Commentationes Mathematicae Universitatis Carolinae

Bogdan Rzepecki

Note on the differential equation $F(t, y(t), y(h(t)), y'(t)) = 0$

Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 4, 627--637

Persistent URL: http://dml.cz/dmlcz/105880

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

19,4 (1978)

NOTE ON THE DIFFERENTIAL EQUATION

$F(t, y(t)), y(h(t)), y'(t)) = 0$

Bogdan RZEPECKI, Poznań

Abstract: We present a result on the existence, uniqueness and continuous dependence on given functions and initial conditions of a solution for the differential equation with deviated argument $F(t, y(t), y(h(t)), y'(t)) = 0$. These facts are a consequence of an application of some fixed-point theorem. This theorem generalizes the well-known Banach principle and is connected with Bielecki's method of changing the norm.

AMS: 34A10

Key words: Differential equation with deviated argument.

---

1. Let $I = [0,a]$, let $(\mathbb{R}^k, \|\cdot\|)$ be a k-dimensional Euclidean space and let $C(I, \mathbb{R}^k)$ denote the space of all continuous functions from an interval I to $\mathbb{R}^k$, with the usual supremum metric.

By (PC) we shall denote the problem of finding the solution of the differential equation with deviated argument

$$F(t,y(t), y(h(t)), y'(t)) = 0$$

(cf. [1],[4],[9]) satisfying the initial condition

$$y(0) = X,$$

where $h: I \longrightarrow I$, $F: I \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ are continuous functions, $X \in \mathbb{R}^k$ and $y(\cdot)$ denotes an unknown function such that $y' \in C(I, \mathbb{R}^k)$.

In this note we present a result on the existence, uniqueness and continuous dependence on given functions and initial conditions of a solution for the (PC) problem. These facts are a consequence of an application of some theorem (given in Sec. 2) of the type of Banach fixed-point principle.

2. Let $(E, |||\cdot|||)$ be a Banach space, let S be a normal cone in E (see e.g. [7]) and let $\preccurlyeq$ denote the partial order generated by the S. Suppose that X is a non-empty set and $d_E: X \times X \longrightarrow S$ is some function. Moreover, let us put $d_E^+(x,y) = |||d_E(x,y)|||$ for x, y in X.

The pair (X, d_E) is called a <u>generalized metric space</u> [7] (cf. also [3],[11]) if for all x, y and z in X the following conditions are satisfied:

1° $d_E(x,y) = \theta$ if and only if $x = y$ (θ denotes the zero of a space E);

2° $d_E(x,y) = d_E(y,x)$;

3° $d_E(x,y) \preccurlyeq d_E(x,z) + d_E(z,y)$.

If, further: every d_E^+-Cauchy sequence in X is d_E^+-convergent in X (i.e., $\lim_{p,q\to\infty} d_E^+(x_p,x_q) = 0$ for a sequence (x_n) in X, implies the existence of an element $x_0 \in X$ such that $\lim_{n\to\infty} d_E^+(x_n,x_0) = 0$), then (X, d_E) is called a <u>complete generalized metric space</u>.

In this section suppose we are given: A - an arbitrary set, (X, d_E) - a generalized metric space, L - a bounded positive linear operator of E into itself with the spectral radius r(L) less than one.

We shall use the following

<u>Lemma</u> (cf. [6]). Let P, R be two transformations defined on the set A with the values in X and such that $P[A] \subset R[A]$. Suppose that $R[A]$ is a complete generalized metric subspace of X, and $d_E(Px,Py) \precsim L(d_E(Rx,Ry))$ for all x, y in A. Then:

(i) for every $u \in R[A]$ the set $P[R_{-1}u]$ contains only one element ($R_{-1}u$ denotes the inverse image of u under R);

(ii) there exists a unique element ξ in $R[A]$ such that $P[R_{-1}\xi] = \xi$, and every sequence of successive approximations $u_{n+1} = P[R_{-1}u_n]$ (n = 1,2,...) is d_E^+-convergent to ξ;

(iii) $Px = Rx$ for all $x \in R_{-1}\xi$;

(iv) if $Px_i = Rx_i$ for i = 1,2, then $Rx_1 = Rx_2$.

