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ON AMALGAMATION OF GRAPHS AND ESSENTIAL SETS OF GENERATORS

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Abstract: The amalgamation of graphs from complete graphs is investigated. Some classes of graphs for which this operation is unambiguous are shown.

Key words: Graph, complete graph, amalgamation.

AMS: 05C99

Introduction. Given two graphs G_1, G_2 with a common (induced) subgraph H we construct the graph (G_1, H, G_2) by amalgamation of G_1, G_2 with respect to H. Recently it was shown that this graph-theoretical operation plays an important role in constructions of "difficult graphs" (such as Ramsey graphs of various types), examples of this are given in [NR]. This paper is devoted to the study of properties of this operation itself.

It is easy to see that every (finite) graph may be obtained by a gradual amalgamation of complete graphs. The basic question we are interested in is whether this procedure is unique. The main result (stated below as Theorem 3) is that every graph which can be constructed from complete graphs of at most 3 different cardinalities is uniquely constructable. This is the best possible as there exists a graph which can be constructed by means of complete graphs of 4 different cardinalities in entirely 2 different ways.

These results solve a question of L. Kučera and J. Nešetřil.

§ 1. Notions and results. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be proper subgraphs of graph $G = (V, E)$. (The term subgraph is used in the sense of [B], i.e. H is a subgraph G if $V(H) \subset V(G)$ and $E(H) = E(G) \cap [V(H)]^2$.) If $V_1 \cup V_2 = V$ and $E_1 \cup E_2 = E$ then the triad (G_1, H, G_2) where $H = (V_1 \cap V_2, E_1 \cap E_2)$ is called the amalgam decomposition of G.

Example: If $G = G_1 + G_2$ then the triad $(G_1, \emptyset, G_2)$ is the amalgam decomposition of G.

If A is the subset of integers $N = \{0,1,\dots\}$ L(A) will denote the smallest class of graphs satisfying the following two conditions:

1. $K_n \in L(A)$ for every $n \in A$. (K_n is the complete graph with n vertices.)

2. If there is an amalgam decomposition (G_1, H, G_2) of the graph G for which $G_1, H, G_2 \in L(A)$ then $G \in L(A)$, too.

If $G \in L(A)$ then we can say that A generates the graph G. Graph G is called k-generated if there is some set A the cardinality of A equals k which generates G.

The integer a is called an essential generator of the graph G if $a \in A$ for every A so that $G \in L(A)$.

Graph is called unambiguous if G is generated by the set of its essential generators.

Remark: Several naturally arising classes of graphs are of the form L(A) for a convenient set A. For example L(1,2) is the class of all trees, L(0,1,2) is the class of all graphs without triangles, L(N) is the class of all graphs.

The following definitions describe some types of essential generators. Three kinds of complete subgraphs are defined and their cardinalities are shown to be essential generators (Theorem 1). Theorem 2 proves that triangulated graphs are unambiguous. The unambiguity of 3-generated graphs results from Theorems 1,2. Simultaneously an example of ambiguous 4-generated graph is presented.

Denote by K(G) the set of all complete subgraphs of the graph G. K(G) always contains the empty graph (denoted by K_o) and one point graphs corresponding to the vertices of the graph G. The term cycle will apply to every simple cycle of the length of at least 4 without chords (see [B], chord of the path will be understood in the same sense). Every simple path without chords will be called the path.

Definition: 1. Complete subgraph $K \in K(G)$ is called the clique if K is the maximal complete subgraph, i.e. there is no $L \in K(G)$, $L \supsetneq K$.

2. Complete subgraph $K \in K(G)$ is called a segment if the equivalence $E_G(K)$ on the set $\{M \in K(G): M \supsetneq K\}$ has at least two classes. The equivalence $E_G(K)$ is generated by the relation $\sim$ defined as follows: $M \sim N$ iff $M \cap N \supsetneq K$. (Equivalently $K \in K(G)$ is a segment if the set $\{M \in K(G): M \supsetneq K\}$ can be divided into two non-empty subsets X,Y so that for eve-

ry pair of complete subgraphs $M \in X$, $N \in Y$ is $M \cap N = K$.)

