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ON THE POINTS OF MULTIPLICITY OF MONOTONE OPERATORS

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Abstract: We prove by a simple method that the set of points at which a monotone operator defined in a separable real Banach space is multivalued can be covered by countably many of Lipschitz hypersurfaces. By the same method we obtain several related results. Applications to singular boundary points of convex sets and normals in non-exposed points of convex sets are given. Our results improve theorems of H. Zarantonello and N. Aronszajn.

Key words: Points of multiplicity of monotone operators, Lipschitz surfaces in Banach spaces, singular boundary points of convex sets, normals in non-exposed points of convex sets.

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1. Introduction. Let X be a real Banach space. If we say that $T:X \longrightarrow X^*$ is a monotone operator, then we mean that T is (possibly) multivalued, the domain D(T) is a nonvoid subset of X and T is monotone (if $y_1 \in Tx_1$, $y_2 \in Tx_2$ then $\langle x_2 - x_1, y_2 - y_1 \rangle \geq 0$). Zarantonello [9] has proved that if X is separable then the points of multiplicity of T form a set of the first category. In the connection with this theorem the following questions arise:

A. For which nonseparable spaces X does the statement of the Zarantonello's theorem hold?

B. For which spaces X any monotone operator $T:X \longrightarrow X^*$ is singlevalued and upper semicontinuous in any point except a set of the first category?

C. If X is separable, is it possible to say something more on the smallness of the set of points of multiplicity of T ?

For the theorems which concern the questions A and B we refer to [7] and [3] where also connections with the differentiation of convex functions are shown and further references are given. Note that in [4] it is proved that the spaces from the question B are precisely Asplund spaces.

The present article concerns the question C. Aronszajn ([1], p. 157, Theorem Z) stated that by the Zarantello's method it is possible to prove that the points of multiplicity of T in the separable case are contained in a set from the class $\mathfrak{A}^{\circ}$. The class $\mathfrak{A}^{\circ}$ of small sets was introduced in [1] in the connection with the study of differentials of convex functions and their generalizations. Any set from $\mathfrak{A}^{\circ}$ is of the first category, but there exist sets of the first category which are contained in no set belonging to $\mathfrak{A}^{\circ}$.

In the second section of the present article we give a simple proof of a theorem (Theorem 1) which includes Theorem Z from [1]. Our method is different from that used in [9] and [1]. In the third section we classify (following [9]) the points of multiplicity of T according to the dimension and codimension of Tx and obtain two theorems (Theorem 2, Theorem 3) using the same main idea as in the proof of Theorem 1. It would be possible to prove Theorem 2 in a more precise form and obtain Theorem 1 as the special case ($n = 1$) of this theorem. We do not proceed in this way since the proof of Theorem 2 is somewhat technical and the proof of Theorem 1 is quite simple. Following [9] we obtain in the fourth section from the results of the third section some theorems concerning singular boundary points of convex sets and normals at non-exposed points of convex sets which generalize some results of [9].

Finally note that no theorem of the present article is the best possible. For example, it is probable that in Theorem 1 it is possible to write " $(c - c)$-hypersurface" instead of "Lipschitz hypersurface". By $(c - c)$-hypersurface we mean a Lipschitz hypersurface for which the function f from the definition is the difference of two Lipschitz convex functions (cf. [8]).

Unfortunately we are able to prove this conjecture in the case of two-dimensional space X only. The Theorem 4 and Theorem 5 can be improved by the method of [8]. These questions will be investigated in a subsequent article.

2. We shall use the following natural definition.

Definition. Let X be a Banach space. We shall say that $A \subset X$ is a Lipschitz surface of dimension n (of codimension n, respectively), if there exists a subspace C of X of codimension n (of dimension n, respectively) and a Lipschitz mapping $f: H \longrightarrow C$, where H is a topological complement of C, such that $A = \{h + f(h);\ h \in H\}$. We shall say that A is a

Lipschitz **surface associated with** C (**with** c, **if** C = **Lin**$\{c\}$). Lipschitz surface of codimension 1 is termed Lipschitz hypersurface.

