representation of ordered commutative semigroups "uniformly" by $\overset{1}{\times}$, $\overset{2}{\times}$, $\overset{3}{\times}$, i.e. given an ordered commutative semigroup $(S,+,\leq)$, there exist collections

$\mathfrak{G} = \{G_i(x) \mid x \in S,\ i = 1,2,3\}$, $\Phi = \{\varphi_{x,y} \mid x,y \in S\}$,
$\Psi = \{\psi_{x,x'} \mid x,x' \in S,\ x \leq x'\}$

such that $\mathfrak{G}$ and Φ fulfil (a),(b),(c),(d) of the theorem and, moreover,

(e) for every $x,x' \in S$, $x \leq x'$, $\psi_{x,x'}$ is a one-to-one mapping of $V(x)$ into $V(x')$ such that it is an isomorphism of $G_i(x)$ onto a summand of $G_i(x')$ for $i = 1,2,3$;

(f) if $x, x' \in S$ and $x \leq x'$ is not fulfilled, then $G_i(x)$ is not isomorphic to any summand of $G_j(x')$, $i, j \in \{1,2,3\}$. (We recall that a subgraph $G = (V,E)$ of $G' = (V', E')$ is said to be its summand if there is no edge in G' joining a vertex of V with a vertex in $V' \setminus V$.)

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