3. Complete graph is called the focus if there is some cycle C in G (disjoint with K) that $K \oplus C$ is a subgraph of G and K is maximal with this property, i.e. for $L \in K(G)$, $L \supsetneq K$ there is no cycle C' so that $L \oplus C'$ is a subgraph of G. Here $K \oplus C$ denotes the graph which can be obtained by the disjoint union of K, C and by adding all edges $\{u,v\}$ where $u \in V(K)$, $v \in V(C)$.

§ 2. Proofs

Theorem 1: If the complete graph K is a clique, segment or focus of the graph G then $|K|$ is an essential generator of G.

Proof: Theorem 1 is valid if G is a complete graph. If G is not complete let us denote (G_1,H,G_2) an amalgam decomposition of the graph G. We are going to prove that every complete subgraph which is clique, segment or focus in the graph G is clique, segment or focus in one of the graphs G_1,H,G_2. Hence each of the three described kinds of generators must be the generator of G_1 or H or G_2. Thus the theorem will be proved. Obviously if M is a complete subgraph of G then M is the complete subgraph of G_1 or G_2.

The proof of this theorem follows from three propositions below:

Proposition 1: If K is clique of the graph G then K is clique of the graph G_1 or G_2.

The proof is obvious.

Proposition 2: If K is the segment of the graph G, then K is the segment of G_1 or G_2 or K is the clique of H.

Proof: Let us denote by $R_1,\dots,R_t$ the classes of equivalence E(K) for the segment K of the graph G. Two cases must be considered.

a) There is G_i and two classes R_j and R_k of equivalence E(K) so that $R_j \cap K(G_i) \neq \emptyset$ and $R_k \cap K(G_i) \neq \emptyset$. Then by definition K is the segment of G_i.

b) If the case a) is not valid, then t = 2 and in case of suitable indexing $R_1 \cap K(G_1) = R_2 \cap K(G_2) = \emptyset$. In this case K is the clique of H.

Proposition 3: If K is the focus of the graph G then K is a focus of G_1 or G_2 or K is a segment of H.

The following lemma will preeede the proof of Proposition 3.

Lemma: Let $\bar{u} = (a,u_1,\dots,u_m,b)$, $\bar{v} = (a,v_1,\dots,v_n,b)$ $m,n \geq 1$ are two disjoint (except a,b) paths of the graph G. Then u_1 is either adjacent to all v_j, $j = 1,\dots,n$ or the cycle of the graph G can be chosen from the union of the vertices of both paths.

Proof of Lemma: Let u_1 be not adjacent to all vertices v_i. If $(a,u_1,\dots,u_m,b, v_n,\dots,v_1,a)$ is not a cycle put
$i_o = \min \{i$: there is j that u_i,v_j are adjacent in $G\}$
$j_o = \min \{ j: u_{i_o}, v_j$ are adjacent in $G\}$. Then
$(a,u_1,\dots,u_{i_o},v_{j_o},\dots,v_1,a)$ is the cycle in G.

Proof of Proposition 3: Let K be the focus of the graph G. Suppose that K is neither focus of G_1 nor G_2 nor segment of H. As K is neither focus of G_1 nor G_2 every cycle with the focus K contains as vertices of $G_1 - H$ as vertices of $G_2 - H$. Let us take the cycle C with the focus K which contains the smallest number of vertices of $G_1 - H$. Denote by a,b two vertices of the cycle C for which the arc between a and b belongs to the graph $G_1 - H$ and $a,b \in V(H)$. Denote this arc $\bar{v} = (a,v_1,\dots,v_m,b)$, $m \geq 1$. The remaining part of the cycle C denote by $\bar{w} = (a,w_1,\dots,w_p,b)$, $p \geq 1$. As K is not segment of the graph H the equivalence $E_H(K)$ has only one class and complete graphs $K \cup \{a\}$, $K \cup \cup \{b\}$ are equivalent. Thus there is a sequence $K \cup \{a\} = = M_o, M_1, \dots, M_t = K \cup \{b\}$ so that $M_i \cap M_{i+1} \supsetneqq K$ for all $i = = 0,\dots,t-1$, $M_i \in K(H)$. Hence there is the shortest path $\bar{u} = (a,u_1,\dots,u_n,b)$, $n \geq 1$ so that $K \cup \{u_i\} \in K(H)$ for all $i = 1,\dots,n$.

