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TOPOLOGICAL SPACES WITHOUT $\varkappa$-ACCESSIBLE DIAGONAL

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Abstract: Spaces which may replace in factorization situations spaces with $G_{\varkappa}$-diagonal are investigated. Problems in special cases are connected with βN and metrizability of compact spaces.

Key words: $\varkappa$-accessible diagonal, factorization of maps on products, cardinal functions, metrizability.

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The following definition was motivated by results concerning factorization of maps on products of spaces. Some basic facts and ideas may be found in [Hu_1].

Definition. We shall say that a topological space X has a (weakly) $\varkappa$-accessible diagonal if there is a net $\{a_\xi \mid \xi < \varkappa\}$ in $X \times X - \Delta_X$ (weakly) converging to diagonal Δ_X.

The fact that X has not (weakly) $\varkappa$-accessible diagonal is denoted by $\varkappa \in \Delta X$ ($\varkappa \in \overline{\Delta} X$, resp.). Thus $\varkappa \in \Delta X$ (or $\varkappa \in \overline{\Delta} X$) iff for any net $\{a_\xi \mid \xi < \varkappa\}$ in $X \times X - \Delta_X$ there is a cofinal set C in $\varkappa$ and a neighborhood U of Δ_X in $X \times X$ such that $U \cap \{a_\xi \mid \xi \in C\} = \emptyset$ ($\overline{U} \cap \{a_\xi \mid \xi \in C\} = \emptyset$, resp.).

Since $\varkappa \in \Delta X$ iff $\operatorname{cof} \varkappa \in \Delta X$ (the same for $\overline{\Delta} X$), it suffices to restrict a consideration to regular cardinals; for them, $\varkappa \in \Delta X$ ($\varkappa \in \bar{\Delta} X$) iff for any $M \subset X \times X$ with $|M - \Delta_X| = \varkappa$ there is a neighborhood U of Δ_X in $X \times X$ with $|M - U| = \varkappa$ ($|M - \bar{U}| = \varkappa$, resp.).

The spaces without $\varkappa$-accessible diagonal ($\varkappa$ regular) generalize spaces X with $\psi(\Delta_X, X \times X) < \varkappa$ (e.g., if X has a G_δ-diagonal (or $\bar{G}_\delta$-diagonal), then $\omega_1 \in \Delta X$ ($\omega_1 \in \bar{\Delta} X$, resp.)).

E. van Douwen after discussion with the author about spaces X with $\omega_1 \in \Delta X$ (Amsterdam 1975) called them "spaces with small diagonal". In the meantime, the author used in several lectures (also in [Hu_2]) the terms "D-spaces, D_1-spaces" for X with $\omega \in \Delta X$, $\omega_1 \in \Delta X$. In this time we are convinced that the term "spaces without $\varkappa$-accessible diagonal" is more justified.

After stating general results we shall restrict our consideration to the cases $\varkappa = \omega$, $\varkappa = \omega_1$. In the sequel, a topological space always means a Hausdorff one, $\varkappa$ denotes a regular infinite cardinal. We shall omit $X \times X$ in $\psi(\Delta_X, X \times X)$ and similar expressions.

1. Observations. (1) $\psi(\Delta_X) < \varkappa \rightarrow \varkappa \in \Delta X$.

(2) $\chi(\Delta_X) = \psi(\Delta_X) = \varkappa \rightarrow \varkappa \notin \Delta X$.

(3) $\varkappa \in \Delta X \rightarrow \varkappa \notin \{\alpha \mid \alpha = \psi(x) = \chi(x)$ for some $x \in X\}$.

(4) If X is compact then (2) and (3) means: $\varkappa = wX$ or $\varkappa \in \{\psi(x) \mid x \in X\} \rightarrow \varkappa \notin \Delta X$.

(5) $\bar{\Delta} X \subset \Delta X$.

The converse implications in (1) - (4) do not hold even for compact X (for $\varkappa = \omega_0$ one may take βN in (1) and 2^{ω_1} in (4)). In (5) the equality holds if Δ_X has a base of closed neighborhoods, which is true e.g. if all neighborhoods of Δ_X form a uniformity.

Proposition 1. The class of spaces without $\varkappa$-accessible diagonal is hereditary, λ-productive for any $\lambda < \varkappa$, and the property is preserved by taking larger topologies.

Clearly, $2^{\varkappa}$ has $\varkappa$-accessible diagonal and hence, we cannot put $\lambda = \varkappa$ in Proposition 1.

In the sequel we shall use the term " $\varkappa$-compactness" in the following meaning: any subset of cardinality at least $\varkappa$ has an accumulation point (i.e., any closed discrete subspace is of cardinality less than $\varkappa$). Any $\varkappa$-compact spaces is pseudo- $\varkappa$-compact in the sense of Isbell. The concept corresponding to pseudo-($\varkappa, \lambda$)-compactness is ($\varkappa, \lambda$)-compactness here: any subset A of cardinality at least $\varkappa$ has a λ-accumulation point x (i.e., for any neighborhood U of x, $|U \cap A| \geqq \lambda$).

