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LIOUVILLE FORMULA FOR SYSTEMS OF LINEAR HOMOGENEOUS ITÔ STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract: Let X(t) be the fundamental matrix solution of Itô equation (1) and D(T) = det X(T). The process D(t) is a solution of (2) and hence given by (6). It is shown that X(t) is regular and a formula for solutions of nonhomogeneous linear Itô equations is derived.

Key words: Linear Itô stochastic equations, Liouville formula, fundamental matrix solutions, variation of constants formula.

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The general system of linear homogeneous Itô stochastic differential equations can be written in the vector form

$$\text{(1)} \qquad dx = A(t)xdt + \sum_{j=1}^{k} B^{(j)}(t)xdw_j,$$

where x is an n-dimensional vector, $A(t)$, $B^{(j)}(t)$, $j = 1,\dots,k$, are matrix functions of the type $n\times n$ defined on $\langle 0,\infty)$, $w_j(t)$ are stochastically independent Wiener processes.

Assume that $\|A(t)\|$, $\|B^{(j)}(t)\|$ are measurable and locally bounded on $\langle 0,\infty)$. A matrix function $X(t)$ of the type $n\times n$ defined on $\langle t_0,\infty)$, $t_0 \geqq 0$ is called a fundamental matrix solution of (1) if the columns of $X(t)$ are solutions of (1) on $\langle t_o, \infty)$ and $X(t_o)$ is the unit matrix. The existence and unicity of the solutions of (1) is proved in [1], [2]. Denote $D(t) = \det X(t)$.

Theorem 1. The process $D(t)$ is a solution of

$$(2)\qquad dD = [\operatorname{tr} A(t) + \frac{1}{2} \sum_{p,q,j} (B_{pp}^{(j)}(t) B_{qq}^{(j)}(t) - B_{pq}^{(j)}(t) B_{qp}^{(j)}(t))]\, D(t)dt + D(t) \sum_j \operatorname{tr} B^{(j)}(t) dw_j \, .$$

Proof. Let $x_i^{(j)}(t)$ be the i-th element of the j-th column of $X(t)$. The determinant $D(t)$ can be written by the well-known formula

$$(3)\qquad D(t) = \sum_{j_1 \dots j_n} \varepsilon(j_1, \dots, j_n) x_1^{(j_1)}(t) \dots x_n^{(j_n)}(t),$$

where the indices $j_1, \dots, j_n$ assume the values of all permutations of $1, \dots, n$, $\varepsilon(j_1, \dots, j_n) = 1$ or -1 if $j_1, \dots, j_n$ is an even or an odd permutation, respectively. Applying the Itô formula to (3) we obtain

$$(4)\qquad dD = \sum_{j_1 \dots j_n} \varepsilon(j_1, \dots, j_n) \Big[\sum_{p=1}^{n} x_1^{(j_1)} \dots x_{p-1}^{(j_{p-1})} \cdot dx_p^{(j_p)} x_{p+1}^{(j_{p+1})} \dots x_n^{(j_n)} + \frac{1}{2} \sum_{p,q} x_1^{(j_1)} \dots x_{p-1}^{(j_{p-1})} \cdot dx_p^{(j_p)} x_{p+1}^{(j_{p+1})} \dots x_{q-1}^{(j_{q-1})} dx_q^{(j_q)} x_{q+1}^{(j_{q+1})} \dots x_n^{(j_n)} \, .$$

Due to (1) we obtain

$$dx_p^{(j_p)} dx_q^{(j_q)} = \sum_j \Big(\sum_k B_{pk}^{(j)} x_k^{(j_p)} \sum_{\ell} B_{q\ell}^{(j)} x_{\ell}^{(j_q)} \Big) dt$$

and equation (4) can be rewritten as

$$(5)\qquad dD = \sum_p \det Q^{(p)} + \frac{1}{2} \sum_{p,q,j} \det R^{(p,q,j)} dt,$$

where $Q^{(p)}$ are matrices of the type $n \times n$ defined by

$$Q^{(p)}_{ij} = x^{(j)}_i \text{ if } i \neq p \text{ and } Q^{(p)}_{pj} = dx^{(j)}_p ,$$

$R^{(p,q,j)}$ are matrices of the type $n \times n$ defined by

$$R^{(o,q,j)}_{u,v} = x^{(v)}_u \text{ if } u \neq p \text{ and } u \neq q,$$

$$R^{(p,q,j)}_{p,v} = \sum_k B^{(j)}_{pk} x^{(v)}_k , \quad R^{(p,q,j)}_{q,v} = \sum_k B^{(j)}_{qk} x^{(v)}_k .$$

Equation (5) can be easily transformed (by using well-known properties of determinants) into

$$dD = D(t)\Big(\sum_p A_{pp}dt + \sum_{p,j} B^{(j)}_{pp} dw_j\Big) +$$

$$+ \frac{1}{2} \sum_{p,q,j} (B^{(j)}_{pp} B^{(j)}_{qq} - B^{(j)}_{pq} B^{(j)}_{qp}) D(t)dt$$

which is the same equation as (2).

Conclusion 1. Let the assumptions of Theorem 1 be fulfilled. If $X(t)$ is the fundamental matrix solution of (1), $X(t_o) = I$ (I is the unit matrix) then

$$\text{(6)} \qquad \det X(t) = D(t) = \exp\Big\{\int_{t_o}^{t} \operatorname{tr} A(\tau)d\tau -$$

$$- \frac{1}{2} \sum_j \int_{t_o}^{t} \operatorname{tr}(B^{(j)}(\tau))^2 d\tau + \sum_j \int_{t_o}^{t} \operatorname{tr} B^{(j)}(\tau) dw_j(\tau)\Big\}.$$

The formula for $D(t)$ follows immediately from (2) and the Itô formula.

