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THE CRITICAL EXPONENT OF OPERATORS WITH CONSTRAINED SPECTRAL RADIUS

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Abstract: Suppose we are given a natural number n and $0 < r < 1$. The proof is given that there is an integer k such that if A is a linear contraction on an n-dimensional Banach space with the spectral radius less or equal to r, then

$$| a^k | < 1.$$

Key words: Spectral radius, norm of iterates.

AMS: 47B99

Let n be an arbitrary but fixed positive integer. Let X_n be an n-dimensional linear space, let $P(X_n)$ be the set of all norms on X_n and let $L(X_n)$ be the algebra of all linear operators on X_n. If $A \in L(X_n)$ and $p \in P(X_n)$, then we shall denote the operator norm of A in the Banach space (X_n, p) by $p(A)$. The spectral radius of $A \in L(X_n)$ will be denoted by $|A|_\sigma$.

If $p \in P(X_n)$, $A \in L(X_n)$, $p(A) = 1$ and $|A|_\sigma < 1$, then it is clear from the spectral radius formula that there is a natural number k such that $p(A^k) < 1$. This led [2] to the following

Definition. The number q is said to be the critical exponent of the Banach space (X_n, p), if the following conditions are satisfied:

(1) If $A \in L(X_n)$ and $p(A) = p(A^q) = 1$ then $|A|_\sigma = 1$;

(2) there exists $B \in L(X_n)$ such that $p(B) = p(B^{q-1}) = 1$ and $|B|_\sigma < 1$.

The first critical exponent to be computed (although not described as such) was that of the n-dimensional l_∞-space; the result, $n^2 - n + 1$, was obtained in 1957 by J. Mařík and V. Pták [1]. Later, Professor V. Pták [2] showed that the critical exponent of the n-dimensional Hilbert space is equal to n.

If $n \geqq 2$, then there is $p \in P(X_n)$ such that $q(X_n, p) = \infty$ (L. Danzer, unpublished), so that no upper bound independent of p for $q(X_n, p)$ can exist. The point of this note is to show that the situation changes if we restrict ourselves to operators with constrained spectral radius.

<u>Theorem</u>. Let $0 < r < 1$. If

$$k = n \cdot \lceil \ln(2^{1/n} - 1)/\ln r \rceil ,$$

$A \in L(X_n)$, $p \in P(X_n)$, $p(A) \leqq 1$ and $|A|_\sigma \leqq r$, then

$$p(A^k) < 1,$$

where $[\,x\,]$ stands for the least natural number greater than x.

Proof: Let r, p and A satisfy the assumptions of the theorem, and let

$$g(x) = x^n - \alpha_1 - \alpha_2 x - \ldots - \alpha_n x^{n-1}$$

be the characteristic polynomial of A. All the roots $\rho_1, \ldots, \rho_n$ of the equation $g(x) = 0$ being less than or equal to r in absolute value,

$$|\alpha_i| = |(-1)^{n-i} \sum_{\substack{e_1+\dots+e_n=n-i+1 \\ e_i \in \{0,1\}}} \varrho_1^{e_1} \dots \varrho_n^{e_n}| \leq \binom{n}{n-i+1} r^{n-i+1}.$$

Thus

$$p(A^n) \leq \sum_{i=1}^{n} |\alpha_i| \leq \sum_{i=1}^{n} \binom{n}{n-i+1} r^{n-i+1} = (1+r)^n - 1.$$

Let s be a positive integer. Since $|A^s|_\sigma = |A|_\sigma^s$, we have $|A^s|_\sigma \leq r^s$ and

$$p(A^{sn}) = p((A^s)^n) \leq (1+r^s)^n - 1.$$

Hence if

(1) $$r^s < 2^{1/n} - 1,$$

then $p(A^{sn}) < 1$. Simple computations show that (1) is equivalent to

$$s \geq [\ln(2^{1/n} - 1)/\ln r],$$

which finishes the proof.

We believe that a table of the actual values of $k(n,r)$ for various values of n and r may be of some interest.

T a b l e .

$r \backslash n$	2	3	4	5	10	15	20
0.1	2	3	4	5	20	30	40
0.2	2	3	8	10	20	30	60
0.3	2	6	8	10	30	45	60
0.5	4	6	12	15	40	75	100
0.7	6	12	20	30	80	135	200
0.8	8	21	32	45	120	210	300
0.9	18	39	64	95	260	435	640

Our results are doubtless capable of refinement for large r. Nevertheless, for $r < 2^{1/n} - 1$ k takes the value n, which is clearly the best possible bound.

R e f e r e n c e s

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