If a cycle C' can be chosen from the union of the paths $\bar{u}$, $\bar{v}$ then this cycle C' belongs to the graph G_1 and $C' \oplus K$ is a subgraph of G_1. Thus K is the focus of G_1, which is a contradiction. Hence by the lemma the vertex v_1 is adjacent to all vertices u_i and because the cycle C possesses no chord the paths $\bar{u}$ and $\bar{v}$ are disjoint (except a,b). Consider two cases.

i) $n \geq 2$. If it is possible to choose a cycle C' from the union of the paths $\bar{u},\bar{w}$ then $C' \oplus K$ is the subgraph G and the cycle C' contains a smaller number of vertices of $G_1 - H$ than the cycle C. Thus by the lemma the vertex w_1 is adjacent to all vertices u_i. But then $C_1 = (a,v_1,u_2,w_1,a)$ is the cycle of G and $C_1 \oplus (K \cup \{u_1\})$ is the subgraph of G. This is the contradiction with the definition of the focus K.

ii) $n = 1$. After the lemma the vertex u_1 is adjacent to all vertices v_i,w_j, $i = 1,\dots,n$, $j = 1,\dots,p$. Thus $C \oplus (K \cup \{u_1\})$ is the subgraph of G, which is a contradiction.

The graph possessing no cycle (cycle without chord of the length at least 4) is called triangulated.

Theorem 2: Every triangulated graph is generated by the set $\{|K| : K$ is clique or segment of $G\}$.

First, we prove the following:

Proposition: Every minimal articulation set of the triangulated graph G is its segment.

(This is a strenghtening of a known fact that every minimal articulation set of the triangulated graph is a complete graph - see [B].)

Proof of the proposition: Let us denote $C_0,\dots,C_q$, $q \geq 1$ the connected component of the graph $G - A$ where A is a minimal articulation set of the graph G. If G is non-connected then A is the empty set and evidently the segment. If G is connected then $A \neq \emptyset$ and by the proof of the theorem in [B] : For every $a \in A$ and for every $i = 0,\dots,q$ the vertex a is adjacent to some vertex in the component C_i. We shall show that in every component C_i there is a vertex u_i adjacent to all vertices $a \in A$.

Denote u a vertex of C_i adjacent to the greatest number of the vertices of A. Suppose that there is $a \in A$ that $\{a,u\}$ is not an edge in G. By the above there is a vertex $v \in C_i$, v adjacent to a. Let us take such v with the minimal distance k from u in the component C_i. By the choice of the vertex v there is the path $\overline{p}$ of the length k which joins vertices u and v in the component C_i. The vertex v cannot be adjacent to all vertices of A which are adjacent to u. Thus there is $b \in A$ that $\{u,b\}$ is an edge and $\{b,v\}$ is not an edge in the graph G. By the application of the lemma from Theorem 1 to the paths (a,b,u) and $(a,\bar{p})$ and by using the fact that $\{b,v\}$ is not an edge we can prove the existence of the cycle in the graph G, which is a contradiction since G is triangulated. Thus the vertex u is adjacent to all vertices of the articulation set A. Evidently the complete graphs $\{u_i\} \cup A$, $\{u_j\} \cup A$, $i \neq j$ are not equivalent in $E(A)$. Thus A is the segment of G.

Proof of Theorem 2: The theorem is obvious for complete graph. To finish the proof of the theorem it is sufficient to take the amalgam decomposition $(C_0 \cup A, A, A \cup \bigcup_{i=1}^{q} C_i)$. It is easy to see that all cliques and all segments of decomposition graphs are cliques and segments of the graph G. Thus Theorem 2 is proved.

Corollary: Triangulated graphs are unambiguous.

Proof: By Theorem 1 cardinalities of the cliques and segments are essential generators. By Theorem 2 these generators are sufficient. Thus triangulated graphs are unambiguous.

Another example of unambiguous graphs are complete k-partite graphs. If G is complementary graph to the equivalence, all classes of which have at least two elements, then all cardinalities of complete subgraphs of G are essential generators of G.