<u>Proof</u>. Fix u in $R[A]$. Suppose that $v_i = P[R_{-1}u]$ for i = 1,2. Then $v_i = Px_i$, where $Rx_i = u$. Hence

$$d_E(v_1,v_2) = d_E(Px_1,Px_2) \precsim L(d_E(Rx_1,Rx_2)) = \Theta$$

and therefore $v_1 = v_2$.

Let us put $Fu = P[R_{-1}u]$ for $u \in R[A]$. For $u_i \in R[A]$ (i = 1,2) with $x_i \in R_{-1}u_i$, we have

$$d_E(Fu_1,Fu_2) = d_E(Px_1,Px_2) \precsim L(d_E(Rx_1,Rx_2)) = \\ = L(d_E(u_1,u_2)).$$

Therefore, applying Theorem II.6.2 from [7, p. 94], we can conclude the proof of (ii). Further, if $\xi \in R[A]$ satisfies (ii) and $x \in R_{-1}\xi$, then $Rx = \xi = F\xi = P[R_{-1}\xi] = Px$.

Now, we prove (iv). Let $Px_i = Rx_i$ (i = 1,2) and $Rx_1 \neq Rx_2$. Obviously, $d_E(Rx_1,Rx_2) \precsim L(d_E(Rx_1,Rx_2))$ and $-d_E(Rx_1,Rx_2) \notin S$. Consequently, by Stečenko theorem [7, th. II.5.4, p.81] we obtain $r(L) \geq 1$. This contradiction completes the proof.

We shall be using the notations of $\mathcal{L}^*$-space, the $\mathcal{L}^*$-product of $\mathcal{L}^*$-spaces and a continuous mapping of $\mathcal{L}^*$-space into $\mathcal{L}^*$-space (see e.g. [8]).

Proposition (of Banach type). Let B be an $\mathcal{L}^*$-space and let L, A, (X, d_E) be as above. Suppose that Q, T are two transformations defined on the set $A \times B$ with the values in X such that for all y in B:

(j) $\{Q(x,y): x \in A\} \subset \{T(x,y): x \in A\}$ and $\{T(x,y): x \in A\}$ is a complete generalized metric subspace of X;

(jj) $d_E(Q(x_1,y), Q(x_2,y)) \lessapprox L(d_E(T(x_1,y), T(x_2,y)))$ for every x_1, x_2 in A;

(jjj) the mapping $T(\cdot, y)$ is one-to-one on A.

Then there exists a unique function $\varphi : B \longrightarrow A$ such that $Q(\varphi(y),y) = T(\varphi(y),y)$ for all y in B. Moreover, if for every fixed x in A the functions $Q(x,\cdot)$, $T(x,\cdot)$ maps continuously $\mathcal{L}^*$-space B into a metric space (X, d_E^+), then the functions $T(\varphi(\cdot),\cdot)$, $Q(\varphi(\cdot),\cdot)$ are continuous from B into (X, d_E^+).

Proof. Let us fix y in B and put $Px = Q(x,y)$, $Rx = T(x,y)$ for x in A. For P and R all the conditions of our Lemma are satisfied. Therefore, by conditions (jjj), there exists exactly one element $\varphi(y)$ in A such that $Q(\varphi(y),y) = T(\varphi(y),y)$.