Conclusion 2. Let the assumptions of Theorem 1 be fulfilled. If $X(t)$ is the fundamental matrix solution of (1), $X(t_o) = I$ then the probability that $X(t)$ is regular for all $t \in \langle t_o, \infty)$ is equal to one.

This conclusion follows directly from formula (6). Conclusion 2 implies that the inverse matrix $X^{-1}(t)$ exists almost everywhere.

Conclusion 3. Let the assumptions of Theorem 1 be fulfilled. If $X(t)$ is the fundamental matrix solution of (1), $X(t_0) = I$ then to every $T > t_0$ and $\alpha \geqq 1$ there exists $c \geqq 0$ such that $E \| X^{-1}(t) \|^{\alpha} \leqq c$ for $t \in \langle 0,T \rangle$.

Proof. If $X^{-1}(t)$ exists then $X^{-1}_{k,\ell} = (-1)^{k+\ell} \det X^{(\ell,k)}/$ $/\det X$ where $X^{(\ell,k)}$ is the submatrix of X corresponding to the element $x_{\ell}^{(k)}$. Since $\det X^{(\ell,k)} = \sum \varepsilon(j_1,\dots,j_{\ell-1},j_{\ell+1},\dots$ $\dots,j_n) \prod_{s \neq \ell} x_s^{(j_s)}$ where $j_1,\dots,j_{\ell-1},j_{\ell+1},\dots,j_n$ are permutations of $1,2,\dots,k-1,k+1,\dots,n$ we can derive an estimate

$$(7) \quad E \left| \frac{\det X^{(\ell,k)}}{\det X} \right|^{\alpha} \leqq ((n-1)!)^{\alpha} \cdot \sum_{j_1,\dots j_{\ell-1},j_{\ell+1},\dots j_n} E \left[\prod_{s \neq \ell} x_s^{(j_s)} \frac{1}{\det X} \right]^{\alpha} \leqq$$

$$\leqq ((n-1)!)^{\alpha-1} \sum_{j_1,\dots j_{\ell-1},j_{\ell+1},\dots j_n} \prod_{s \neq \ell} \sqrt[n]{E \left| x_s^{(j_s)} \right|^{\alpha n}} \sqrt[n]{E \frac{1}{|\det X|^{\alpha n}}}$$

where E is the mathematical expectation. It is proved in [2] that to every $T > t_0$, $\alpha \geqq 1$ there exists $C \geqq 0$ such that $E \| x(t) \|^{\alpha n} \leqq C$ for $t \in \langle t_0,T \rangle$ where $x(t)$ is a solution of (1) fulfilling $\| x(t_0) \| \leqq 1$. Using (6) we obtain that also $E \frac{1}{|\det X|^{\alpha n}} \leqq C_1$ for $t \in \langle 0,T \rangle$. Inequality (7) implies $E \left| \frac{\det X^{(\ell,k)}}{\det X} \right|^{\alpha} \leqq ((n-1)!)^{\alpha} C^{\frac{n-1}{n}} C_1^{\frac{1}{n}}$ and the statement of

Conclusion 3 easily follows.

Theorem 2. Let $A(t)$, $B^{(j)}(t)$, $w_j(t)$, $j = 1,\dots,k$ fulfil the conditions of Theorem 1 and let $\alpha(t)$, $\beta_j(t)$, $j = 1,\dots,k$ be n-dimensional vector functions defined on $\langle 0,\infty)$ such that $\|\alpha(t)\|$, $\|\beta_j(t)\|^2$ are locally integrable. Denote by $X(t)$ the fundamental matrix solution of (1), $X(t_0) = I$. If x_0 is a nonstochastic vector then the process

$$x(t) = X(t)x_0 + X(t)\int_{t_0}^{t} X^{-1}(\tau)(\alpha(\tau) -$$

$$- \sum_j B^{(j)}(\tau)\beta_j(\tau))d\tau + X(t)\int_{t_0}^{t} X^{-1}(\tau) \sum_j \beta_j(\tau)dw_j(\tau)$$

is the solution of the nonhomogeneous Itô equation

$$dx = A(t)xdt + \sum_{j=1}^{k} B^{(j)}(t)xdw_j + \alpha(t)dt + \sum_{j=1}^{k} \beta_j(t)dw_j$$

fulfilling $x(t_0) = x_0$.

Proof. With respect to Conclusion 2 the process $X^{-1}(\tau)$ exists and the integrals converge. Denote $J_1(t) = X(t)x_0$, $J_2(t) = X(t)\int_{t_0}^{t} X^{-1}(\tau)(\alpha(\tau) - \sum_j B^{(j)}(\tau)\beta_j(\tau))d\tau$ and

$J_3(t) = X(t)\int_{t_0}^{t} X^{-1}(\tau) \sum_j \beta_j(\tau)dw_j(\tau)$. The process $J_1(t)$ is evidently the solution of (1) fulfilling $J_1(t_0) = x_0$. Using the Itô formula we obtain that $J_2(t)$ is the solution of

$$dJ_2 = AJ_2\,dt + \sum_j B^{(j)}J_2\,dw_j + (\alpha - \sum_j B^{(j)}\beta_j)dt$$

fulfilling $J_2(t_0) = 0$ and the process $J_3(t)$ is the solution of

$$dJ_3 = AJ_3\, dt + \sum_j B^{(j)} J_3\, dw_j + \sum_j B^{(j)} \beta_j\, dt + \sum_j \beta_j dw_j$$

fulfilling $J_3(t_o) = 0$.

Remark. The theorems and the conclusions are valid even if $A(t)$, $B(t)$, $\alpha(t)$, $\beta_j(t)$ are nonanticipative stochastic processes fulfilling the above conditions with probability 1.

R e f e r e n c e s

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