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ON CONTRACTIVE MAPPINGS IN METRIC SPACES

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Abstract: A number of authors have defined contractive type mappings on a complete metric space X which are generalizations of the well known Banach's contraction, and which have the property that each of such mappings has a unique fixed point. In this paper we shall prove the further generalizations of the Banach contraction mapping principle.

Key words: Generalized contractions, fixed point principle.

AMS: 47H10

The purpose of this paper is to consider the operators T on a metric space (X,ρ) which are not necessarily continuous. First of all we recall the following definitions.

Let T be a mapping of a metric space X into itself. The space X is said to be T-orbitally complete iff every Cauchy sequence of the form $\{T^{n_i}(x) \mid i = 1,2,\dots\}$, $x \in X$, converges in X, where $T^1(x) = Tx$ and $T^n x = T(T^{n-1}x)$ for $n = 2,3,\dots$. The mapping T is said to be orbitally continuous iff $\lim_{i\to\infty} T^{n_i}x = u$ implies $\lim_{i\to\infty} T(T^{n_i}x) = Tu$ for each $x \in X$.

Theorem 1. Let $T:X \longrightarrow X$ be a mapping on X and let X be a T-orbitally complete metric space. If T satisfies the following condition: for every $x,y \in X$, there exist real numbers $\alpha_i(x,y) = \alpha_i$, $\beta(x,y) = \beta$ such that, $\alpha_1 + \alpha_2 + \alpha_3 > \beta$ and $(\beta - \alpha_2 \geqq 0, \sup_{x,y} (\beta - \alpha_2)(\alpha_1 + \alpha_3)^{-1} = \lambda_1 \in [0,1))$ or $(\beta - \alpha_3 \geqq 0, \sup_{x,y} (\beta - \alpha_3)(\alpha_1 + \alpha_2)^{-1} = \lambda_2 \in [0,1))$, and

$$(1) \quad \alpha_1 \rho[Tx,Ty] + \alpha_2 \rho[x,Tx] + \alpha_3 \rho[y,Ty] + \alpha_4 \min\{\rho[x,Ty], \rho[y,Tx]\} \leqq \beta \rho[x,y];$$

then for each $x \in X$, the sequence $(T^n x)$ converges to a fixed point of T.

<u>Proof</u>. Let $x \in X$ be arbitrary. We shall show that the sequence of iterates

$$(2) \quad x_0 = x, \quad x_n = T(x_{n-1}), \quad n = 1,2,3,\ldots,$$

at x is a Cauchy sequence. Since $x_{k-1} = x_k$ for some $k \in N$ immediately implies that (x_n) is the Cauchy's sequence, we can suppose that $x_{n-1} \neq x_n$ for each $n \in N$. By (1) for $x = x_{n-1}$ and $y = x_n$ we have

$$\alpha_1 \rho[x_n, x_{n+1}] + \alpha_2 \rho[x_{n-1}, x_n] + \alpha_3 \rho[x_n, x_{n+1}] + \alpha_4 \min\{\rho[x_{n-1}, x_{n+1}], 0\} = \alpha_1 \rho[x_n, x_{n+1}] + \alpha_2 \rho[x_{n-1}, x_n] + \alpha_3 \rho[x_n, n_{n+1}] \leqq \beta \rho[x_{n-1}, x_n]$$

i.e.

$$\rho[x_n, x_{n+1}] \leqq \frac{\beta - \alpha_2}{\alpha_1 + \alpha_3} \rho[x_{n-1}, x_n] \leqq \lambda_1 \rho[x_{n-1}, x_n] .$$

Proceeding in this manner we obtain

$$\rho[x_n, x_{n+1}] \leqq \lambda_1 \rho[x_{n-1}, x_n] \leqq \ldots \leqq \lambda_1^n \rho[x, Tx] .$$

Hence for any $s \in N$ one has

$$\rho[x_n,x_{n+s}] \leq \sum_{i=1}^{n+s-1} \rho[x_i,x_{i+1}] \leq \lambda_1^n(1-\lambda_1)^{-1}\rho[x,Tx].$$

Since $\lim_{n\to\infty}\lambda_1^n(1-\lambda_1)^{-1} = 0$, it follows that (2) is a Cauchy sequence. X being T-orbitally complete, there is some $\xi \in X$ such that $\xi = \lim_{n\to\infty} T^n x$. To prove $T\xi = \xi$, consider the following inequalities, for $x = T^n x$, and $y = \xi$:

$$\alpha_1\rho[T^{n+1}x,T\xi] + \alpha_2\rho[T^n x,T^{n+1}x] + \alpha_3\rho[\xi,T\xi] +$$

$$+ \alpha_4\min\{\rho[T^n x,T\xi],\rho[T^{n+1}x,\xi]\} \leq \beta\rho[T^n x,\xi].$$

Hence, letting n tend to infinity, it follows $\rho[\xi,T\xi] = 0$, i.e. $T\xi = \xi$, which concludes the proof.