Now, we consider the mapping $y \longmapsto \varphi(y)$. Suppose that (y_n) is a sequence in B converging to y_0, and $Q(x,\cdot)$, $T(x,\cdot)$ (x is fixed in A) are continuous on B. Let $\varepsilon > 0$ be such that $r(L) + \varepsilon < 1$. Further, let us denote by $\|\cdot\|_\varepsilon$

the norm equivalent to $|||\cdot|||$ such that $\| L \|_{\varepsilon} \leq r(L) + \varepsilon$ (see [7, p. 15]) ($\| L \|_{\varepsilon}$ is the norm of operator L generated by $\|\cdot\|_{\varepsilon}$).

We have

$$d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),y_o)) = d_E(Q(\varphi(y_n),y_n),$$
$$Q(\varphi(y_o),y_o)) \preccurlyeq L(d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),y_n))) +$$
$$+ d_E(Q(\varphi(y_o),y_n), Q(\varphi(y_o),y_o)) \preccurlyeq$$
$$\preccurlyeq L(d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),y_o))) + L(d_E(T(\varphi(y_o),y_o),$$
$$T(\varphi(y_o),y_n))) + d_E(Q(\varphi(y_o),y_n), Q(\varphi(y_o),y_o)),$$

hence

$$\| d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),y_o)) - L(d_E(T(\varphi(y_n),y_n),$$
$$T(\varphi(y_o),y_o)))\|_{\varepsilon} \leq M \| L \|_{\varepsilon} \cdot \| d_E(T(\varphi(y_o),y_n), T(\varphi(y_o),$$
$$y_o))\|_{\varepsilon} + M \| d_E(Q(\varphi(y_o),y_n), Q(\varphi(y_o),y_o))\|_{\varepsilon},$$

where M is some constant. Therefore

$$\| d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),y_o))\|_{\varepsilon} \leq \| d_E(T(\varphi(y_n),y_n),$$
$$T(\varphi(y_o),y_o)) - L(d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),y_o)))\|_{\varepsilon} +$$
$$+ \| L \|_{\varepsilon} \cdot \| d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),y_o))\|_{\varepsilon} \leq M \| L \|_{\varepsilon} \cdot$$
$$\cdot \| d_E(T(\varphi(y_o),y_n), T(\varphi(y_o),y_o))\|_{\varepsilon} + M \| d_E(Q(\varphi(y_o),y_n),$$
$$Q(\varphi(y_o),y_o))\|_{\varepsilon} + (r(L) + \varepsilon) \cdot \| d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),$$
$$y_o))\|_{\varepsilon}$$ and consequently

$$\lim_{n\to\infty} \| d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),y_o))\|_{\varepsilon} \leq (r(L) + \varepsilon) \cdot$$
$$\cdot \lim_{n\to\infty} \| d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),y_o))\|_{\varepsilon} .$$

Since $r(L) + \varepsilon < 1$, so

$$\lim_{n\to\infty} \| d_E(T(\varphi(y_n),y_n), T(\varphi(y_o),y_o))\|_{\varepsilon} = 0,$$

which completes the proof.

This Proposition generalizes the well-known Banach fixed-point principle and is connected with the Bielecki's method [2] of changing the norm in the theory of differential equations. If we put above: $E = \mathbb{R}^1$, $S = [0,\infty)$, $0 \le k < 1$ and $Lx = k \cdot x$ for $x \in \mathbb{R}^1$, we get the result from [10].

Note, finally, that [5] a non-negative and non-zero matrix $M = [a_{ij}]$ $(1 \le i,\ j \le k)$ has the spectral radius $r(M)$ less than one if and only if

$$\begin{vmatrix} 1 - a_{11} & - a_{12} & \cdots & - a_{1i} \\ - a_{21} & 1 - a_{22} & \cdots & - a_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ - a_{i1} & - a_{i2} & \cdots & 1 - a_{ii} \end{vmatrix} > 0$$

for all $i = 1,2,\dots,k$. Let us remark that there exists a positive constant p_0 such that $r(p \cdot M) < 1$ for every $0 < p \le p_0$.