<u>Theorem 1</u>. Let X be a separable Banach space and $(c_n) \subset X$ a complete sequence. Let $T: X \longrightarrow X^*$ be a monotone operator and B the set of all points at which T is multivalued. Then there exists a sequence of Lipschitz hypersurfaces $(H_{nk})_{n,k=1}^{\infty}$ such that H_{nk} is associated with c_n and $B \subset \bigcup_{n,k=1}^{\infty} H_{nk}$.

<u>Proof</u>. For positive integers n, K and rationals $r < s$ let $B(n,K,r,s)$ be the set of all $x \in D(T)$ for which there exist $a(x) \in Tx$, $b(x) \in Tx$ such that $\langle c_n, a(s) \rangle < r$, $\langle c_n, b(x) \rangle > s$, $\| a(x) \| < K$ and $\| b(x) \| < K$. Since (c_n) is complete, obviously $B = \bigcup B(n,K,r,s)$. Let n,K,r,s be fixed and H be a topological complement of Lin $\{c_n\}$. Let

$$z_i \in B(n,K,r,s),\ z_i = h_i + y_i c_n,\ h_i \in H, \quad i = 1,2.$$

We can suppose without any loss of generality that $y_2 \geq y_1$. From the monotonicity of T then

$$0 \leq \langle z_2 - z_1, a(z_2) - b(z_1) \rangle = \langle h_2 - h_1, a(z_2) - b(z_1) \rangle + \\ + \langle (y_2 - y_1) c_n, a(z_2) - b(z_1) \rangle .$$

Therefore

$$|y_2 - y_1| \leq 2K (s - r)^{-1} \| h_2 - h_1 \| .$$

Since for every Lipschitz **function** defined on a subset of a metric space there **exists a** Lipschitz **extension** on the whole space (see e.g. **[5]**), there **exists a** Lipschitz function f on H **such** that $B(n,K,r,s) \subset \{f(h) c_n + h : h \in H\}$ and thus $B(n,K,r,s)$ is contained in a Lipschitz hypersurface associated with c_n. Since the system of all $B(n,K,r,s)$ is countable, it is easy to finish the proof.

Note. It is easy to see that Theorem 1 includes Theorem Z from [1].

3. If we write Banach space (Hilbert space) we mean real Banach space (Hilbert space). The open ball of the centre c and radius r is denoted by $B(c,r)$.

In the subsequent we shall use the following well-known facts concerning extensions of Lipschitz mappings.

Theorem E. Let X, Y be metric spaces, $A \subset X$ and $f: A \to Y$ a Lipschitz mapping. In the following cases there exists a Y-valued Lipschitz extension of f defined on X:

(i) $Y = R^n$

(ii) X, Y are Hilbert spaces

(iii) $X = R$ and Y is a Banach space

Note. For (i) see e.g. Mc Shane [5], for (ii) see e.g. Minty [6]. The proof of (iii) is easy.

We shall need also the following simple lemmas which we state without a proof.

Lemma 1. Let X be a separable Banach space and $D \subset X$ a countable dense subset of X. Let $\mathcal{S}$ be the system of all n-dimensional subspaces of X which are generated by n vectors from D. For any $S \in \mathcal{S}$ denote by N_S a countable dense subset of S^* and by $\mathcal{L}_S$ the countable subset of $X^*/S^{\perp}$ which corresponds to N_S in the natural isomorphism $S^* \cong X^*/S^{\perp}$. Put $\mathcal{L} = \{\mathcal{L}_S : S \in \mathcal{S}\}$ and $\mathcal{V} = \{S^{\perp} ; S \in \mathcal{S}\}$.

Then

(i) Any n-dimensional subspace $P \subset X^*$ has a topological complement $V \in \mathcal{V}$.

(ii) If P, V are as in (i), $c \in X^*$ and $\varepsilon > 0$, then there exists $t \in (c + P) \cap B(c, \varepsilon)$ such that $L = t + V \in \mathcal{L}$.