This proof is made under the assumption that $\beta - \alpha_2 \geq 0$ ($\Longrightarrow \alpha_1 + \alpha_3 > 0$). We can also prove the Theorem when $\beta - \alpha_3 \geq 0$ ($\Longrightarrow \alpha_1 + \alpha_2 > 0$) in a similar way, using the fact that distance is a symmetric function.

<u>Theorem 2</u>. Let $T:X\longrightarrow X$ be an orbitally continuous mapping on a metric space X which satisfies the following conditions

$$(3)\quad \alpha_1\rho[Tx,Ty] + \alpha_2\rho[x,Tx] + \alpha_3\rho[y,Ty] +$$

$$+ \alpha_4\min\{\rho[x,Ty],\rho[y,Tx]\} < \beta\rho[x,y],$$

whenever $x \neq y$ and $\alpha_1 + \alpha_2 + \alpha_3 \geq \beta$ and $\beta - \alpha_2 > 0 \vee \beta - \alpha_3 > 0$ (α_i, β are real constants). If for some $x_0 \in X$ the sequence $\{T^n x_0\}$ has a cluster point $\xi \in X$, then ξ is a fixed point of T.

<u>Proof</u>. If $T^{r-1}x_0 = T^r x_0$ for some $r \in N$, then $T^n x_0 = T^r x_0 = \xi$ for all $n \geq r$, and the assertion follows. Assume

now that $T^{r-1}x_0 \neq T^r x_0$ for all $r \in N$, and let $\lim_{i\to\infty} T^{n_i}x_0 =$ $= \xi$. Then for $T^{n-1}x_0$, $T^n x_0 \in X$, by (3)

$$\alpha_1 \rho[T^n x_0, T^{n+1}x_0] + \alpha_2 \rho[T^{n-1}x_0, T^n x_0] + \alpha_3 \rho[T^n x_0,$$
$$T^{n+1}x_0] + \alpha_4 \min\{\rho[T^{n-1}x_0, T^{n+1}x_0], 0\} =$$
$$= (\alpha_1 + \alpha_3)\rho[T^n x_0, T^{n+1}x_0] + \alpha_2 \rho[T^{n-1}x_0, T^n x_0] <$$
$$< \beta \rho[T^{n-1}x_0, T^n x_0]$$

i.e.

$$\rho[T^n x_0, T^{n+1}x_0] < \frac{\beta - \alpha_2}{\alpha_1 + \alpha_3} \rho[T^{n-1}x_0, T^n x_0] \leq \rho[T^{n-1}x_0, T^n x_0].$$

Hence

$$\rho[T^n x_0, T^{n+1}x_0] < \rho[T^{n-1}x_0, T^n x_0] .$$

Therefore, $\{\rho[T^n x_0, T^{n+1}x_0]\}$ is a decreasing and hence convergent sequence of positive real numbers. Since

$\lim_i \rho[T^{n_i}x_0, T^{n_i+1}x_0] = \rho[\xi, T\xi]$ and $\{\rho[T^{n_i}x_0, T^{n_i+1}x_0]\} \subseteq$ $\subseteq \{\rho[T^n x_0, T^{n+1}x_0]\}$,

it follows that

$$\lim_n \rho[T^n x_0, T^{n+1}x_0] = \rho[\xi, T\xi] . \tag{4}$$

Also, as $\lim_i T^{n_i+1}x_0 = T\xi$, $\lim_i T^{n_i+2}x_0 = T^2\xi$ and

$$\{\rho[T^{n_i+1}x_0, T^{n_i+2}x_0]\} \subseteq \{\rho[T^n x_0, T^{n+1}x_0]\} ,$$

by (4)

$$\rho[T\xi, T^2\xi] = \rho[\xi, T\xi] . \tag{5}$$

Suppose that $\rho[\xi, T\xi] > 0$. Then by (3) we have

$$\rho[T\xi, T^2\xi] < \rho[\xi, T\xi].$$

which contradicts (5). This proves that $T\xi = \xi$. The proof is complete.

The above proof is made under the assumption that $\beta - \alpha_2 > 0$ ($\Longrightarrow \alpha_1 + \alpha_3 > 0$). We can also prove the Theorem when $\beta - \alpha_3 > 0$ ($\Longrightarrow \alpha_1 + \alpha_2 > 0$) in a similar way, using the fact that distance is a symmetric function.

The results were presented on lectures together with examples and connections with previously obtained theorems (see [1] and the references there), while the author was visiting the Charles University, January 1978.

R e f e r e n c e

[1] TASKOVIČ M.: Some results in the fixed point theory, Publ. Inst. Math. 20(34), 1976, 231-242.

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