3. Let us denote:

by Φ_0 - the space of all continuous functions from I into I, with the usual supremum metric ϱ;

by Φ - the some non-empty subspace of Φ_0

by $\mathcal{F}$ - the set of all continuous functions $F = (f_1,\dots,f_k)$ from $I \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k$ into $\mathbb{R}^k$ satisfying the following conditions:

$$|w_i - \overline{w}_i + \mu_i(f_i(t,u,v,w) - f_i(t,u,v,\overline{w}))| \le \sum_{j=1}^{k} N_{ij} |w_j - \overline{w}_j|,$$

$$|f_i(t,u,v,w) - f_i(t,\overline{u},\overline{v},w)| \le \sum_{j=1}^{k} M_{ij}(|u_j - \overline{u}_j| + |v_j - \overline{v}_j|)$$

$(i = 1,2,\dots,k)$

for all $t \in I$ and $u = (u_1,\dots,u_k)$, $\overline{u} = (\overline{u}_1,\dots,\overline{u}_k)$, $v = (v_1,\dots,v_k)$, $\overline{v} = (\overline{v}_1,\dots,\overline{v}_k)$, $w = (w_1,\dots,w_k)$, $\overline{w} = (\overline{w}_1,\dots,\overline{w}_k)$ in

$\mathbb{R}^k$, where $\mu_i \neq 0$, $M_{ij} \geq 0$, $N_{ij} \geq 0$ $(1 \leq i, j \leq k)$ are constants.

In the sequel we shall deal with the set $\mathcal{F}$ as an $\mathcal{L}^*$-space endowed with the almost uniform convergence. Moreover, $\mathcal{F} \times \Phi \times \mathbb{R}^k$ considered as an $\mathcal{L}^*$-product of the spaces $\mathcal{F}$, Φ and $\mathbb{R}^k$.

It is easy to verity that (PC) problem is equivalent to the equation

$$(+) \qquad F(t, X + \int_0^t z(s)ds,\ X + \int_0^{h(t)} z(s)ds,\ z(t)) = 0.$$

In particular, if $z \in C(I, \mathbb{R}^k)$ is a solution of (+) then the function $t \longmapsto X + \int_0^t z(s)ds$ is a solution of (PC). We shall prove the following

<u>Theorem</u>. Let $\sup_{h \in \Phi} \sup_{t \in I} (h(t) - t) < \infty$. Suppose that there exists a constant $p > 0$ such that the matrix

$$(*) \quad [N_{ij}] + p^{-1}(1 + \exp(p \cdot \sup_{h \in \Phi} \sup_{t \in I} (j(t) - t))) \cdot$$
$$\cdot [|\mu_i| \cdot M_{ij}] \qquad (1 \leq i, j \leq k)$$

has spectral radius less than 1. Then, for an arbitrary $F \in \mathcal{F}$, $h \in \Phi$ and $X \in \mathbb{R}^k$ there exists a unique function $y_{(F,h,X)}(\cdot)$ satisfying the (PC) problem on I. Moreover, the function

$$(F,h,X) \longmapsto y_{(F,h,X)}$$

maps continuously $\mathcal{L}^*$-space $\mathcal{F} \times \Phi \times \mathbb{R}^k$ into $C(I, \mathbb{R}^k)$.

<u>Proof</u>. Let $\mathfrak{X}$ denote the set of all continuous functions from I to $\mathbb{R}^k$. Let us put: $E = \mathbb{R}^k$, $S = \{(q_1, \dots, q_k) \in \mathbb{R}^k : q_i \geq 0 \text{ for } 1 \leq i \leq k\}$. Obviously, $X \preceq Y$ for $X = (x_1, \dots, x_k)$, $Y = (y_1, \dots, y_k)$ in $\mathbb{R}^k$ means $x_i \leq y_i$ for every $i = 1, 2, \dots, k$. In $\mathfrak{X}$ we define the distance functions d_E,

d_E^+: for $z = (z_1,\dots,z_k)$, $w = (w_1,\dots,w_k)$ in $\mathcal{X}$ we put

$$d_E(z,w) = (\varrho(z_1,w_1), \varrho(z_2,w_2),\dots, \varrho(z_k,w_k))$$

$$\text{and } d_E^+(z,w) = \| d_E(z,w) \| .$$

Then $(\mathcal{X}, d_E)$ is a complete generalized metric space.