Theorem 1. If X is a $\varkappa$-compact space, then it has not $\varkappa$-accessible diagonal iff any continuous $f: \prod_I X_i \longrightarrow X$, $\prod_I X_i$ $\varkappa$-compact, depends on less than $\varkappa$ coordinates.

Proof. Suppose first that $\varkappa \in \Delta X$, $\prod_I X_i$ is $\varkappa$-compact, $f: \prod_I X_i \longrightarrow X$ is continuous not depending on less than $\varkappa$ coordinates. Then $|\{i \in I \mid fx_i \neq fy_i$ for some x_i, $y_i \in \prod_I X_i$ with $pr_{I-(i)}x_i = pr_{I-(i)}y_i\}| \geqq \varkappa$ (denote this subset of I by J). There are a neighborhood U of Δ_X and a $J' \subset J$ with $|J'| = \varkappa$, $\{\langle fx_i, fy_i\rangle \mid i \in J'\} \cap U = \emptyset$. Let x be an accumulation point of $\{x_i \mid i \in J'\}$ in $\prod_I X_i$, V its canonical neighborhood such that $f(V) \times f(V) \subset U$. There is an $i \in J'$ such that $x_i \in V$, $pr_i(V) = X_i$; consequently, $y_i \in V$, and $\langle fx_i, fy_i\rangle \in U$, which is a contradiction.

Suppose now that $\varkappa \notin \Delta X$, i.e., there exists a set $A = \{\langle x_\xi, y_\xi\rangle \mid \xi < \varkappa\}$ in $X \times X - \Delta_X$ converging to Δ_X. Put X_{-1} to be the set $A \cup \Delta_X$ with the following topology: A is an open discrete subspace of X_{-1}, neighborhoods of points from Δ_X are traces on X_{-1} of their neighborhoods in $X \times X$. It is almost self-evident that X_{-1} is $\varkappa$-compact. Now, $X_{-1} \times 2^{\varkappa}$ is $\varkappa$-compact and the following map $f: X_{-1} \times 2^{\varkappa} \longrightarrow X$ is continuous and does not depend on less than $\varkappa$ coordinates:

$$f(\langle x_\xi, y_\xi\rangle, \{k_\xi\}_{\xi<\varkappa}) = \begin{cases} x_\xi & \text{if } k_\xi = 0, \\ y_\xi & \text{if } k_\xi = 1, \end{cases}$$

$$f(\langle x, x\rangle, \{k_\xi\}_{\xi<\varkappa}) = x.$$

In the first part of the proof, $\varkappa$-compactness of X was not used, but we must realize that by investigating factorizations of f we are interested only in $f(\prod_I X_i)$. Hence, the restriction on X in Theorem 1 is no restriction if we want $\prod_I X_i$ to be $\varkappa$-compact.

The most general condition which may be posed on $\prod_I X_i$ in the above factorization theorems is pseudo-$\varkappa$-compactness ([NU] for uncountable $\varkappa$, [Hu_1] for $\varkappa = \omega$). In that case the situation is more complicated, and we know only the following result:

Theorem 2. Each of the following conditions implies the next one:

(1) X has not weakly $\varkappa$-accessible diagonal (i.e., $\varkappa \in \in \bar{\Delta} X$).

(2) Any continuous $f: \prod_I X_i \longrightarrow X$, $\prod_I X_i$ pseudo-$\varkappa$-compact, depends on less than $\varkappa$ coordinates.

(3) X has not $\varkappa$-accessible diagonal (i.e., $\varkappa \in \Delta X$).

Proof is similar to the preceding one. (See [Hu_1] for details of (1) $\Longrightarrow$ (2).) To prove (2) $\Longrightarrow$ (3), one may take in the proof of Theorem 1 the subspace $A \cup (\bar{A} \cap \Delta_X)$ of X_{-1} as the new X_{-1}; if A converges to Δ_X, then this new X_{-1} is pseudo-$\varkappa$-compact. The remaining procedure is the same.

The implication (2) $\Longrightarrow$ (1) is not true in general. Clearly, if $\Delta X = \bar{\Delta} X$, then all the three conditions are equivalent. We do not know whether (3) $\Longrightarrow$ (2) (in fact, we do not know any example of a pseudo-$\varkappa$-compact space X with $\varkappa \in \Delta X - \bar{\Delta} X$).

In the second part of the proof of Theorem 1 we used the index set of cardinality $\varkappa$; in such a case we may prove more:

Theorem 3. If X has not $\varkappa$-accessible diagonal, then any continuous map $f: Y \longrightarrow X$, where Y is a $(\varkappa, \varkappa)$-compact subspace of a $\varkappa$-fold product $\prod_{\varkappa} X_\xi$, depends on less than $\varkappa$ coordinates.