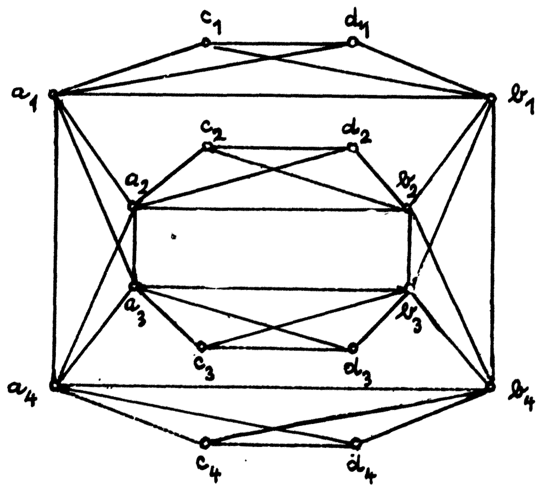
The following theorem proves the unambiguity of 3-generated graphs.

Theorem 3: Every 3-generated graph is unambiguous. There exists a 4-generated graph which is ambiguous.

Proof: Let G be a 3-generated graph with the generators a,b,c, $a < b < c$. If G is triangulated then G is unambiguous by Theorem 2. If G is not triangulated then there is a cycle in G and thus G possesses some focus F. This focus is contained in some segment P, $P \supsetneq F$ and P is again contained in some clique K, $K \supsetneq P$. Thus $|F| = a$, $|P| = b$, $|K| = c$. This proves that a,b,c are essential generators and thus the graph G is unambiguous.

The example of the 4-generated graph which is ambiguous is given in Figure 1.

Fig.1:



Denote G_1 subgraph of G on the set $\{a_1,a_2,a_3,a_4,b_4,c_4,d_4\}$

G_2 " " $\{a_1,a_2,a_3,b_1,\dots,b_4,$
$c_1,d_1,c_2,d_2,c_3,d_3\}$

G_3 " " $\{a_1,\dots,a_4\}$

G_4 " " $\{a_4,b_4,c_4,d_4\}$

Denote G_5 subgraph of G on the set $\{a_1,a_2,a_3,b_2,b_3,b_4,c_2,$
$d_2,c_3,d_3\}$

G_6 " " $\{a_1,b_1,\dots,b_4,c_1,d_1\}$

Then $G = (G_1,G_1\cap G_2,G_2)$, $G_1 = (G_3,G_3\cap G_4,G_4)$, $G_2 = (G_5,G_5\cap$ $\cap G_6,G_6)$. Evidently $G_3,G_4,G_5,G_6,G_1\cap G_2,G_3\cap G_4,G_5\cap G_6 \in$ $\in L(0,1,3,4)$.

Thus $G\in L(0,1,3,4)$.

Denote H_1 subgraph of G on the set $\{a_1,\dots,a_4,b_3,b_4,c_3,c_4,$
$d_3,d_4\}$

H_2 " " $\{a_1,a_2,b_1,\dots,b_4,c_1,c_2,$
$d_1,d_2\}$

Then $G = (H_1,H_1\cap H_2,H_2)$, evidently $H_1,H_2,H_1\cap H_2\in L(0,1,2,4)$. Thus $G\in L(0,1,2,4)$.

Now we prove that $G\notin L(0,1,4)$. Let $G = (G_1,H,G_2)$ be an amalgam decomposition of G that $\{a_1,\dots,a_4\}\subset V(G_1)$, $\{b_1,\dots$ $\dots,b_4\}\subset V(G_2)$. If there is only one index i that $a_i\in V(G_2)$ then $\{b_j : j\neq i\}\subset V(G_1)$ and thus G_1 contains the clique of the cardinality 3 .

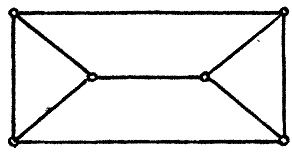
If there are just two indexes i,j that $a_i,a_j\in V(G_2)$ then G_2 contains the clique of the cardinality 2 .

If there are just three indexes i,j,k that $a_1,a_j,a_k\in V(G_2)$ then G_2 contains the clique of the cardinality 3 .

One of these three cases must come. Thus $G\notin L(0,1,4)$.

<u>Remark</u>: We believe that the following conjecture is true: Every ambiguous graph contains a subgraph, as shown in Figure 2.

Fig. 2:



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