Let us put $B = \mathcal{F} \times \Phi \times \mathbb{R}^k$. For $y = (y_1,\dots,y_k) \in \mathcal{X}$, $F = (f_1,\dots,f_k) \in \mathcal{F}$, $h \in \Phi$ and $X \in \mathbb{R}^k$ we define on I:

$$T_i(y,(F,h,X))(t) = y_i(t) \cdot \exp(-pt),$$

$$Q_i(y,(F,h,X))(t) = (y_i(t) + \mu_i \cdot f_i(t, X + \int_0^t y(s)ds, X + \int_0^{h(t)} y(s)ds, y(t))) \cdot \exp(-pt)$$

a

$$(y,(F,h,X))(t) = (T_1(y,(F,h,X))(t),\dots,T_k(y,(F,h,X))(t)),$$

$$Q(y,(F,h,X))(t) = (Q_1(y,(F,h,X))(t),\dots,Q_k(y,(F,h,X))(t)).$$

Obviously, T and Q map the set $\mathcal{X} \times B$ into $\mathcal{X}$ and

$$\{Q(y,\eta) : y \in \mathcal{X}\} \subset \{T(y,\eta) : y \in \mathcal{X}\}, \{T(y,\eta) : y \in \mathcal{X}\} = \mathcal{X}$$

for each $\eta \in B$.

Denote by L a linear operator generated by the matrix (*). Let us fix $\eta = (F,h,X) \in B$, where $F = (f_1,\dots,f_k)$. First, observe that the mapping $T(\cdot,\eta)$ is one-to-one on $\mathcal{X}$. Further, for $1 \le i \le k$, $t \in I$ and $z = (z_1,\dots,z_k)$, $w = (w_1,\dots,w_k)$ in $\mathcal{X}$

$$|Q_i(z,\eta)(t) - Q_i(w,\eta)(t)| \le (\sum_{j=1}^{k} N_{ij} | z_j(t) - w_j(t)| + |\mu_i| \cdot \sum_{j=1}^{k} M_{ij} \cdot \int_0^t |z_j(s) - w_j(s)| ds + |\mu_i| \cdot \sum_{j=1}^{k} M_{ij} \cdot \int_0^{h(t)} |z_j(s) - w_j(s)| ds) \cdot$$

$$\cdot \exp(-pt) \le \sum_{j=1}^{k} N_{ij} \cdot \varrho(T_j(z,\eta), T_j(w,\eta)) +$$

$$+ |\mu_i| \cdot \exp(-pt) \cdot \left(\int_0^t e^{ps} ds + \int_0^{h(t)} e^{ps} ds \right) \cdot$$

$$\cdot \sum_{j=1}^{k} M_{ij} \cdot \rho(T_j(z,\eta), T_j(w,\eta)) \leq$$

$$\leq \sum_{j=1}^{k} (N_{ij} + p^{-1}(1 + \exp(pC)) \cdot |\mu_i| \cdot M_{ij}) \cdot \rho(T_j(z,\eta),$$

$$T_j(w,\eta)),$$

where $C = \sup_{h \in \Phi} \sup_{t \in I} (h(t) - t)$. Hence $d_E(Q(z,\eta), Q(w,\eta)) \preceq$ $\preceq L(T(z,\eta), T(w,\eta))$ for $\eta \in B$ and $z, w \in \mathfrak{X}$.