Proof. Suppose that an f from our theorem does not depend on less than $\varkappa$ coordinates. Then we can find points x_ξ, y_ξ in Y for $\xi < \varkappa$ with $pr_\eta\, x_\xi = pr_\eta\, y_\xi$ for all $\eta \in \xi$

and $fx_\xi \neq fy_\xi$. Thus for a cofinal C in $\varkappa$ and a neighborhood U of Δ_X we have $U \cap \{\langle fx_\xi, fy_\xi \rangle \mid \xi \in C\} = \emptyset$. Let $x \in Y$ be a $\varkappa$-accumulation point of $\{x_\xi \mid \xi \in C\}$, V its canonical neighborhood such that $f(V \cap Y) \times f(V \cap Y) \subset U$. There is a $\xi \in C$ such that $x_\xi \in V$ and $pr_\eta V = X_\eta$ provided $\eta \geq \xi$; hence, $y_\xi \in V$ - a contradiction.

From the results of the second section we shall see that Theorem 3 is not valid for more than $\varkappa$-fold products; if $2^\omega = \omega_1$, $X = \beta N$, then X may be embedded into $[0,1]^{\omega_1}$ and the identity 1_X does not depend on countably many coordinates.

It is not difficult to show that if X is compact, then $\varkappa \in \Delta X$ iff $X \times X - \Delta_X$ is $(\varkappa, \varkappa)$-compact.

At the end of the first part we shall remark that if X is a scattered compact space, then $\Delta X = [|X|^+, \longrightarrow[$. Indeed, if A is an infinite subset of X, x_0 is a complete accumulation point of A with the least order, U is a closed neighborhood of x_0 with $U \cap \{x \mid \text{order of } x \geq \text{order of } x_0\} = (x_0)$, then $U \cap A$ converges as a well-ordered net of type $|A|$ to x_0.

2. In this part we shall be interested in the case $\varkappa = \omega$. The earlier results have now simpler formulations, mainly for compact spaces:

<u>Theorem 4</u>. The following are equivalent for a compact space X:

(1) X has not ω-accessible diagonal.

(2) $X \times X - \Delta_X$ is countably compact.

(3) Any continuous map $f: \prod_I X_i \longrightarrow X$, $\prod_I X_i$ pseudocompact (or compact), depends on finitely many coordinates.

(4) Any continuous map $f: Y \longrightarrow X$, where Y is a countably compact subspace of a countable product, depends on finitely many coordinates.

If X has not ω-accessible diagonal, then it has no convergent nontrivial sequence and, hence, nondiscrete metrizable spaces, infinite dyadic compact spaces, infinite Eberlein compact spaces, infinite scattered compact spaces, infinite supercompact spaces [DM] have ω-accessible diagonal. The space βN with doubled N has ω-accessible diagonal and no convergent nontrivial sequence.

It seems that for compact spaces, only finite ones have not ω-accessible diagonal. The next result shows that there are many nontrivial compact spaces without ω-accessible diagonal. The result appeared in [Hu_1].

Theorem 5. If any countable discrete set in a completely regular space X is C^*-embedded in X, then X has not weakly ω-accessible diagonal.

Proof. Suppose $\{\langle x_n, y_n \rangle\}_\omega \subset X \times X - \Delta_X$. If one of the points x_n, y_n appears infinitely many times, e.g. all x_n equal to x_o, then for suitable neighborhoods U, V of x_o, $\overline{V} \subset \operatorname{int} U$, $\overline{U}$ misses infinitely many of y_n's, the set $X \times (X - \overline{V}) \cup (U \times U)$ is a neighborhood of Δ_X the closure of which misses infinitely many of $\langle x_n, y_n \rangle$'s. In the other case we can choose a subsequence $\{\langle u_n, v_n \rangle\}$ of $\{\langle x_n, y_n \rangle\}$ such that the sets $\{u_n\} = A$, $\{v_n\} = B$ are disjoint and discrete in X; moreover, we may suppose that $A \cup B$ is discrete (if $B \subset \bar{A}$, then there is infinite $A_1 \subset A$ with $\bar{A}_1 \cap B = \emptyset$ because A is C^*-embedded). Then $\bar{A}^{\beta X} \cap \bar{B}^{\beta X} = \emptyset$ and, consequently, $\bar{A}^{\beta X} \times \bar{B}^{\beta X}$ is separated from $\Delta_{\beta X}$ in $\beta X \times \beta X$.

It does not suffice to suppose that any countable subset of X contains a C^*-embedded infinite subset: put X to be βN with doubled N.