<u>Lemma 2</u>. Let X^* be a separable Banach space and $D \subset X^*$ a countable dense subset of X^*. Let $\mathcal{V}$ be the system of all n-dimensional subspaces of X^* which are generated by n vectors from D. Denote $\mathcal{L} = \{d + V;\ d \in D,\ V \in \mathcal{V}\}$. Then

(i) Any subspace $P \subset X^*$ of codimension n has a topological complement $V \in \mathcal{V}$.

(ii) If P, V are as in (i), $c \in X^*$ and $\varepsilon > 0$, then there exists $t \in (c + P) \cap B(c, \varepsilon)$ such that $L = t + V \in \mathcal{L}$.

<u>Note</u>. Lemma 2 holds for a general separable Banach space X^* but we shall use it for dual spaces and it is the reason for our notation.

<u>Theorem 2</u>. Let X be a separable Banach space and $T: X \longrightarrow X^*$ a monotone operator. For any positive integer n denote by B_n the set of all points x for which the convex cover of Tx is at least n-dimensional. Then B_n can be covered by countably many Lipschitz surfaces of codimension n.

<u>Theorem 3</u>. Let X be a Banach space with the separable dual space X^* and $T: X \longrightarrow X^*$ a monotone operator. For any positive integer n denote by B_n the set of all points x for which the convex closure of Tx contains a ball of codimension n (a relatively open ball in an affine manifold of codimension n). Then B_n has σ-finite n-dimensional Hausdorff measure and if X is a Hilbert space or n = 1, then B_n can

be covered by countably many Lipschitz surfaces of dimension n.

Note. The part of Theorem 3 which concerns Hausdorff measure is the consequence of Theorem 3 from [9] in the case X is a Hilbert space.

Proof of Theorem 2 and Theorem 3. Since any monotone operator has a maximal monotone extension we can suppose in the proof of Theorem 2 (Theorem 3, respectively) that for $x \in B_n$ the set Tx contains a ball of dimension n (of codimension n, respectively). (Remind that if T is a maximal monotone operator then Tx is a convex closed set for any $x \in D(T)$ (cf. [2]).)

If we put in the subsequent proof $\alpha = n$ ($\alpha = \infty - n$, respectively) we obtain the proof of Theorem 2 (Theorem 3, respectively). Instead of subspace of codimension n we shall write $(\infty - n)$-dimensional subspace. By Lemma we shall mean Lemma 1 if $\alpha = n$ and Lemma 2 if $\alpha = \infty - n$.

For any $x \in B_n$ choose $c_x \in Tx$, a rational $r_x > 0$ and an α-dimensional subspace P_x of X^* such that $B(c_x, r_x) \cap \cap (c_x + P_x) \subset Tx$. Choose further a rational M_x such that $\| c_x \| \leq M_x$. Let $\mathcal{V}$ and $\mathcal{L}$ be the systems from Lemma. Choose by Lemma, (i), a topological complement $V_x \in \mathcal{V}$ of P_x. Let π_x be the projection of X^* onto P_x parallel to V_x and q_x a rational such that $\| \pi_x \| < q_x$. Choose further by Lemma, (ii), $t_x \in X^*$ and $L_x \in \mathcal{L}$ corresponding to c_x and $\varrho_x = = r_x/2$.

Let $B(r,M,V,q,L)$ be the set of all points $x \in B_n$ for which $r_x = r$, $M_x = M$, $V_x = V$, $q_x = q$, $L_x = L$. Obviously

$B_n = \bigcup B(r,M,V,q,L)$ and the family of the sets $B(r,M,V,q,L)$ is countable.

Let r,M,V,q,L be fixed. If we put $Z = {}^{\perp}V$, Z is α-dimensional subspace of X. Denote by W a topological complement of Z.