Fix y in $\mathfrak{X}$. Let $\eta_m = (F_m, h_m, X_m) \in B$ $(m = 0,1,\ldots)$, where $F_m = (f_1^{(m)}, \ldots, f_k^{(m)})$ and $X_m = (x_1^{(m)}, \ldots, x_k^{(m)})$. For $1 \leq i \leq K$, $n \geq 1$ and $t \in I$, we obtain

$$|Q_i(y,\eta_n)(t) - Q_i(y,\eta_o)(t)| \leq |\mu_i| \cdot \sum_{j=1}^{k} M_{ij}(2|x_j^{(n)} -$$

$$- x_j^{(o)}| + \sup_{t \in I} \left| \int_0^{h_n(t)} y_j(s)ds - \int_0^{h_o(t)} y_j(s)ds \right|) +$$

$$+ |\mu_i| \cdot \sup_{t \in I} |f_i^{(n)}(t, X_o + \int_0^t y(s)ds, X_o + \int_0^{h_o(t)} y(s)ds, y(t)) -$$

$$- f_i^{(o)}(t, X_o + \int_0^t y(s)ds, X_o + \int_0^{h_o(t)} y(s)ds, y(t))|$$

hence

$$\| d_E(Q(y,\eta_n), Q(y,\eta_o)) \| \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

if $\lim_{n\to\infty} \| X_n - X_o \| = 0$, $\lim_{n\to\infty} \rho(h_n, h_o) = 0$ and $\lim_{n\to\infty} F_n = F_o$ almost uniformly. Finally, $Q(y, \cdot)$ is continuous from B into $(\mathfrak{X}, d_E^+)$.

Consequently, the Proposition given in Sec. 2 is applicable to the mapping T and Q. Hence there exists a unique continuous function $\varphi : B \longrightarrow C(I, \mathbb{R}^k)$ such that $\varphi(\eta)$ $(\eta \in B)$ satisfies the equation (+) on I. This completes the proof of our theorem.

Remark. Suppose that for each $h \in \Phi$ we have: $h(t) \leq t$

on I. Then $C = \sup_{\lambda \in \Phi} \sup_{t \in I} (h(t) - t) \leq 0$, and therefore $\exp(pC) \leq 1$ for every $p > 0$. Consequently, there exists $p > 0$ such that the matrix (*) has spectral radius less than 1 if $r([N_{ij}]) < 1$.

References

[1] S. ABIAN - A.B. BROWN: On the solution of the differential equation f(x,y,y′) = 0, Amer. Math. Monthly 66(1959), 192-199.

[2] A. BIELECKI: Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinaires, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 4(1956), 261-264.

[3] L. COLLATZ: Funktionalanalysis und Numerische Mathematik, Berlin-Göttingen-Heidelberg 1964.

[4] R. CONTI: Sulla resoluzione dell′ equazione F(t,x,x′) = = 0, Ann. Mat. Pura Appl. 48(1959), 97-107.

[5] F.R. GANTMACHER: The theory of matrices [in Russian], Moscow 1966.

[6] K. GOEBEL: A coincidence theorem, Bull. Acad. Polon. Sci,, Sér. Sci. Math. Astronom. Phys. 16(1968), 733-735.

[7] M.A. KRASNOSELSKIĬ - G.M. VAĬNIKKO - P.P. ZABREĬKO - J.A.B. RUTICKIĬ - V.JA. STECENKO: Approximate solution of operator equations [in Russian], Moscow 1969.

[8] C. KURATOWSKI: Topologie. V I. Warszawa 1958.

[9] G. PULVIRENTI: Equazioni differenziali in forma implicita in uno spazi di Banach, Ann. Mat. Pura Appl. 56(1961), 177-191.

[10] B. RZEPECKI: On the Banach principle and its application to the theory of differential equation, Comment. Math. 19(1977), 355-363.

[11] B. RZEPECKI: A generalization of Banach's contraction theorem, to appear in Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 26.

Institute of Mathematics
A. Mickiewicz University
Matejki 48/49, 60-769 Poznań
POLAND

(Oblatum 12.12.1977)