There are compact spaces without ω-accessible diagonal containing a set having no C^*-embedded (in X) infinite subset (e.g. the compactification of N from the Example 5.22 [W] obtained as a quotient of βN along an idempotent permutation).

Corollary. (1) If D is a discrete space, then no subspace of βD has weakly ω-accessible diagonal.

(2) No extremally disconnected space has ω-accessible diagonal.

In (2) we may put basically disconnected or moreover F-spaces instead of extremally disconnected spaces. The class of spaces without ω-accessible diagonal is bigger than that of F-spaces because the former class is finitely productive (or use the example just before Corollary). We do not know whether any compact space without ω-accessible diagonal can be embedded into a countable (hence finite) product of F-spaces.

Theorem 6. If X is an infinite compact space without ω-accessible diagonal, then $|X| \geq 2^{\omega_1}$.

Proof follows from a theorem of Čech and Pospíšil because X contains an infinite compact subspace Y without isolated points (since X is not scattered) and $\chi(x,Y) \geq \omega_1$

for any $x \in Y$.

As follows from results in [MŠ], the last Theorem can be improved under MA: If X is an infinite compact space without ω-accessible diagonal, then $|X| \geq 2^{2^{\omega}}$.

It is an open problem whether there exists a compact space of cardinality 2^{ω} without ω-accessible diagonal. We are not sure that one can use the Fedorčuk's construction of a compact space of cardinality 2^{ω} and without convergent nontrivial sequences.

At the end of this part we want to stress the fact that if a compact space without ω-accessible diagonal is embedded into a countable product, then it can be embedded into a finite subproduct. This result is related to a recent deeper but more special result by V. Malyhin (unpublished): If βN is embedded into a countable product then it can be embedded into one member of the product.

3. The case $\varkappa = \omega_1$ has in a sense "opposite" problems than the countable case. We do not know whether there are nonmetrizable compact spaces without ω_1-accessible diagonal (or pseudo-ω_1-compact spaces without both G_δ-diagonal and ω_1-accessible diagonal). This is important to know because up to now we do not know whether the factorization result in Theorem 1 is a generalization of the known result (the range has G_δ-diagonal).

E. van Douwen proved that any compact linearly ordered space without ω_1-accessible diagonal is metrizable, and D. Lutzer improved this for Lindelöf instead of compact (oral communications).

Under CH we are able to prove similar results:

Theorem 7. (CH) A compact space is metrizable iff it has not ω_1-accessible diagonal and one of the following conditions holds:

(a) $dX = \omega$

(b) $tX = \omega$

(c) $wX \leqq 2^{\omega}$ or $|X| \leqq 2^{\omega}$

(d) $|C(X)| \leqq 2^{\omega}$

Proof. Suppose X is a compact space without ω_1-accessible diagonal. Then (c) clearly implies metrizability of X. Since (a) $\Longrightarrow$ (d), it will suffice to prove that (d) implies metrizability and (b) $\Longrightarrow$ (a). Under (d), $X \hookrightarrow [0,1]^{2^{\omega}}$, thus by Theorem 3, $X \hookrightarrow [0,1]^{\omega}$. Suppose now that $tX = \omega$. If X is not separable, then there is a set $A = \{x_\xi \mid \xi < \omega_1\}$ such that $x_\eta \notin \overline{\{x_\xi \mid \xi < \eta\}}$ for all $\eta < \omega_1$. Since $tX = \omega$, we have $\overline{A} = \bigcup_{\eta<\omega_1} \overline{\{x_\xi \mid \xi < \eta\}}$ and, by preceding considerations, all $\overline{\{x_\xi \mid \xi < \eta\}}$ are metrizable. Hence $|\overline{A}| \leqq 2^{\omega}$ and $\overline{A}$ is metrizable, hence hereditarily separable - a contradiction.

Questions. (1) Is there a compact nonmetrizable space without ω_1-accessible diagonal? Under CH, this question is equivalent to the following one: Is there a nonmetrizable compactification X of the discrete space ω_1 such that X has not ω_1-accessible diagonal? (Any separable subspace of X must be metrizable.)

If one can prove that any compact space without ω_1-accessible diagonal is first countable, then it is metrizable without using CH ($X \times X$ has not ω_1-accessible diagonal, the quotient of $X \times X$ along Δ_X has not ω_1-accessible diagonal).

(2) Has βN always a convergent net of type ω_1? Equivalently: Is there always an ultrafilter on N that can be expressed as a union of strictly increasing family of ω_1 filters? (Our conjecture: it is consistent with ZFC that there is no such ultrafilter on N (perhaps under MA + $\neg$ CH ?)).

At the end we want to remark that I. Juhász has recently come to a similar problem: Is there a compact space X with $\chi(X) > \omega$ and with no convergent nontrivial net of type ω_1 ? This question is related to the problem of omitting ω_2 by compact spaces (see [J_2]).

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