Let now x_1, x_2 be from $B(r,M,V,q,L)$, $x_1 = z_1 + w_1$, $x_2 = z_2 + w_2$, $z_1,z_2 \in Z$, $w_1,w_2 \in W$. Put $z = z_2 - z_1$ and choose $h \in X^*$ such that $\|h\| = 1$ and $\langle z,h\rangle > \|z\|/2$. If $h = v + p$, $v \in V$, $p \in P_{x_2}$, then $\langle z,p\rangle = \langle z,h\rangle > \|z\|/2$ and $\|p\| < q$. Put $u = t_{x_2} - (r/2q)p$. Then we have $\|u - c_{x_2}\| \leq \|c_{x_2} - t_{x_2}\| + (r/2q)\|p\| < r$ and therefore $u \in Tx_2$. Hence from the monotonicity of T it follows that $0 \leq \langle x_2 - x_1, u - t_{x_1}\rangle = \langle w_2 - w_1, u - t_{x_1}\rangle + \langle z, u - t_{x_1}\rangle = \langle w_2 - w_1, u - t_{x_1}\rangle + \langle z, t_{x_2} - t_{x_1}\rangle - (r/2q)\langle z,p\rangle$. Since $t_{x_2} - t_{x_1} \in V$, $\langle z,p\rangle > 1/2\,\|z_2 - z_1\|$ and $\|u - t_{x_1}\| \leq 2(M + r)$ we obtain

$$\|z_2 - z_1\| \leq \frac{8q(M + r)}{r}\|w_2 - w_1\| .$$

Using well known facts concerning Hausdorff measure and Theorem E it is easy to finish the proof.

Theorem 3 can be further generalized in the following way.

<u>Theorem 3*</u>. Let X be a Banach space and $Y \subset X^*$ a separable closed weak* dense subspace of X^*. Let $T: X \longrightarrow Y$ be a monotone operator. For any positive integer n denote by B_n the set of all points x for which the convex closure of Tx contains a ball of codimension n. Then B_n has σ-finite n-dimensional Hausdorff measure and if X is a Hilbert space

or $n = 1$, then B_n can be covered by countably many Lipschitz surfaces of dimension n.

Proof. It is sufficient to write Y instead of X^* in the proof of Theorem 3. The crucial point is the existence of h. From this reason we suppose that Y is weak* dense in X^*.

4. Zarantonello in [9] obtained some theorems concerning singular elements of convex sets as corollaries of his theorems on monotone operators. Since our theorems generalize and make Zarantonello's results on monotone operators more precise, we obtain by the same method as in [9] some new theorems concerning singular elements of convex sets.

We use the terminology of [9]:

Definition. Let K be a convex subset of a Banach space X and $x \in K$. We shall say that $y \in X^*$ is a normal of K at x if $\langle x,y \rangle = \sup_{t \in K} \langle t,y \rangle$. The set $V_K x$ of all normals of K at x is called the vertex of K at x. If $y \in X^*$ we mean by the face perpendicular to y the set $F_K y$ of all $x \in K$ for which y belongs to the vertex of K at x.

It is the well known easy fact that the operators $y \longrightarrow F_K y$ (the support mapping of K) and $x \longrightarrow V_K x$ are monotone. Therefor our Theorem 2, Theorem 3, Theorem 3* yield immediately the following theorems.

Theorem 4. Let X be a separable Banach space and $K \subset X$ a convex set. Then the set of points $x \in K$ having a vertex containing a ball of dimension n can be covered by countably many Lipschitz surfaces of codimension n.

Proof. It is sufficient to use Theorem 2 for $Tx = V_K x$.

Theorem 5. Let X be a Banach space with the separable dual space X^*, and K a convex subset of X. Then the set of normals to K at faces which are at least n-dimensional can be covered by countably many Lipschitz surfaces of codimension n.

Proof. It is sufficient to use Theorem 2 for $Ty = F_K y$.

Theorem 6. Let X be a Banach space with the separable dual space X^* and K a convex subset of X. Then the set of points $x \in K$ having a vertex containing a ball of codimension n has σ-finite n-dimensional Hausdorff measure and if X is a Hilbert space or n = 1 then this set can be covered by countably many Lipschitz surfaces of dimension n.

Proof. It is sufficient to use Theorem 3 for $Tx = V_K x$.

Theorem 7. Let X be a separable Banach space and K a convex subset of X. Then the set of normals to K at faces which contain a ball of codimension n has σ-finite n-dimensional Hausdorff measure and if X is a Hilbert space then this set can be covered by countably many Lipschitz surfaces of dimension n.

Proof. It is sufficient to use Theorem 3 for $Tx = V_K x$. Here we use that X is weak*dense in X^{**}.

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